Finite Difference Solutions to the Lees-Dorodnitsyn Compressible Boundary Layer Equations

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May 12, 2020

2.29 Final Project

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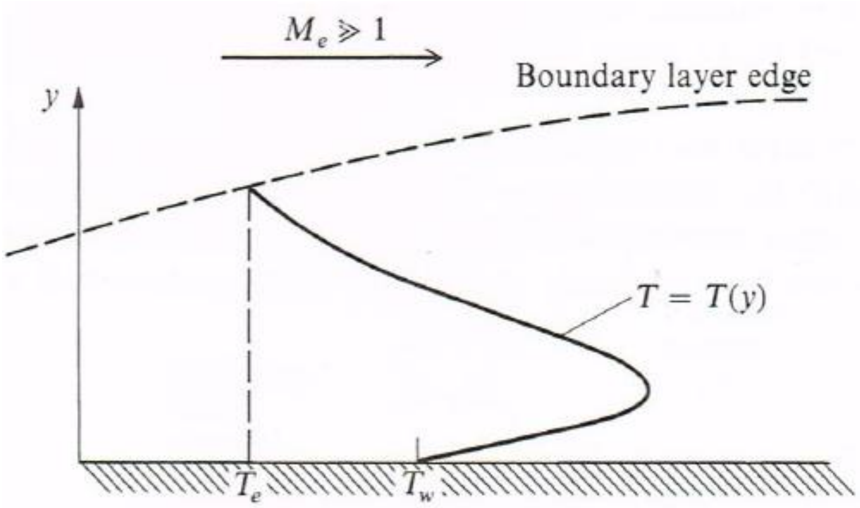
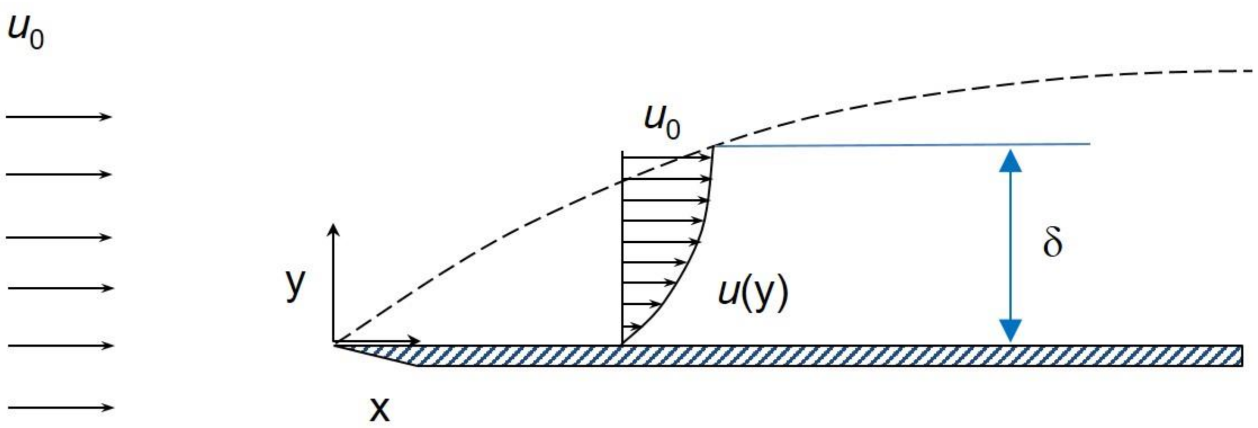
Motivation

- Compressible, and specifically hypersonic, boundary layers can have complex velocity and temperature profiles
- Lees-Doroditsyn transformation results in decreased need for boundary layer scaling
- Boundary layer analysis is necessary to assess surface heating, skin friction, and external flow displacement effects
- Developing robust methods for modeling hypersonic boundary layers





TSL L-D Coordinate Transformation

$$\xi = \int_0^x \rho_e u_e \mu_e \, dx \qquad \eta = \frac{u_e}{\sqrt{2\xi}} \int_0^y \rho \, dy$$

$$(Cf'')' + ff'' = \frac{2\xi}{u_e}\left[(f')^2 - \frac{\rho_e}{\rho}\right]\frac{du_e}{d\xi} + 2\xi\left(f'\frac{\partial f'}{\partial \xi} - \frac{\partial f}{\partial \xi}f''\right)$$

$$\frac{\partial p}{\partial \eta} = 0$$

$$\left(\frac{C}{Pr}g'\right)' + fg' = 2\xi\left[f'\frac{\partial g}{\partial \xi} + \frac{f'g}{h_e}\frac{\partial h_e}{\partial \xi} - g'\frac{\partial f}{\partial \xi} + \frac{\rho_e u_e}{\rho h_e}f'\frac{du_e}{d\xi}\right] - C\frac{u_e^2}{h_e}(f'')^2$$

$$C = \frac{\rho\mu}{\rho_e\mu_e}, g = \frac{h}{h_e}, \text{ and, } f' = \frac{u}{u_e}$$

Lees-Dorodnitsyn Equations

- For 2D, Laminar, Compressible Boundary Layers
- Effectively Parabolic
- 5th Order system of coupled equations
- Introduce F, U, S, H, and Q to simplify into 5 - 1st order equations
    - $F = f = \frac{\varphi}{\sqrt{2\xi}}$ (stream function relation)
    - $U = \frac{\partial f}{\partial \eta} = \frac{u}{u_e}$ (velocity relation)
    - $S = \frac{\partial U}{\partial \eta} = \frac{\partial^2 f}{\partial \eta^2}$ (shear stress relation)
    - $H = g = \frac{h}{h_e} = \frac{T}{T_e} = \frac{\rho_e}{\rho}$ (enthalpy/temperature relation)
    - $Q = \frac{\partial g}{\partial \eta} = \frac{\partial H}{\partial \eta}$ (heat transfer relation)

Flow Parameters

FLUID MODELS

- Viscosity Sutherland's Law
  - $\mu = \frac{\mu_{ref}\left(\frac{T}{T_{ref}}\right)^{\frac{3}{2}}(T_{ref}+S_{ref})}{T+S_{ref}}$
- Calorically Perfect Gas
  - $h = c_p T$
  - $\gamma = 1.4$
- Constant Pr
  - 0.71 (Van Driest), 0.75, or 1
- Isothermal wall

SPECIFIED PARAMETERS

- ξ_{max}
- η_e
- $T_e \rightarrow h_e$
- $Ma_e(\xi) \rightarrow u_e$
- $\frac{h_w}{h_e} \rightarrow h_w$
- $P_e = \rho_e$

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Boundary Conditions

- 5 Required for a 5th Order system:
- Wall:
  - $F_{wall} = 0$
  - $U_{wall} = 0$
  - $G_{wall} = G_{specified}$
- Edge:
  - $U_{edge} = 1$
  - $G_{edge} = 1$
- Initial condition @ $\xi = 0$
  - 1D Flat plate similarity solution (calculated with Newton-Raphson method)

Residual Form

$$\mathrm{F}' - U = 0$$

$$U' - S = 0$$

$$(CS)' + FS + \frac{2\xi}{u_e}[H - (U)^2]\frac{du_e}{d\xi} + 2\xi\left(\frac{\partial F}{\partial \xi}S - U\frac{\partial U}{\partial \xi}\right) = 0$$

$$\left(\frac{C}{Pr}Q\right)' + FQ - 2\xi\left[U\frac{\partial H}{\partial \xi} + \frac{UH}{h_e}\frac{\partial h_e}{\partial \xi} - Q\frac{\partial F}{\partial \xi} + \frac{Hu_e}{h_e}U\frac{du_e}{d\xi}\right] + C\frac{u_e^2}{h_e}(S)^2 = 0$$

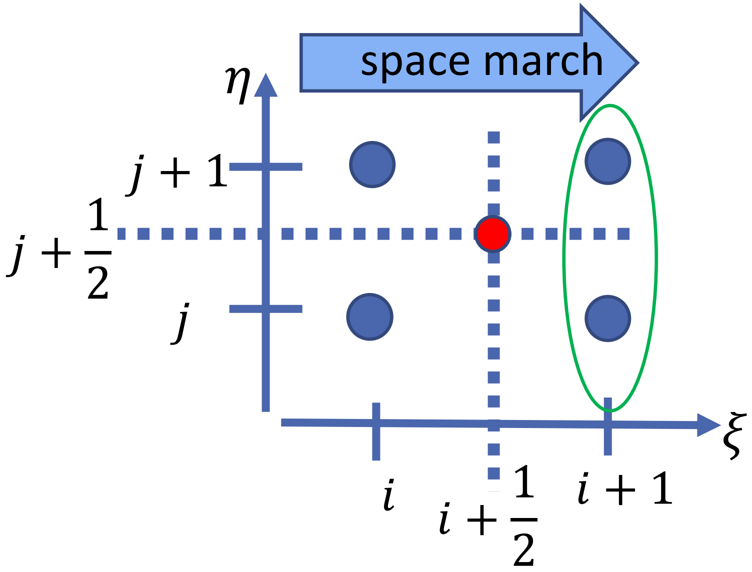
$$H' - Q = 0$$

F, U, S, H, Q are unknowns

Finite Difference Stencil

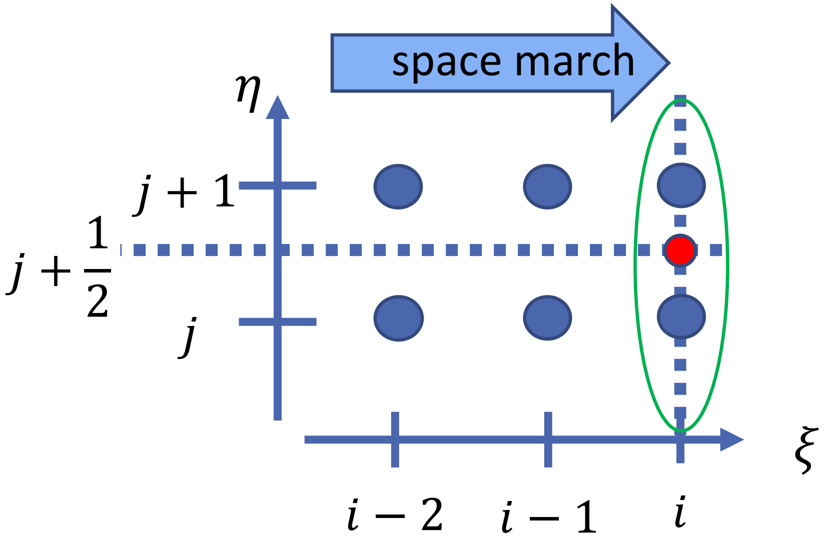
BOX SCHEME

- Space-march in ξ direction
- 2nd Order
- Centered Difference in both directions
- Similar to Crank Nicholson

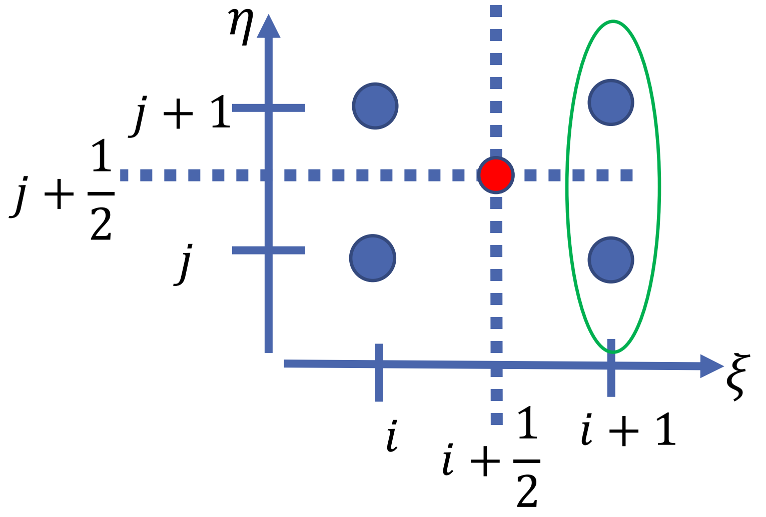


3-POINT BACKWARD DIFFERENCE

- Space-march in ξ direction
- 2nd Order
- Centered Difference in η
- 3-pt Backward Difference in ξ



Box Stencil

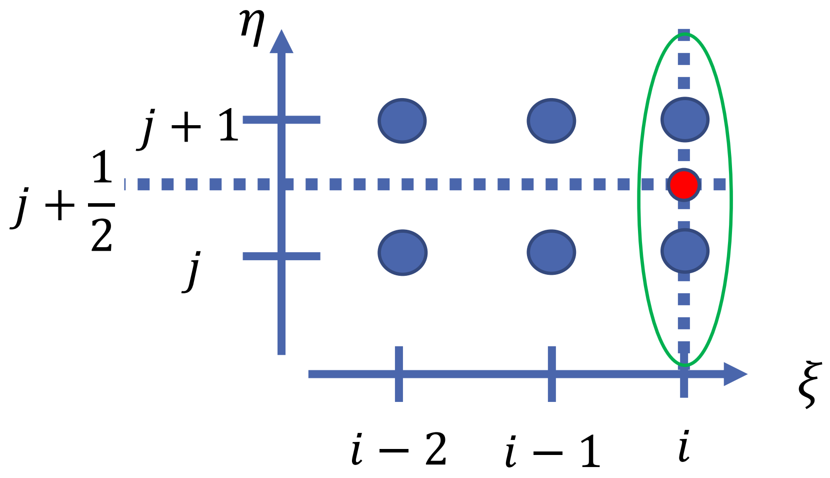


$$X = X_{i+\frac{1}{2}}^{j+\frac{1}{2}} = \frac{1}{4}\left(X_i^j + X_i^{j+1} + X_{i+1}^j + X_{i+1}^{j+1}\right)$$

$$X' = {X'}_{i+\frac{1}{2}}^{j+\frac{1}{2}} = \frac{\left(X_{i+\frac{1}{2}}^{j+1} - X_{i+\frac{1}{2}}^{j}\right)}{\Delta\eta} = \frac{\left(\frac{1}{2}\left(X_{i+1}^{j+1} + X_i^{j+1}\right) - \frac{1}{2}\left(X_{i+1}^{j} + X_i^{j}\right)\right)}{\Delta\eta}$$

$$\frac{\partial X}{\partial \xi} = \left.\frac{\partial X}{\partial \xi}\right|_{i+\frac{1}{2}}^{j+\frac{1}{2}} = \frac{\left(X_{i+1}^{j+\frac{1}{2}} - X_i^{j+\frac{1}{2}}\right)}{\Delta\xi} = \frac{\left(\frac{1}{2}\left(X_{i+1}^{j+1} + X_{i+1}^{j}\right) - \frac{1}{2}\left(X_i^{j+1} + X_i^{j}\right)\right)}{\Delta\xi}$$

3pt - Backward Stencil

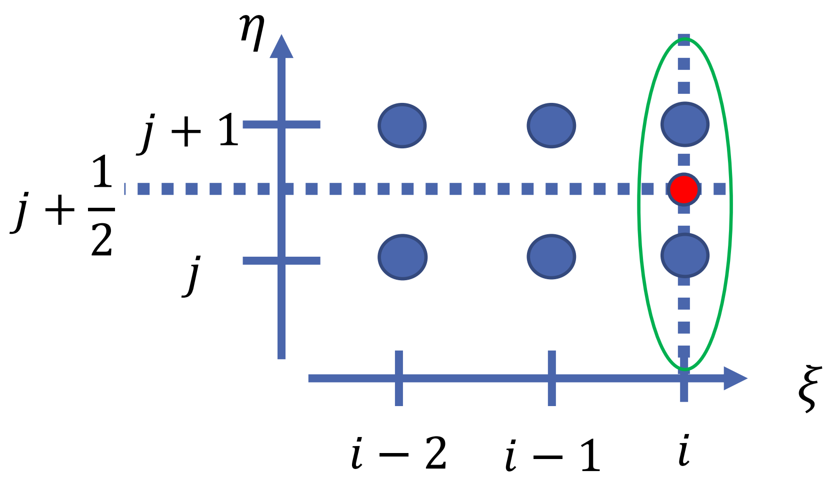


$$X = X_i^{j+\frac{1}{2}} = \frac{1}{2}\left(X_i^j + X_i^{j+1}\right)$$

$$X' = X'^{j+\frac{1}{2}}_i = \frac{\left(X_i^{j+1} - X_i^j\right)}{\Delta\eta}$$

$$\frac{\partial X}{\partial \xi} = \frac{\partial X}{\partial \xi}\bigg|_i^{j+\frac{1}{2}} = \frac{\left(\frac{3}{2}X_i^{j+\frac{1}{2}} - 2X_{i-1}^{j+\frac{1}{2}} + \frac{1}{2}X_{i-2}^{j+\frac{1}{2}}\right)}{2\Delta\xi} = \frac{\left(\frac{3}{2}\left(X_i^{j+1} + X_i^j\right) - 2\left(X_{i-1}^{j+1} + X_{i-1}^j\right) + \frac{1}{2}\left(X_{i-1}^{j+1} + X_{i-1}^j\right)\right)}{4\Delta\xi}$$

3pt - Backward Stencil



Finite difference approximations:

$$X = X_i^{j+\frac{1}{2}} = \frac{1}{2}\left(X_i^j + X_i^{j+1}\right)$$

$$X' = X'^{j+\frac{1}{2}}_i = \frac{\left(X_i^{j+1} - X_i^j\right)}{\Delta\eta}$$

$$\frac{\partial X}{\partial \xi} = \left.\frac{\partial X}{\partial \xi}\right|_i^{j+\frac{1}{2}} = \frac{\left(\frac{3}{2}X_i^{j+\frac{1}{2}} - 2X_{i-1}^{j+\frac{1}{2}} + \frac{1}{2}X_{i-2}^{j+\frac{1}{2}}\right)}{2\Delta\xi} = \frac{\left(\frac{3}{2}\left(X_i^{j+1} + X_i^j\right) - 2\left(X_{i-1}^{j+1} + X_{i-1}^j\right) + \frac{1}{2}\left(X_{i-1}^{j+1} + X_{i-1}^j\right)\right)}{4\Delta\xi}$$

Analytical Evaluations:

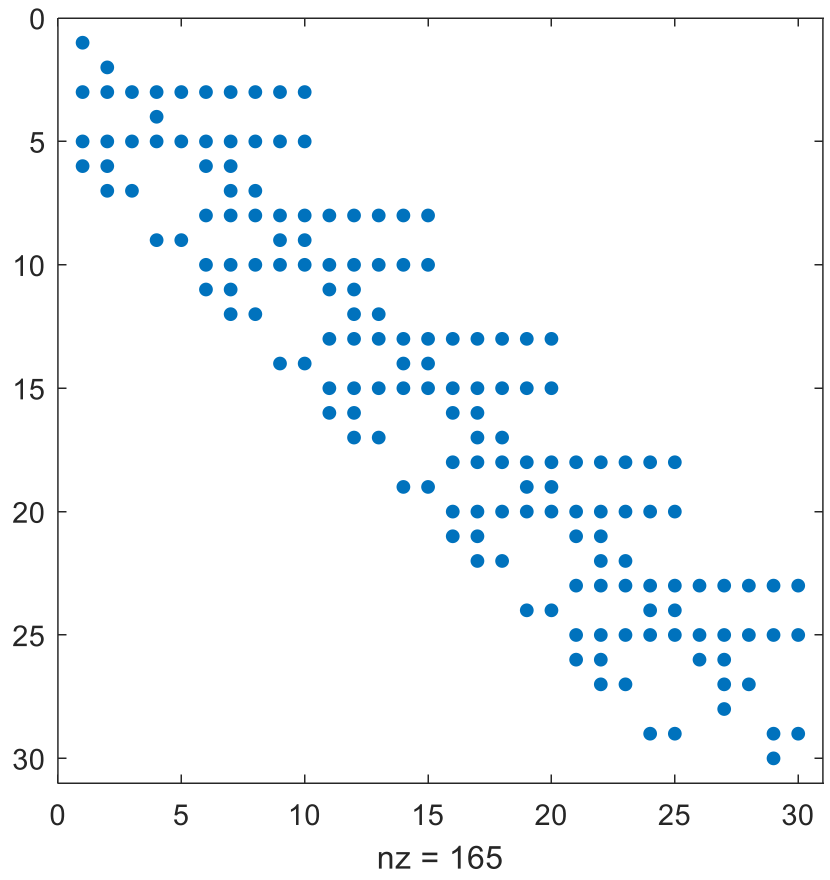
$$\xi = \xi_i$$
$$u_e = u_{e_i} = u_e(\xi_i)$$
$$h_e = h_{e_i} = h_e(\xi_i)$$

$$\frac{\partial h_e}{\partial \xi} = \left.\frac{\partial h_e}{\partial \xi}\right|_i = \frac{\partial h_e}{\partial \xi}(\xi_i)$$

$$\frac{\partial u_e}{\partial \xi} = \left.\frac{\partial u_e}{\partial \xi}\right|_i = \frac{\partial u_e}{\partial \xi}(\xi_i)$$

Newton-Raphson Solver

- Necessary for Non-linear system
- 10 – Diagonal Sparse Matrix
- Analytically calculated jacobian
- Utilized MATLAB built in matrix solver
- Quadratic convergence
- Convergence criteria based on magnitude of max residual



Matrix pattern

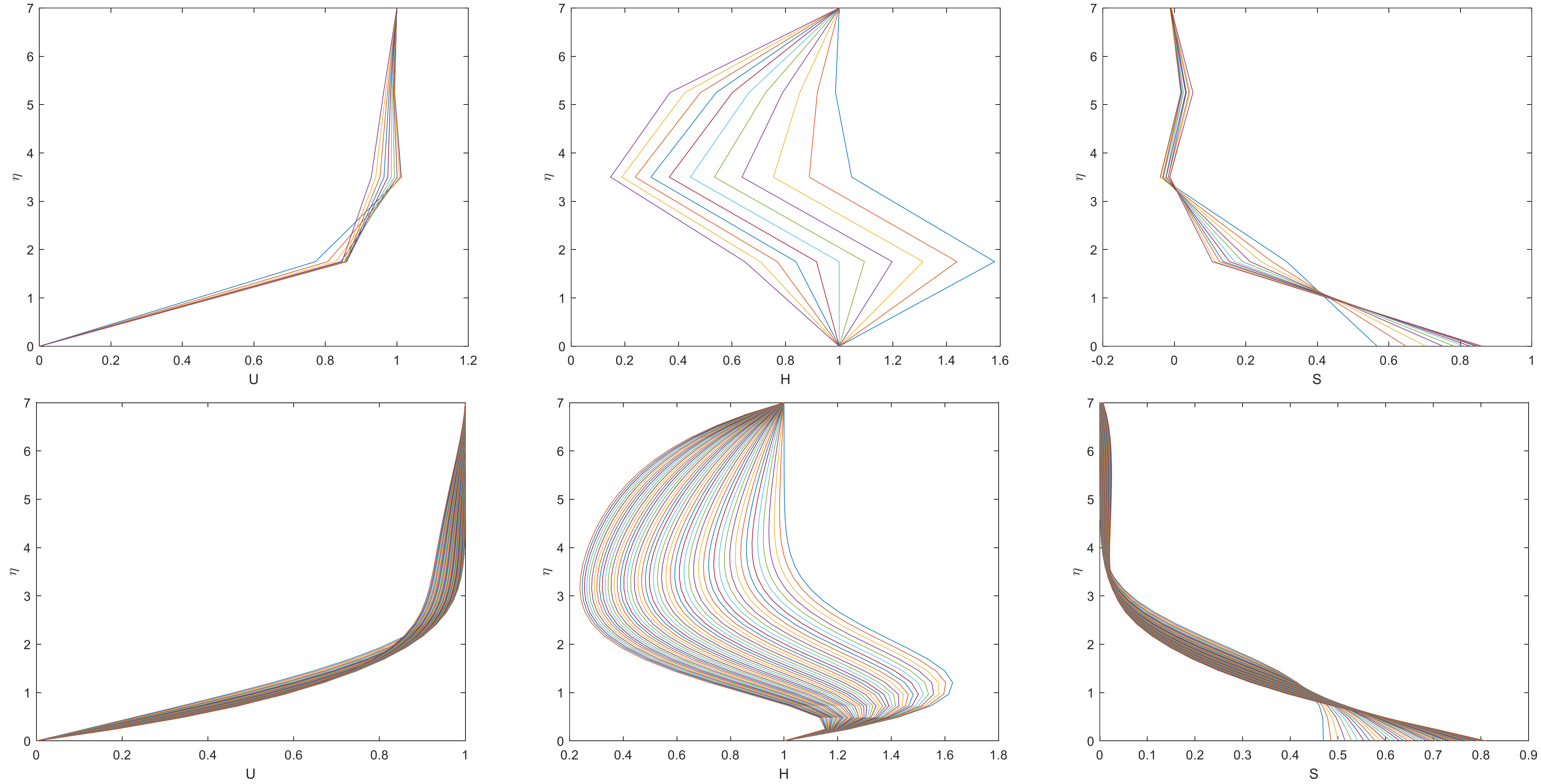
Newton vs. MATLAB FSolve

- Compared the Newton-Raphson Method against MATLAB's nonlinear solver
- Fsolve is a quasi-Newton method
- Assumed both solution methods had similar accuracies

	Newton-Raphson	MATLAB FSolve
Runtime	6-7 sec	116 sec
# of Iterations	3-4	2
Implementation Complexity	~650 lines	~125 lines
# of Function Evaluations	525	243

(Solved for 30 x 30 system)

Selected Results

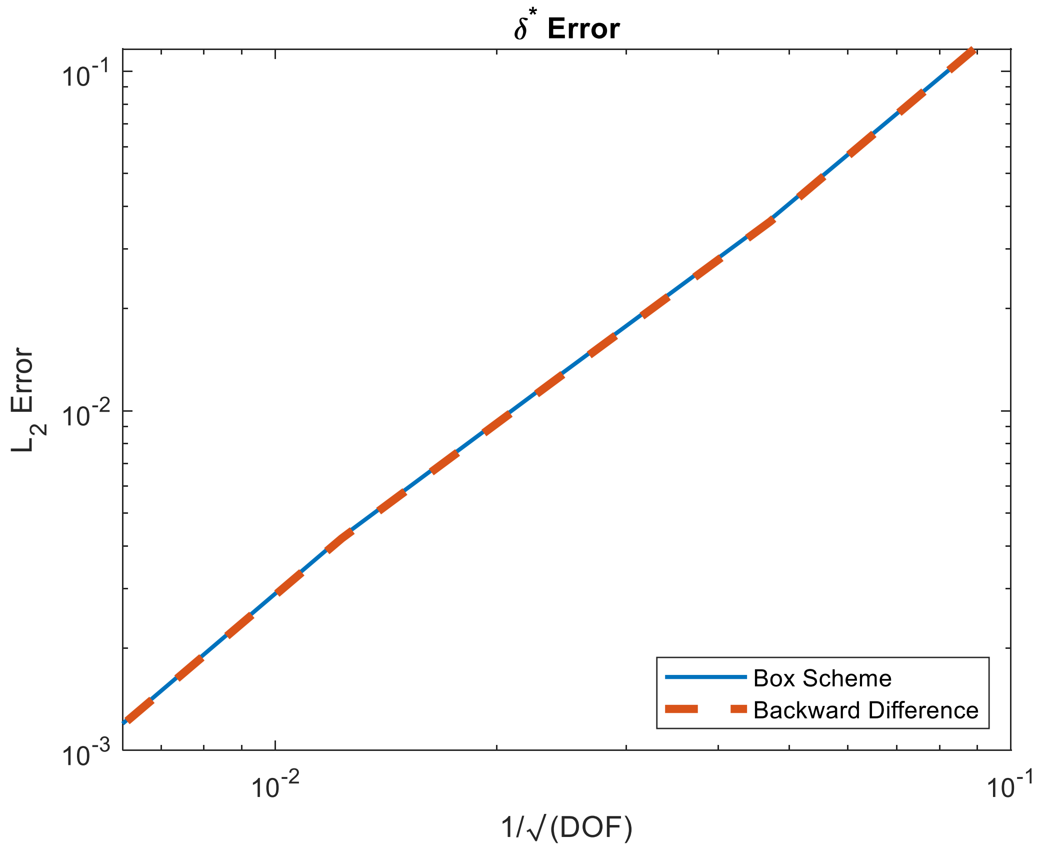


Error Analysis

- L2 Error Analysis of $\delta^*(\xi)$, displacement thickness

$$\delta^* = \int_0^{\eta_e} \left(1 - \frac{\rho}{\rho_e} U\right) \frac{\partial y}{\partial \eta}\, d\eta = \int_0^{\eta_e} \left(1 - \frac{\rho}{\rho_e} U\right) \frac{H\sqrt{2\xi}}{u_e \rho_e}\, d\eta$$

- Compared to solution with fine resolution
- Roughly 2nd order convergence, as expected
- Comparable performance of Box Scheme and 3-pt scheme



Conclusions

- Newton-Raphson (in this case) is significantly preferable to FSolve
- Stability is not a significant concern for the backwards difference scheme
- Oscillations are not present in the box scheme
- There is no clear advantage to using the box scheme instead of the 3-pt backwards difference
- Further sampling of the solution space may be necessary to determine overall behavior of solution methods

Thank you

Questions?

Appendix

Analytical Evaluations:

$$\xi = \xi_{i+\frac{1}{2}}$$
$$u_e = u_{e_{i+\frac{1}{2}}} = u_e\left(\xi_{i+\frac{1}{2}}\right)$$
$$h_e = h_{e_{i+\frac{1}{2}}} = h_e\left(\xi_{i+\frac{1}{2}}\right)$$

$$\frac{\partial h_e}{\partial \xi} = \left.\frac{\partial h_e}{\partial \xi}\right|_{i+\frac{1}{2}} = \frac{\partial h_e}{\partial \xi}\left(\xi_{i+\frac{1}{2}}\right)$$
$$\frac{\partial u_e}{\partial \xi} = \left.\frac{\partial u_e}{\partial \xi}\right|_{i+\frac{1}{2}} = \frac{\partial u_e}{\partial \xi}\left(\xi_{i+\frac{1}{2}}\right)$$

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$$\xi = \xi_i$$
$$u_e = u_{e_i} = u_e(\xi_i)$$
$$h_e = h_{e_i} = h_e(\xi_i)$$

$$\frac{\partial h_e}{\partial \xi} = \left.\frac{\partial h_e}{\partial \xi}\right|_{i} = \frac{\partial h_e}{\partial \xi}(\xi_i)$$
$$\frac{\partial u_e}{\partial \xi} = \left.\frac{\partial u_e}{\partial \xi}\right|_{i} = \frac{\partial u_e}{\partial \xi}(\xi_i)$$

Chapman Rubesin factor

- $C = C(g)$
- $C_{i+\frac{1}{2}}^{j+\frac{1}{2}} = C\left(g_{i+\frac{1}{2}}^{j+\frac{1}{2}}\right)$
- $C_{i+\frac{1}{2}}^{j+\frac{1}{2}} = \frac{1}{4}\left[C\left(g_i^j\right)+C\left(g_{i+1}^j\right)+C\left(g_i^{j+1}\right)+C\left(g_{i+1}^{j+1}\right)\right]$

Similarity Equations

<u>2 - Coupled 2nd and 3rd Order Diff. Eqns</u>

$$(Cf'')' + ff'' = 0$$

$$\left(\frac{C}{Pr}g'\right)' + fg' + C\frac{u_e^2}{h_e}(f'')^2 = 0$$

$$f = F$$

$$g = H$$

<u>5 - Coupled 1st Order ODEs (Drela Notation)</u>

$$\mathrm{F}' = U$$

$$U' = S$$

$$(CS)' + FS = 0$$

$$H' = Q$$

$$\left(\frac{C}{Pr}Q\right)' + FQ + C\frac{u_e^2}{h_e}(S)^2 = 0$$

Where $\phi' = \frac{\partial\phi}{\partial\eta}$