Dynamic, Data-Driven Reduced-Order Models

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Reduced Order Modelling

(ROM)

- Cheaper alternative to full-model solve

- Built **offline**, used **online**

Peherstorfer (2016)

Dynamic Mode Decomposition (DMD) (1)

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k)$$

Discrete system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$$

Approximate, discrete, locally linear dynamical system

Kutz, et. al

Dynamic Mode Decomposition (DMD) (2)

Tall, skinny snapshot matrices

$$\mathbf{X} = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{m-1} \\ | & | & & | \end{bmatrix},$$

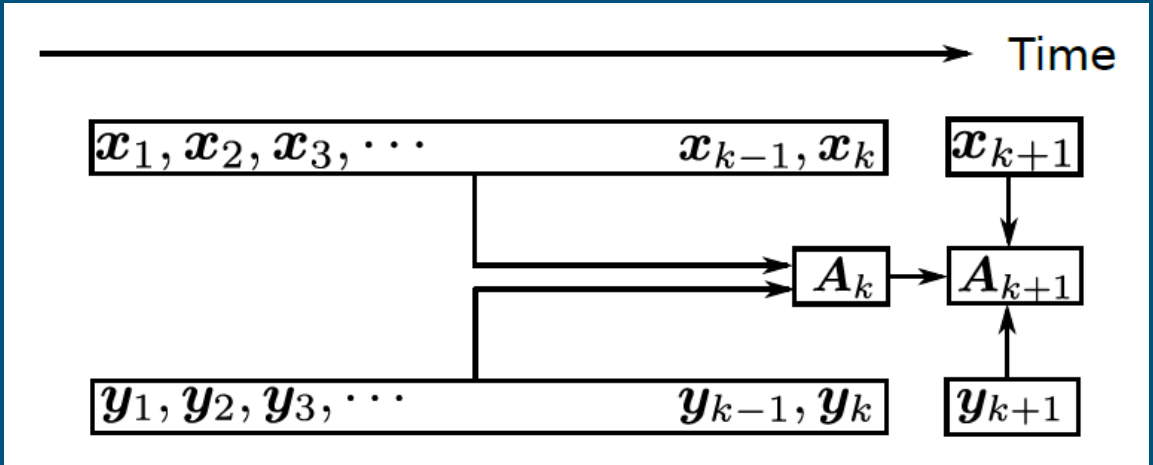
$$\mathbf{X}' = \begin{bmatrix} | & | & & | \\ \mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_{m} \\ | & | & & | \end{bmatrix}.$$

$$||\mathbf{X}' - \mathbf{A}\mathbf{X}||_F$$

$$\mathbf{A} = \mathbf{X}'\mathbf{X}^{\dagger}$$

Kutz, et. al

Online DMD



Zhang, et al. 2017

Online DMD

(1)

Rank-1 Updates

$$\boldsymbol{P}_k = (\boldsymbol{X}_k \boldsymbol{X}_k^T)^{-1}$$

Is $X_k X_k^T$ invertible?

Zhang, et al. 2017

Online DMD

(1)

Rank-1 Updates

$$\boldsymbol{P}_k = (\boldsymbol{X}_k \boldsymbol{X}_k^T)^{-1} \qquad \gamma_{k+1} = \frac{1}{1 + \boldsymbol{x}_{k+1}^T \boldsymbol{P}_k \boldsymbol{x}_{k+1}}$$

$$\boldsymbol{P}_{k+1} = (\boldsymbol{P}_k^{-1} + \boldsymbol{x}_{k+1}\boldsymbol{x}_{k+1}^T)^{-1} = \boldsymbol{P}_k - \gamma_{k+1}\boldsymbol{P}_k\boldsymbol{x}_{k+1}\boldsymbol{x}_{k+1}^T\boldsymbol{P}_k$$

$$\boldsymbol{A}_{k+1} = \boldsymbol{A}_k + \gamma_{k+1}(\boldsymbol{y}_{k+1} - \boldsymbol{A}_k\boldsymbol{x}_{k+1})\boldsymbol{x}_{k+1}^T\boldsymbol{P}_k$$

Zhang, et al. 2017

Online DMD

(2)

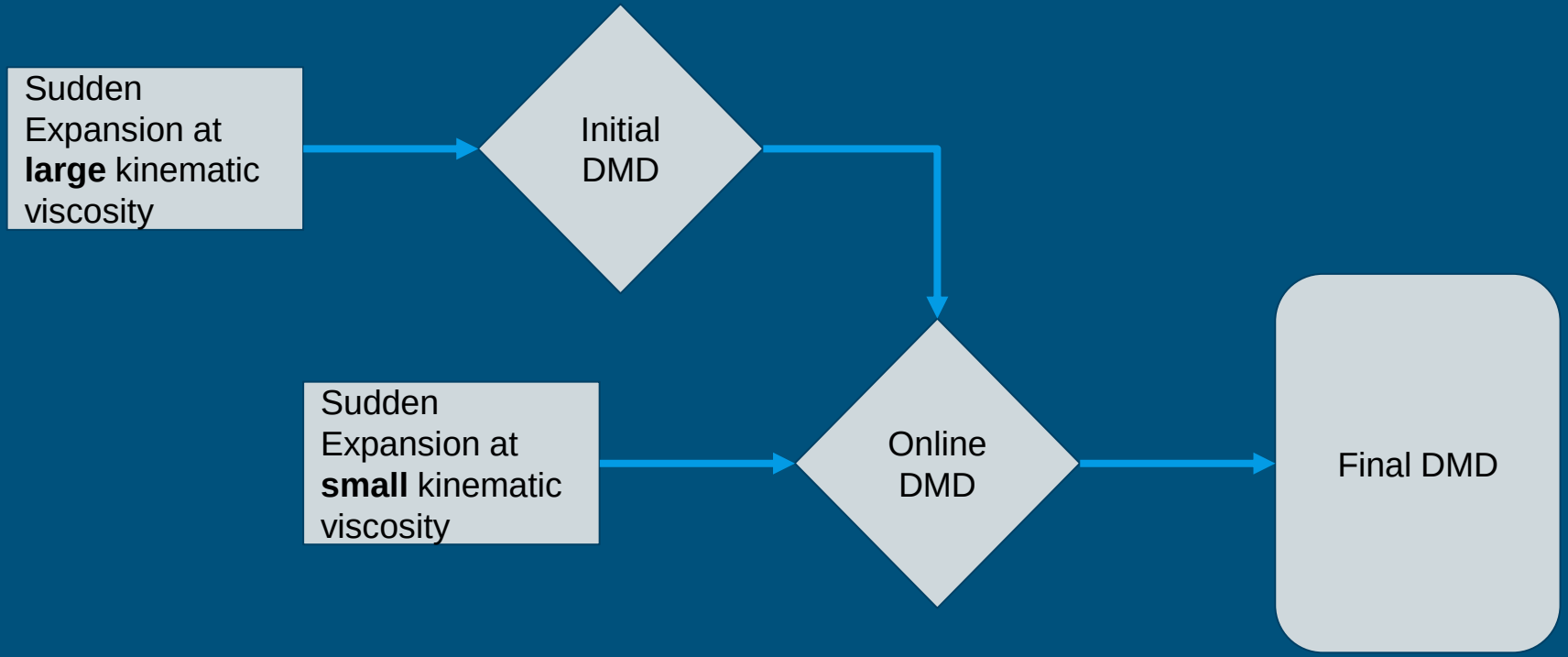
Weighted Online DMD

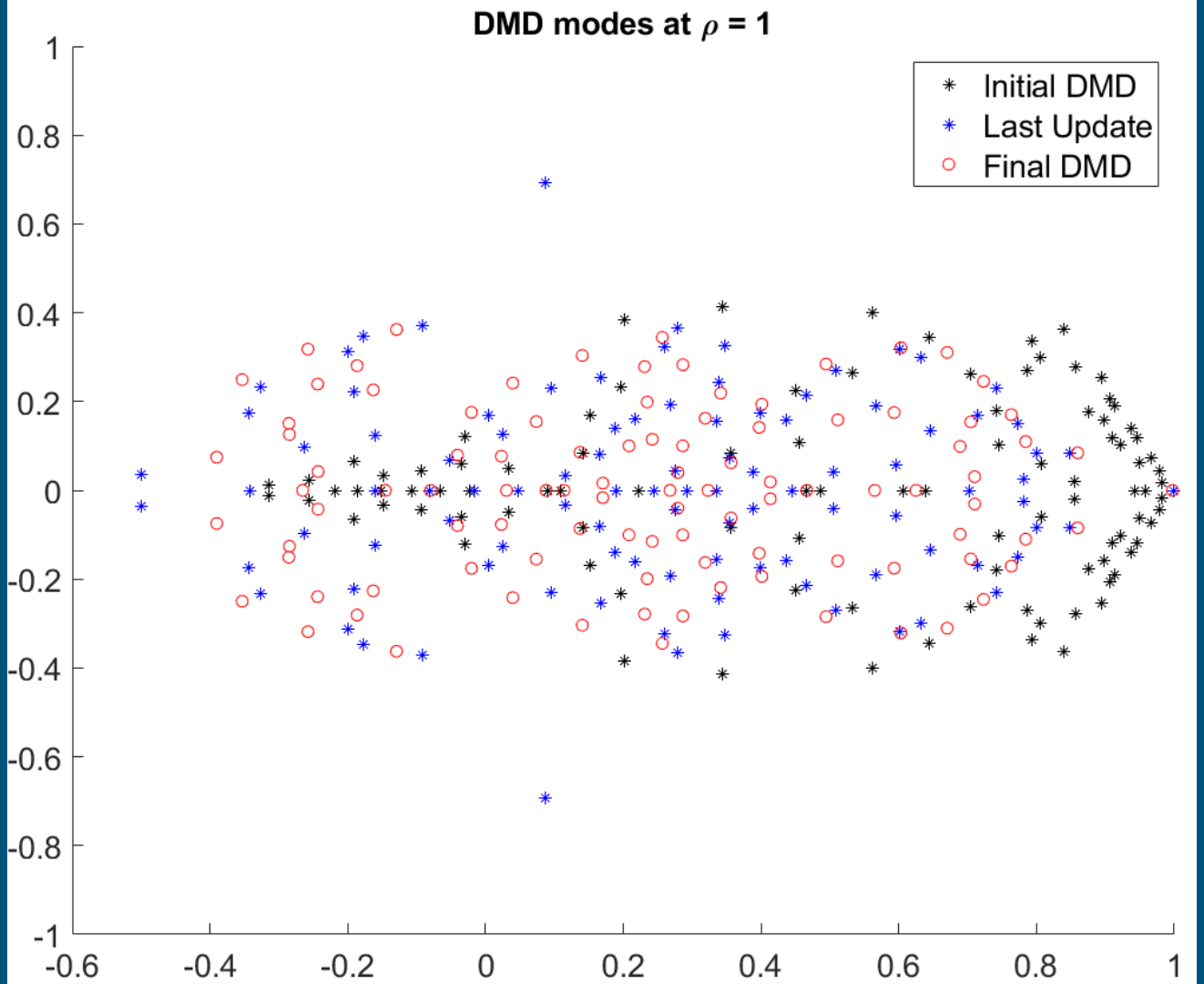
$$\tilde{\boldsymbol{X}}_k = \begin{bmatrix} \sigma^{k-1}\boldsymbol{x}_1 & \sigma^{k-2}\boldsymbol{x}_2 & \cdots & \boldsymbol{x}_k \end{bmatrix}$$

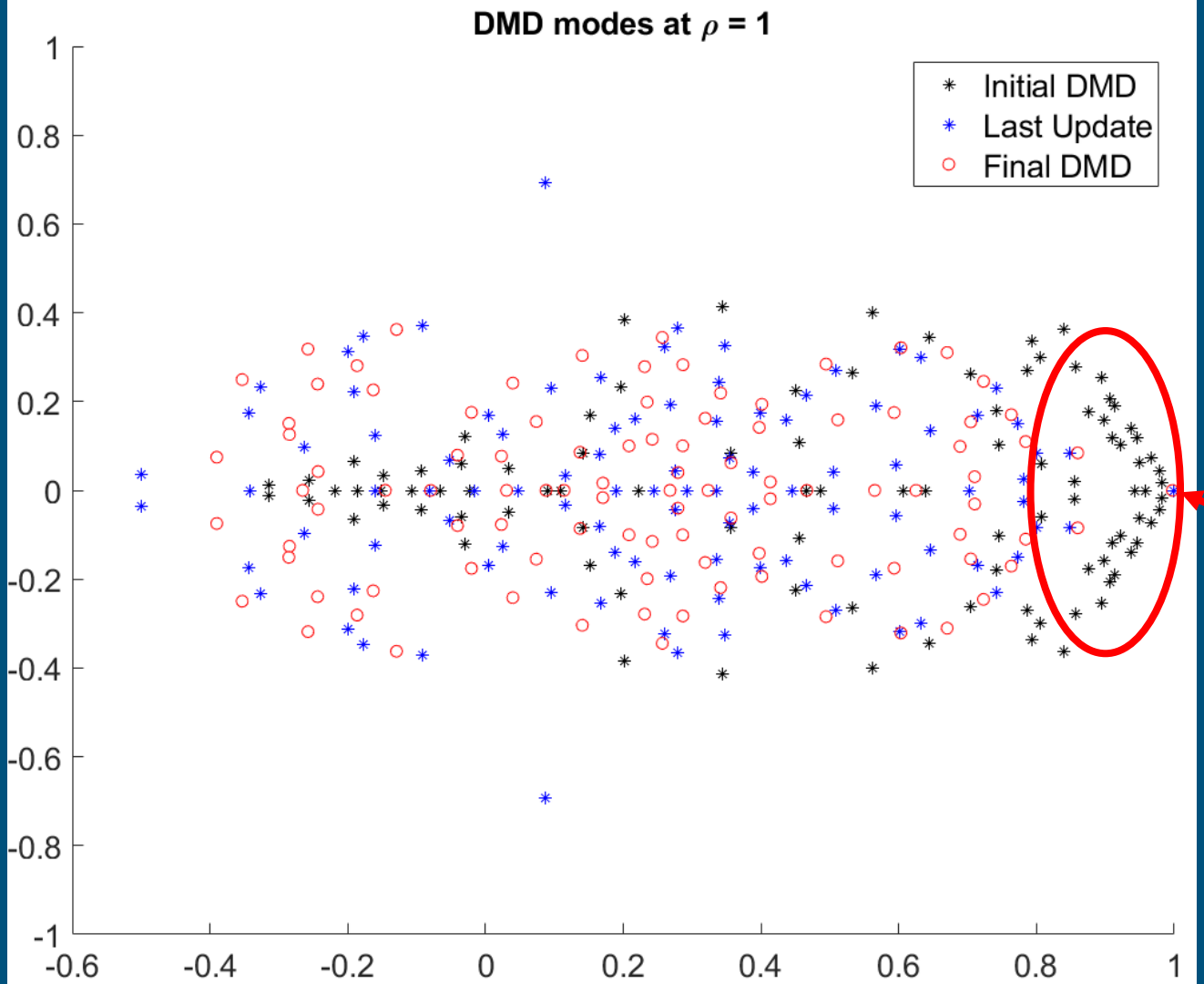
$$\tilde{\boldsymbol{Y}}_k = \begin{bmatrix} \sigma^{k-1}\boldsymbol{y}_1 & \sigma^{k-2}\boldsymbol{y}_2 & \cdots & \boldsymbol{y}_k \end{bmatrix}$$

Zhang, et al. 2017

MATLAB implementation

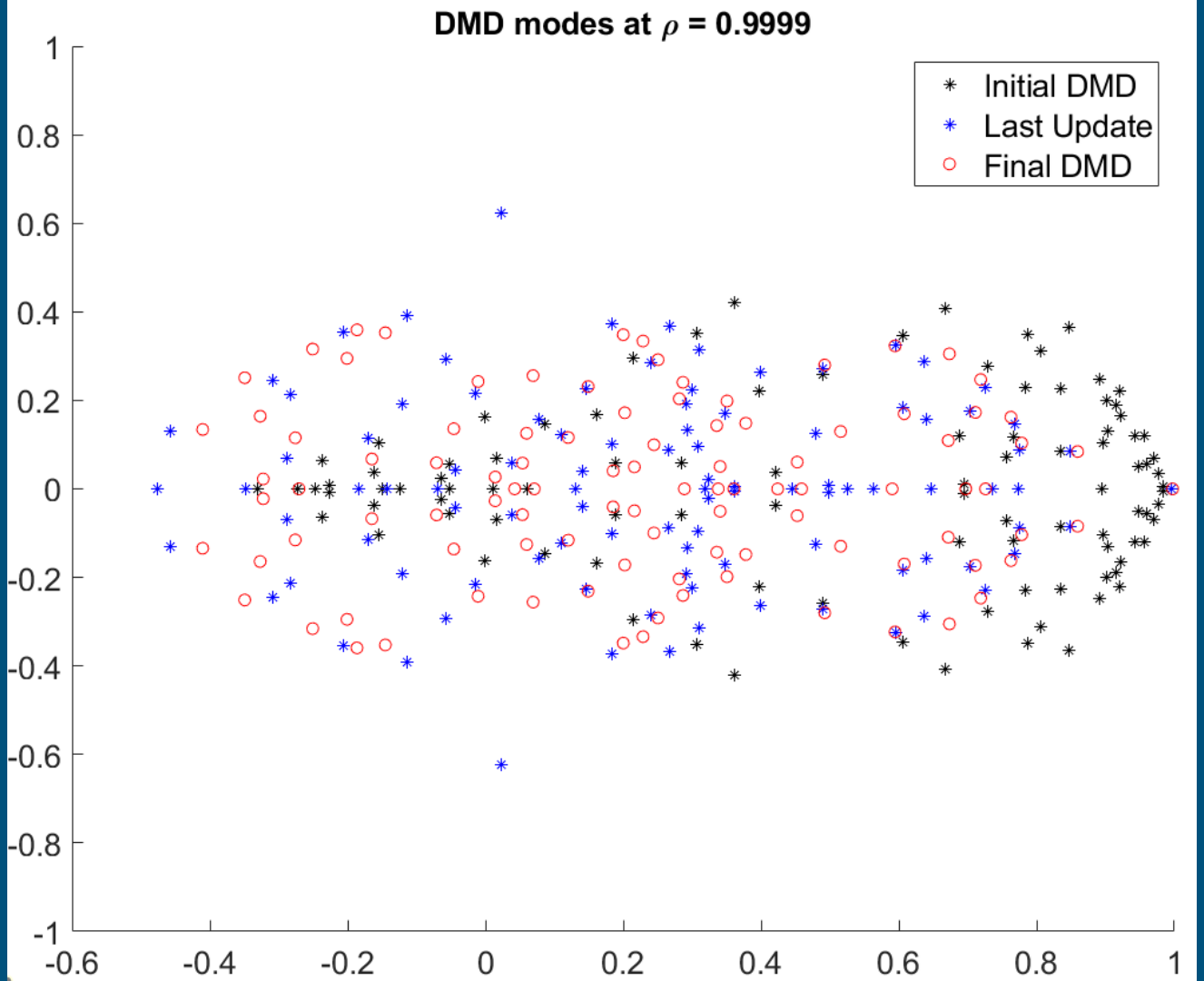



DMD modes at ρ = 1
Initial DMD
Last Update
Final DMD
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


DMD modes at ρ = 1
Initial DMD
Last Update
Final DMD
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Results

	Mode Error
Initial DMD	6.5098
Online Updated DMD	5.6892


DMD modes at ρ = 0.9999
Initial DMD
Last Update
Final DMD
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Results

	Error (Rho = 1)	Error (Rho = 0.9999)
Initial DMD	6.5098	6.8395
Online Updated DMD	5.6892	5.8391

Discussion/Future Work

- Limitations of needing rank(X_k) > n

  Anqi Bao, et al. (2019)

- Compressed-sensing DMD

  Kutz (2016)

References

Hao Zhang, Clarence W. Rowley, Eric A. Deem and Louis N. Cattafesta. Online dynamic mode decomposition for time-varying systems, 2017

J. Nathan Kutz, Steve L. Brunton, Bingni W. Brunton and Joshua L. Proctor. Dynamic mode decomposition: Data-driven modeling of complex systems, 2016

Benjamin Peherstorfer and Karen Wilcox. Dynamic data-driven reduced-order models, 2015

Anqi Bao, Eduardo Gildin, Abhinav Narasingam and Joseph S. Kwon. Data-driven model reduction for coupled flow and geomechanics based on DMD methods, 2019

Questions?

Extra slides

Aspect	Standard	Batch	Mini-batch	Streaming	Online	Windowed
Computational time	$\mathcal{O}(mn^2)$	$\mathcal{O}(kn^2)$	$\mathcal{O}(wn^2)$	$\mathcal{O}(r^2n)$	$4n^2$	$8n^2$
Memory	mn	kn	wn	$\mathcal{O}(rn)$	$2n^2$	$wn + 2n^2$
Store past snapshots	Yes	Yes	Yes	No	No	Yes
Track time variations	No	No	Yes	Yes	Yes	Yes
Real-time DMD matrix	No	Yes	Yes	Yes	Yes	Yes
Exact DMD matrix	Yes	Yes	Yes	No	Yes	Yes

Table 1: Characteristics of the various DMD algorithms considered. Relevant parameters are state dimension n, total number of snapshot pairs $m \gg n$, window size w such that $n < w \ll m$, low rank $r < n$, and discrete time $k > n$. Computational time denotes the required floating-point multplies for one iteration (computing the DMD matrix).

Online DMD

(2)

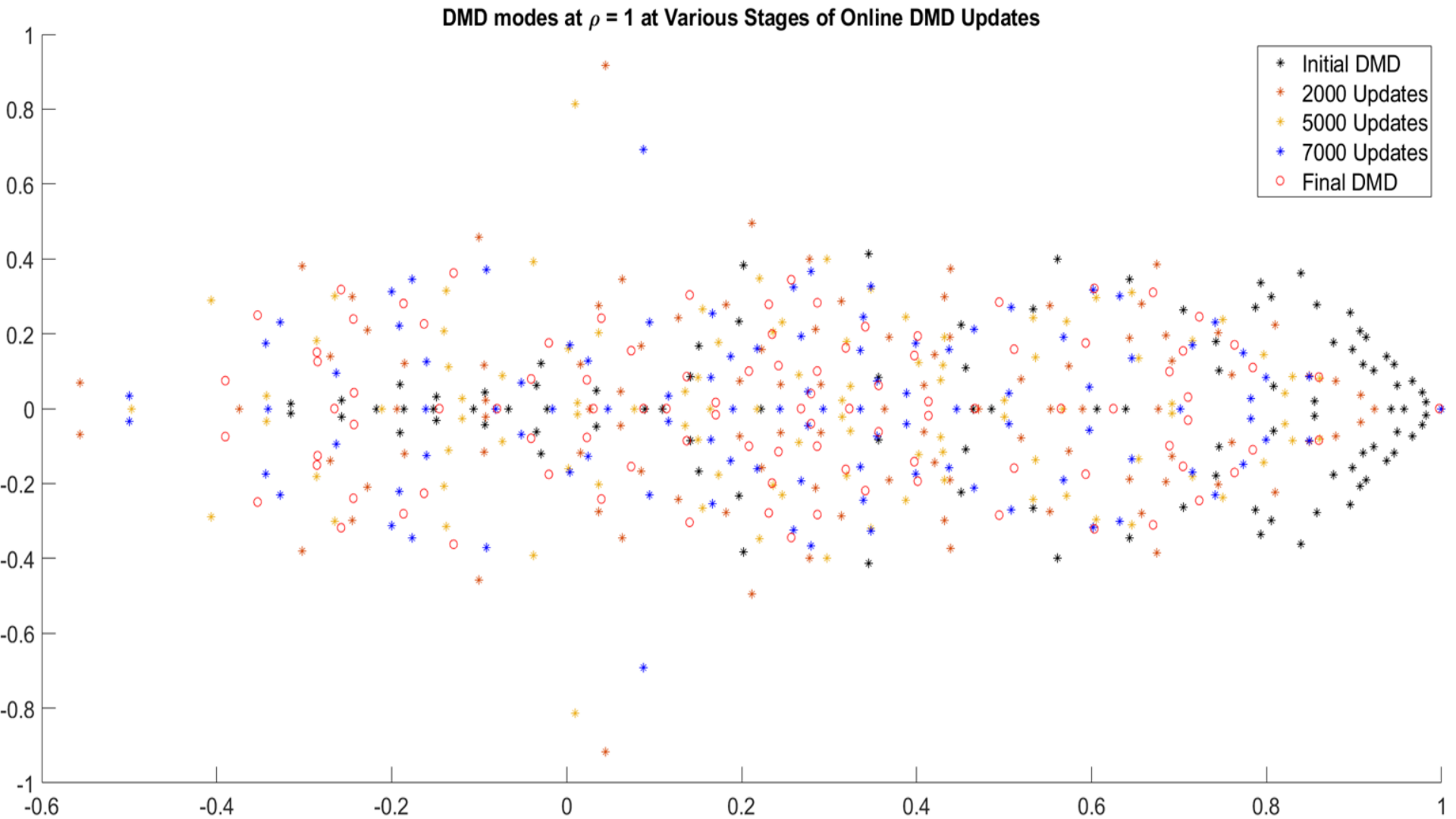
Weighted Online DMD

$$\hat{\boldsymbol{P}}_{k+1} = \frac{1}{\rho}(\hat{\boldsymbol{P}}_k - \gamma_{k+1}\hat{\boldsymbol{P}}_k \boldsymbol{x}_{k+1}\boldsymbol{x}_{k+1}^T \hat{\boldsymbol{P}}_k)$$

$$\gamma_{k+1} = \frac{1}{1 + \boldsymbol{x}_{k+1}^T \hat{\boldsymbol{P}}_k \boldsymbol{x}_{k+1}}$$

$$\boldsymbol{A}_{k+1} = \boldsymbol{A}_k + \gamma_{k+1}(\boldsymbol{y}_{k+1} - \boldsymbol{A}_k \boldsymbol{x}_{k+1})\boldsymbol{x}_{k+1}^T \hat{\boldsymbol{P}}_k$$

Zhang, et al. 2017


DMD modes at ρ = 1 at Various Stages of Online DMD Updates
Initial DMD
2000 Updates
5000 Updates
7000 Updates
Final DMD
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Results

	Mode Error
Initial DMD	6.5098
1 Update	6.4331
2000 Updates	5.4332
5000 Updates	5.3270
7000 Updates	5.6892

Results

	Error (Rho = 1)	Error (Rho = 0.9999)
Initial DMD	6.5098	6.8395
1 Update	6.4331	6.6461
2000 Updates	5.4332	6.0202
5000 Updates	5.3270	6.1574
7000 Updates	5.6892	5.8391