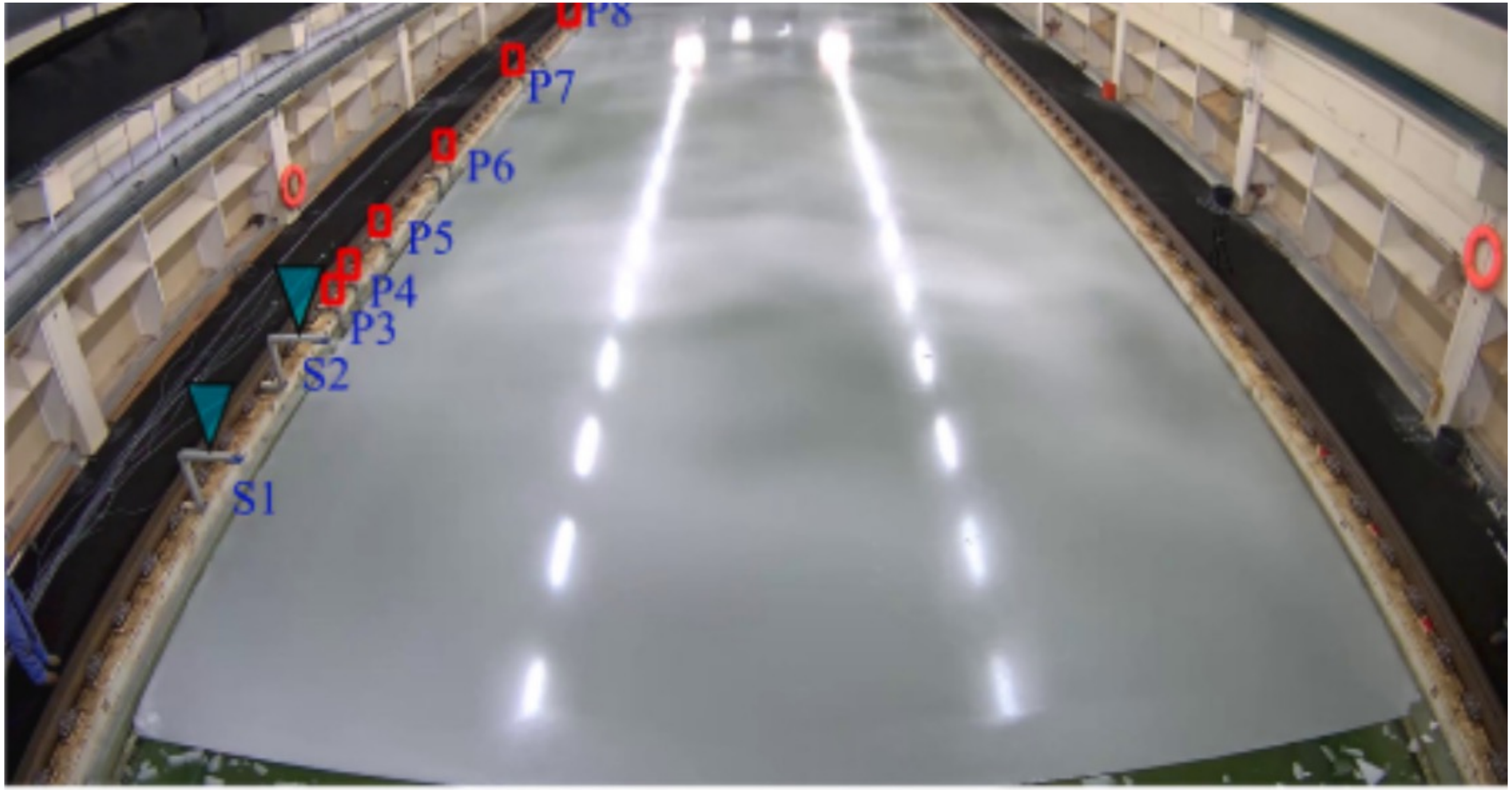
Boundary Element Method for Floating Ice Sheet

2.29 Final Project

Max Pierce

Ice-Covered Wave Tank



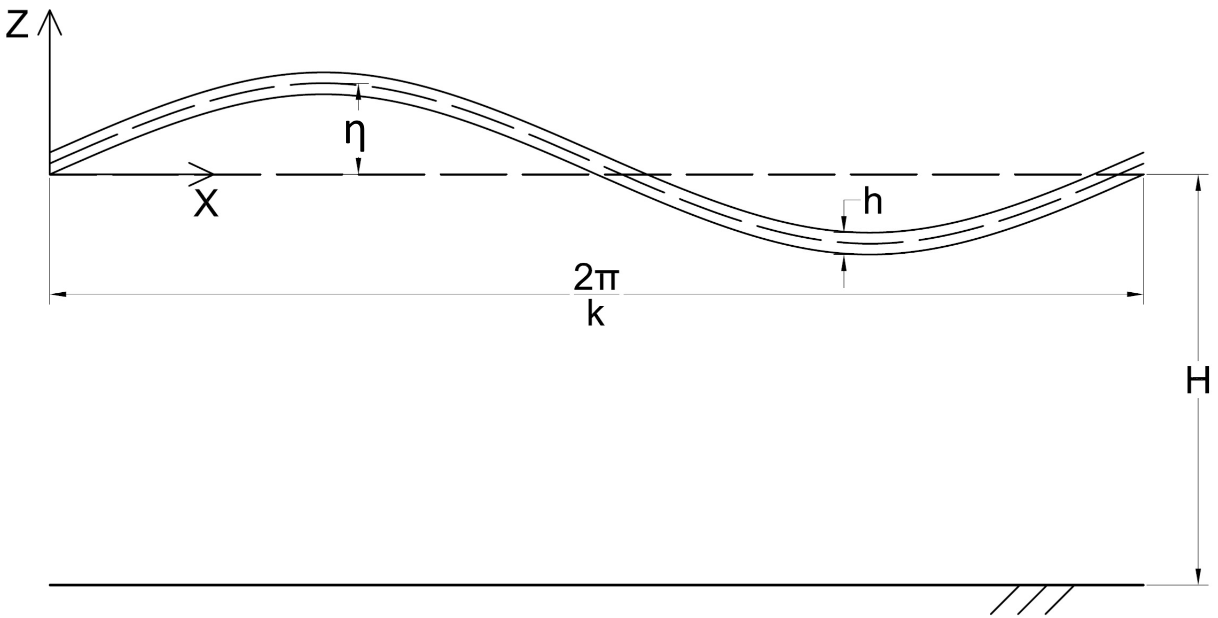
Li, Deniz, & Lubbad. (2021) *CRST*

Objectives

- Modify existing numerical wave tank (“open water”) to model a thin, continuous ice covering (“ice-covered”)
- Discuss difficulties resulting from inherent instabilities and stiff equations

Model Assumptions

Fluid
- Potential flow, $\vec{V} = \nabla\phi$
- Nominally 2D
- Linear waves
- Finite depth



Structure
- Thin
- Homogeneous
- Small deflections

→ Euler-Bernoulli Beam Model

Boundary Value Problem

$S_F: \boldsymbol{\phi} = \boldsymbol{\phi}^s(\boldsymbol{t})$

$S_P: \boldsymbol{\phi}_n = -\boldsymbol{\phi}_x(\boldsymbol{t})$

$S_B: \boldsymbol{\phi}_n = \mathbf{0}$

- Governing Equation: $\nabla^2\phi = 0$
- Boundary Conditions

Kinematic	$\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial z}\Big\rvert_{z=0} = 0$	on $\boldsymbol{r} \in S_F$
Dynamic	$\frac{\partial\phi^s}{\partial t} + \eta + \tilde{D}\frac{\partial^4\eta}{\partial x^4} = -P_a$	on $\boldsymbol{r} \in S_F$
Body Kinematic	$\frac{\partial\phi}{\partial n} = 0$	on $\boldsymbol{r} \in S_B$
	$\frac{\partial\phi}{\partial n} = \vec{U}_p \cdot \vec{n}$	on $\boldsymbol{r} \in S_P$

- BVP solved using quadratic boundary element method

$$[A]\begin{bmatrix}\phi_z \\ \phi\end{bmatrix} = \begin{bmatrix}\phi \\ -\phi_x\end{bmatrix}$$

- Solved using GMRES
- Time marching using 4th-order Runge Kutta

Saw-Tooth Instability

- Numerical noise of wavelength $2\Delta x$ inherent to the BVP

$$\delta\eta_j \sim \delta A(-1)^j$$

- Considering single mode of Fourier decomposition inserted into exact spatial derivative:

$$\frac{\partial^4\left(f(t)e^{ikx}\right)}{\partial x^4} = k^4 f(t)e^{ikx}$$

- Inviscid assumption – no physical damping

Smoothing

- Use knowledge of saw-tooth instability to derive specific smoothing scheme

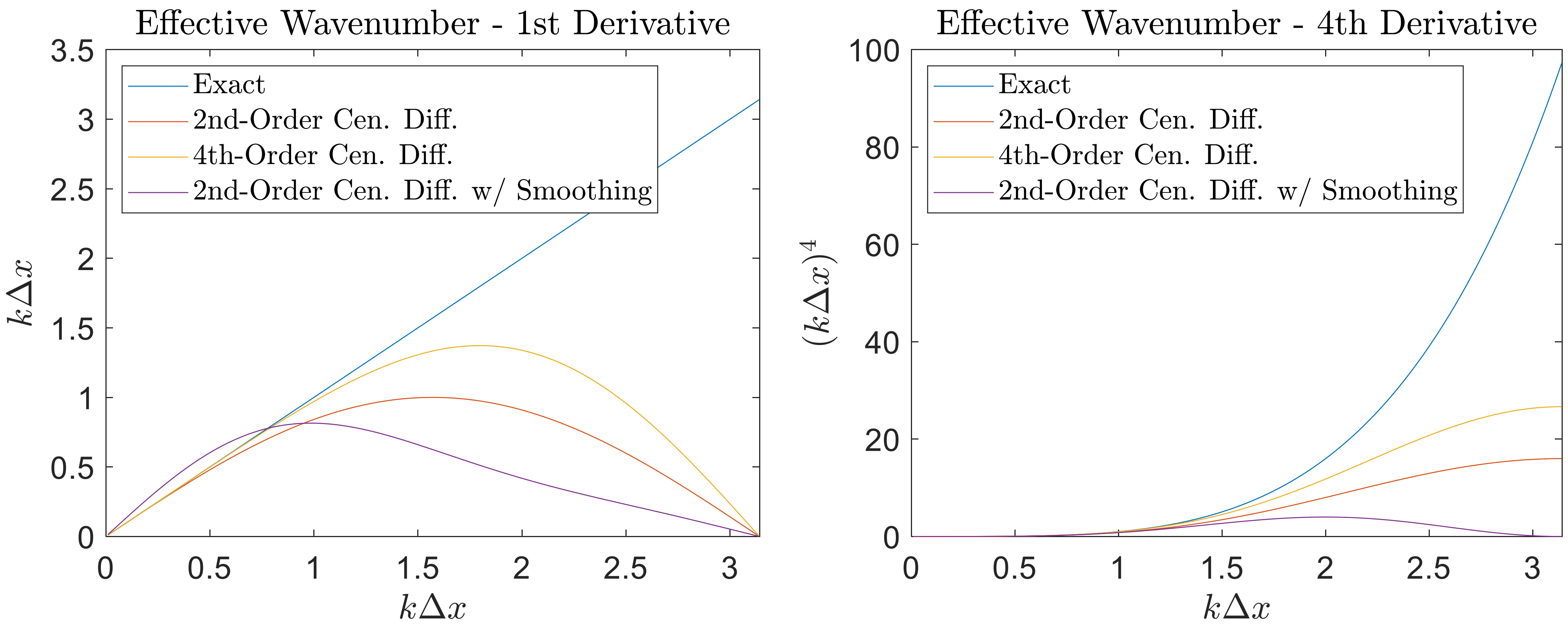
$h(x) = (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + (-1)^j \, (b_0 + b_1 x + \cdots + b_{n-1} x^{n-1})$

- 5-Point Scheme

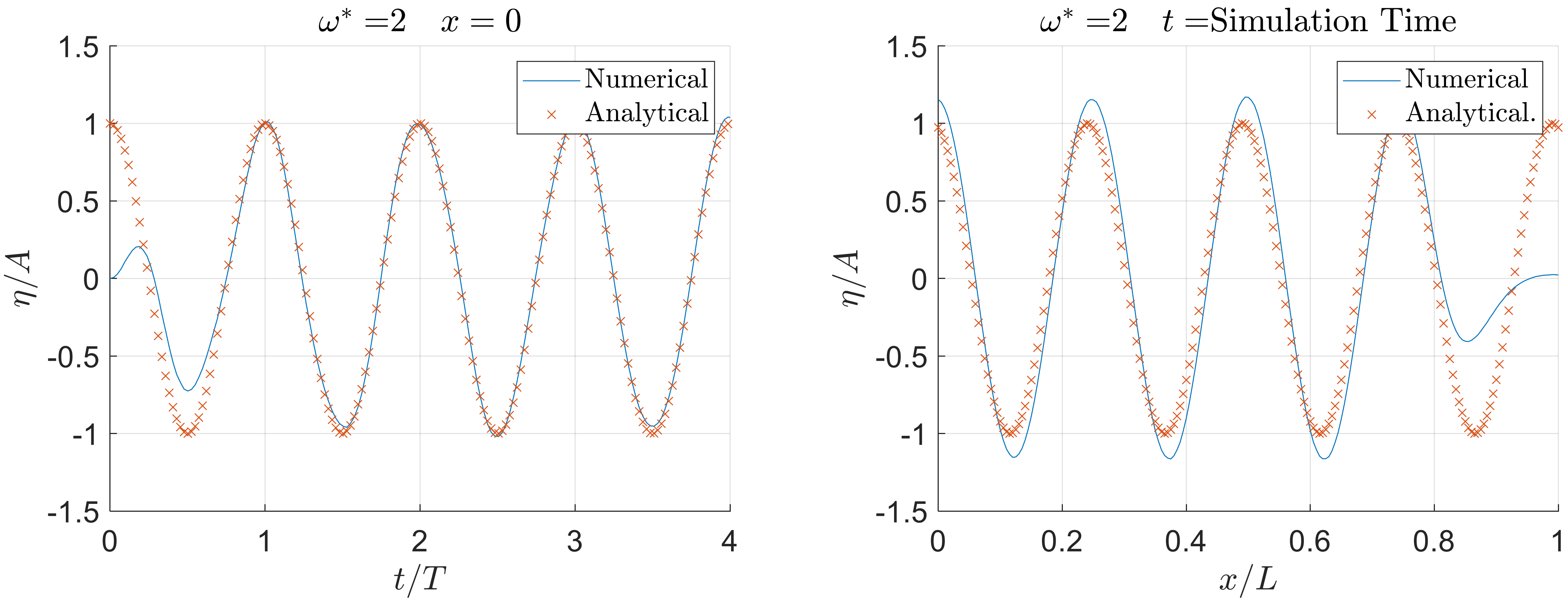
$$\overline{f_j} = \frac{1}{16}\left(-f_{j-2} + 4f_{j-1} + 10f_j + 4f_{j+1} - f_{j+2}\right)$$

Longuet-Higgins, M. & Cokelet, E. (1976) *R. Soc. Proc. A*

Effective Wavenumber with Smoothing



Open Water Verification



Modifications to Include Ice-Covering

- Velocity potential update changes to resemble hyperbolic PDE

$$\begin{cases} \dfrac{\partial \phi^s}{\partial t} = -\eta - \widetilde{D}\,\dfrac{\partial^4 \eta}{\partial x^4} \\ \dfrac{\partial \eta}{\partial t} = \left.\dfrac{\partial \phi}{\partial z}\right|_{z=0} \end{cases}$$

- Adopted 2nd-order upwind scheme to evaluate the fourth derivative to take advantage of numerical diffusion

Stiff System of Equations

- Time step sized in accordance with open water leads to loss of convergence
- Reducing time step significantly helped resulting in ~1/2 wavelength of realistic solution

- Adopt deferred-correction time marching scheme to gain stability of implicit method

Next Steps

- Von Neumann stability analysis $\begin{cases} \dfrac{\partial \phi^s}{\partial t} = -\cancelto{0}{\eta} - \widetilde{D}\dfrac{\partial^4 \eta}{\partial x^4} \\ \dfrac{\partial \eta}{\partial t} = \left.\dfrac{\partial \phi}{\partial z}\right|_{z=0} \end{cases}$

- Isolate the flexural wave

Acknowledgements & References

Thanks to João Seixas de Medeiros

<u>References</u>

Li, H., Gedikli, E., & Lubbad, R. 2021. Laboratory study of wave-induced flexural motion of ice floes. *CRST 182.*

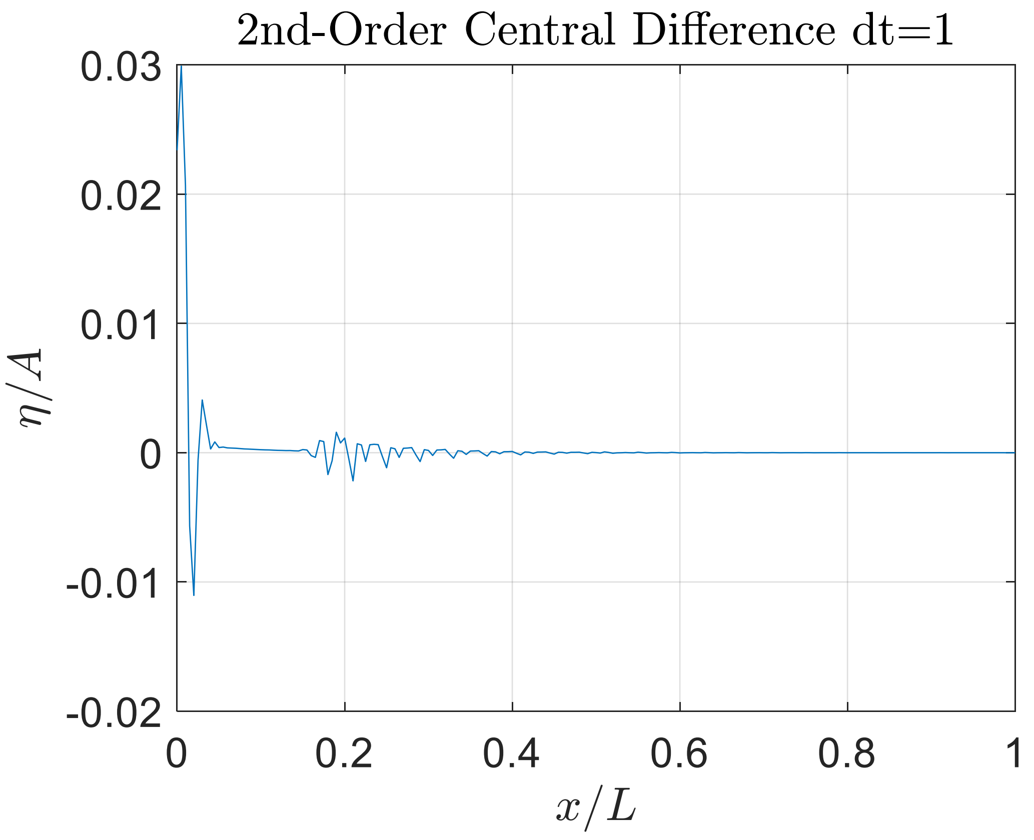
Longuet-Higgins, M. S. & Cokelet, E. D. 1976. The Deformation of Steep Surface Waves on Water. I. A Numerical Method of Computation. *Proc. R. Soc. A. 350*, 1-26.

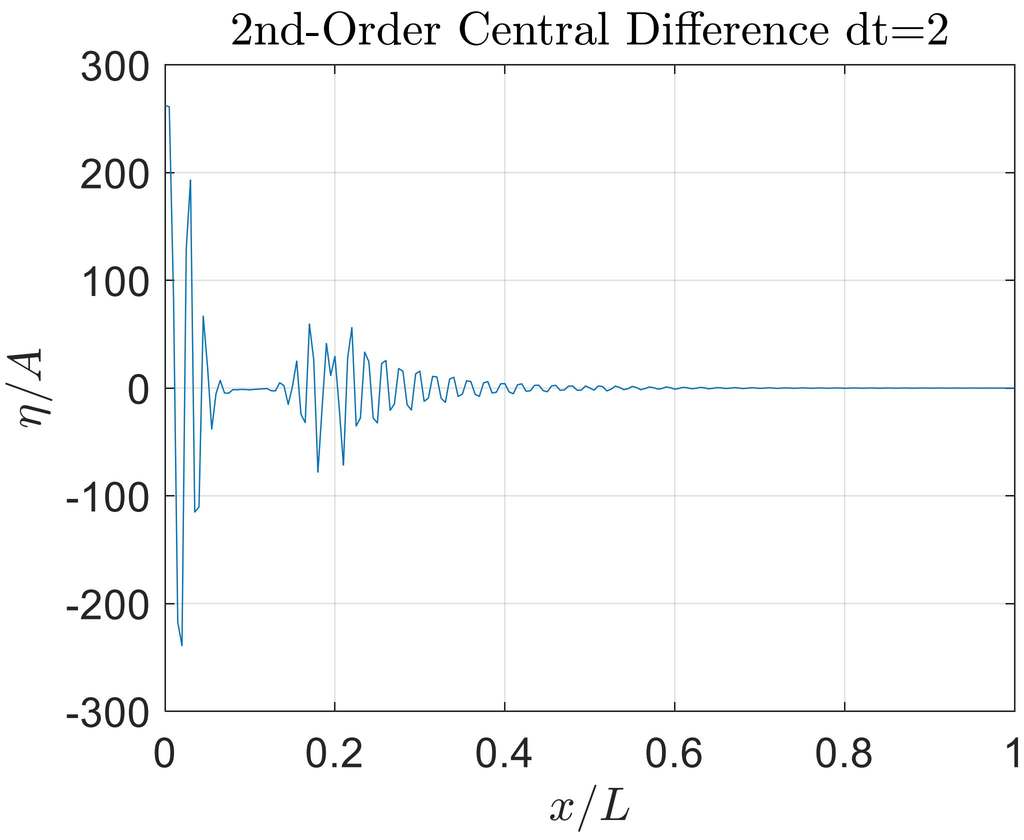
Acknowledge Joao

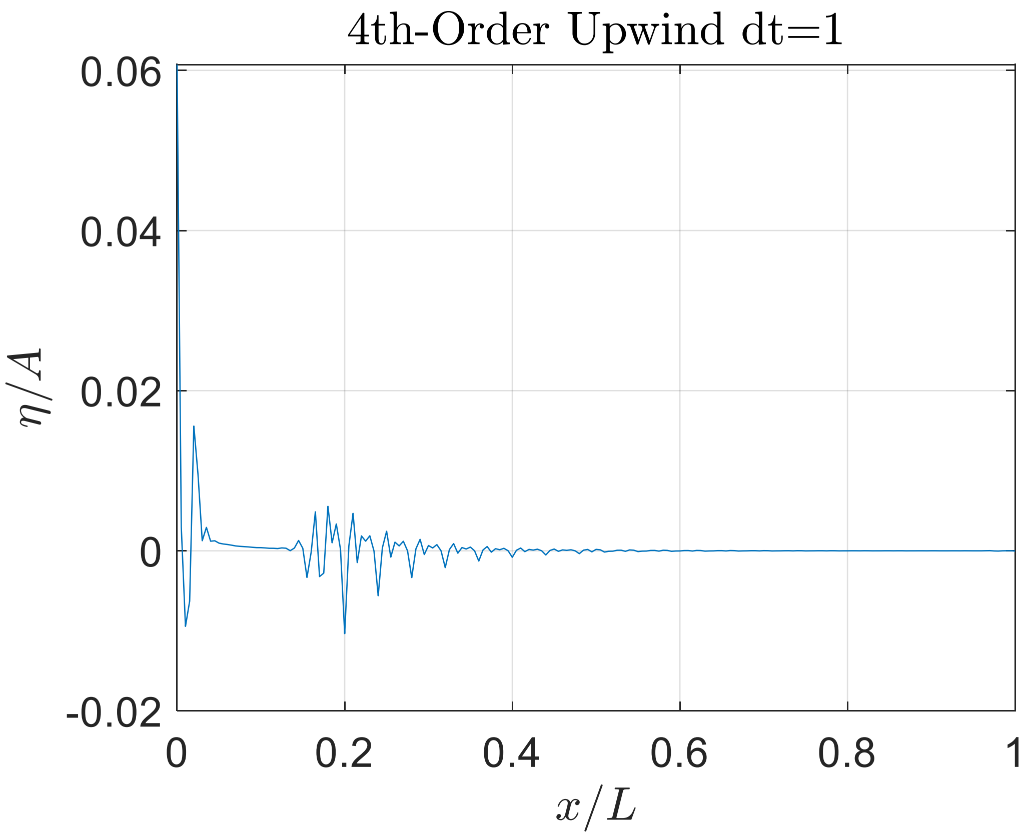
- $y_j(x_j) = \cos(x_j) + \delta y(-1)^j U_j\ (0,1)$

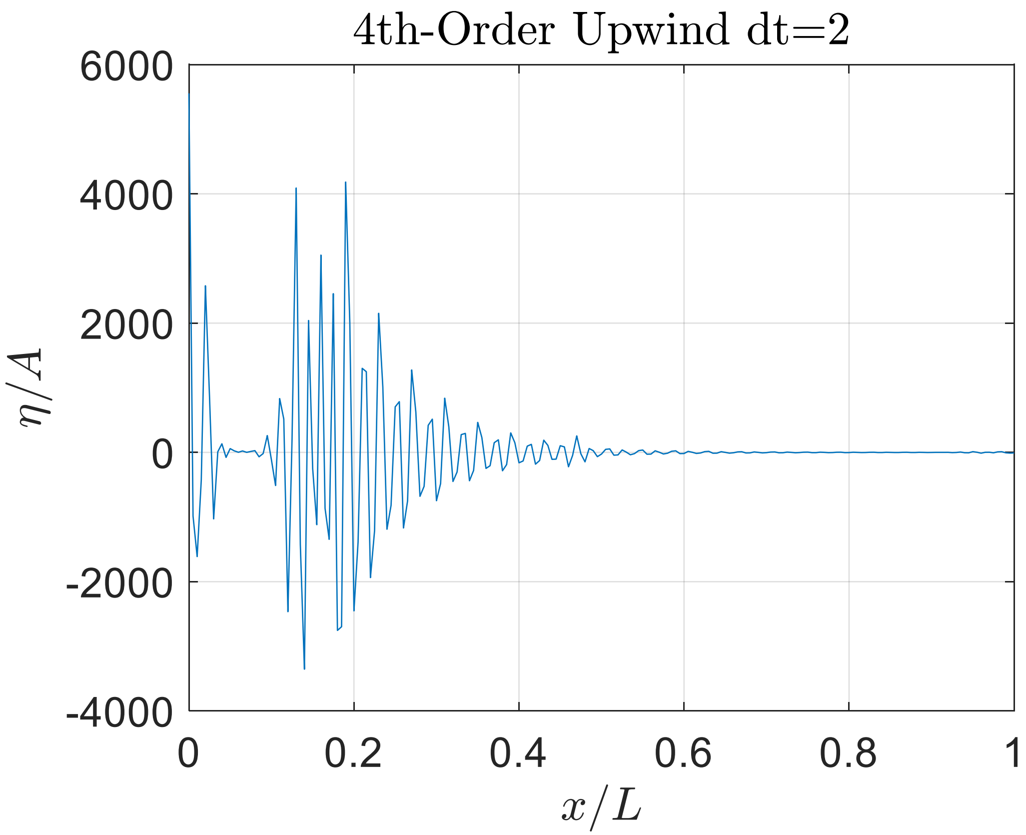
Vector Norms of Amplitude-Normalized Free Surface Elevation Showing Instability

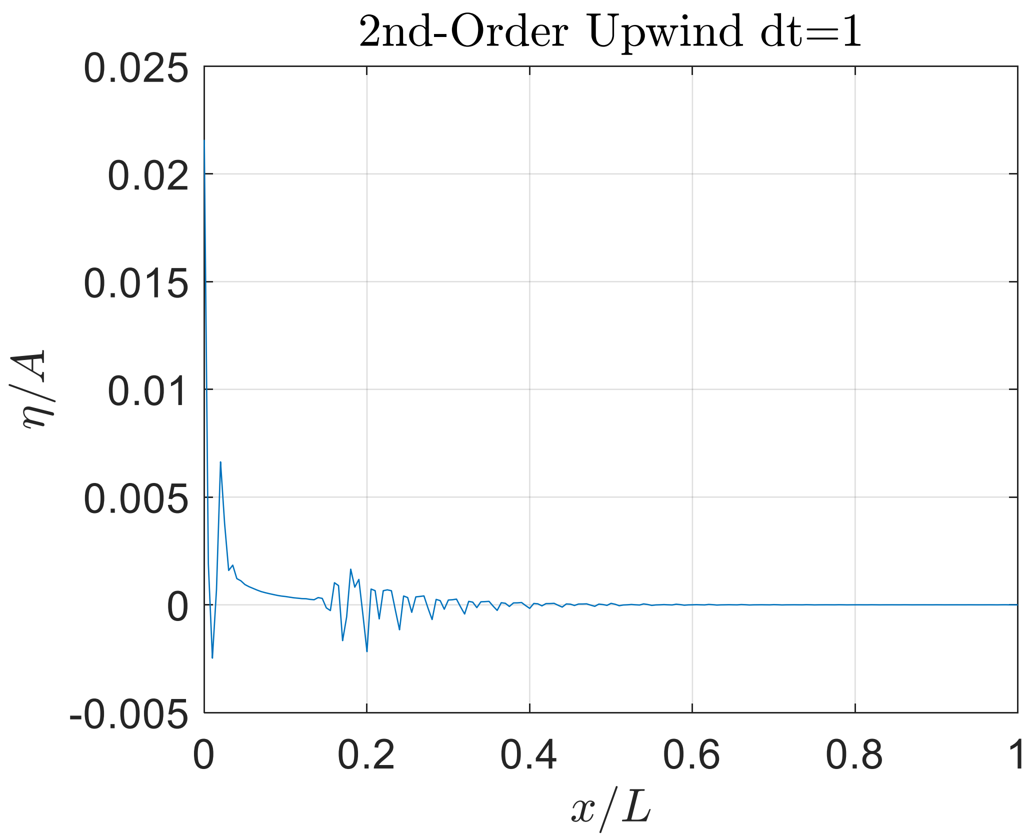
dt	2nd-Order Cen. Diff.	4th-Order Cen. Diff.	2nd-Order Upwind	4th-Order Upwind
1	1.14×10^{-4}	6.38×10^{-4}	5.96×10^{-5}	1.67×10^{-4}
2	1.52	888	0.567	32.0

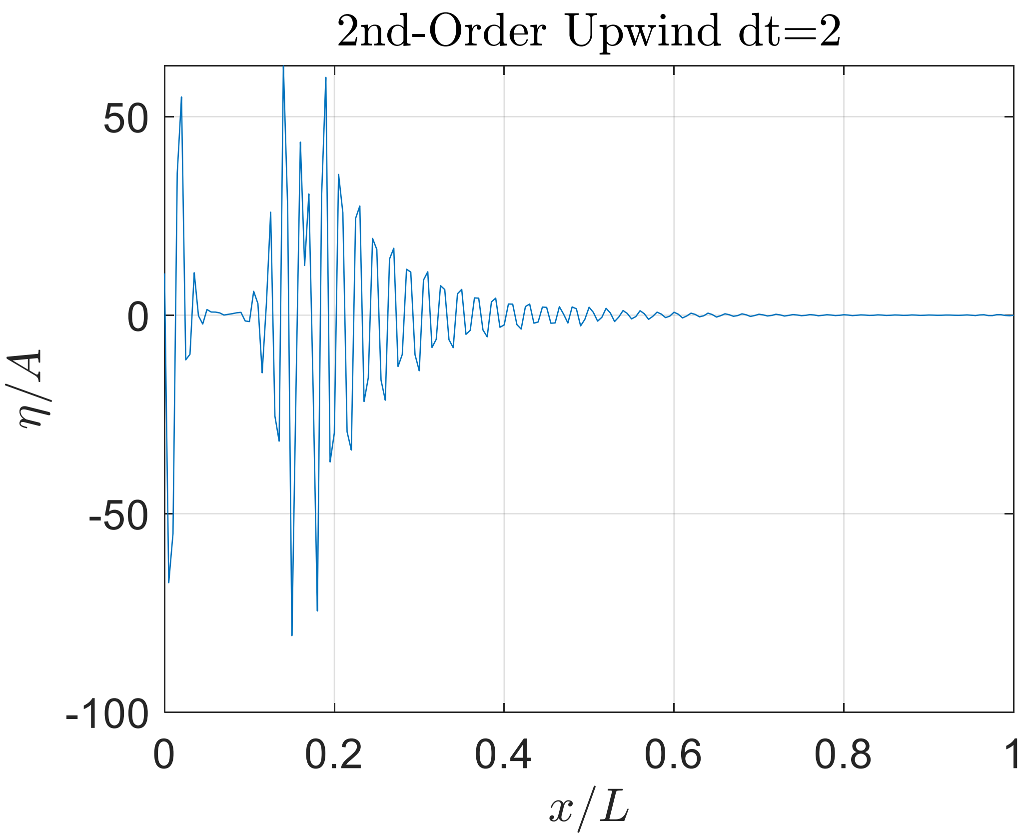

2nd-Order Central Difference dt=1
0.03
0.02
0.01
0
-0.01
-0.02
η/A
0
0.2
0.4
0.6
0.8
1
x/L

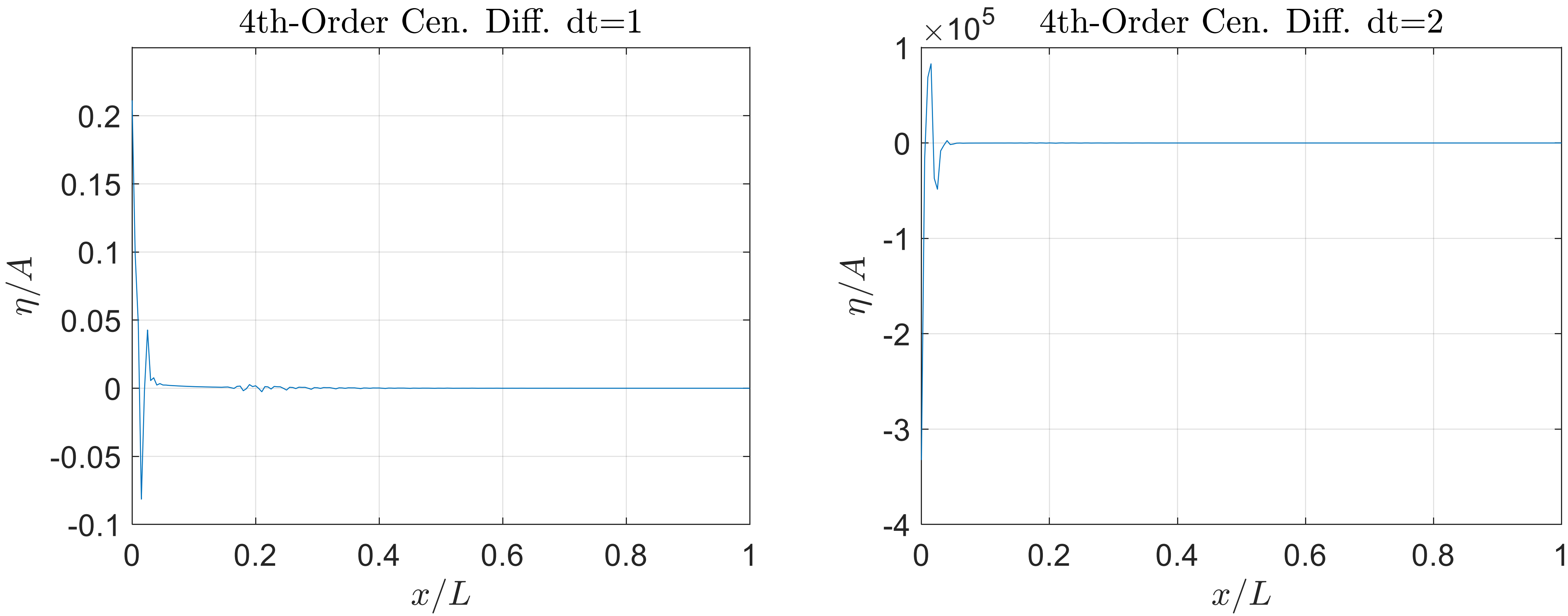

2nd-Order Central Difference dt=2
300
200
100
0
-100
-200
-300
η/A
0
0.2
0.4
0.6
0.8
1
x/L


4th-Order Upwind dt=1
0.06
0.04
0.02
0
-0.02
η/A
0
0.2
0.4
0.6
0.8
1
x/L


4th-Order Upwind dt=2
6000
4000
2000
0
-2000
-4000
η/A
0
0.2
0.4
0.6
0.8
1
x/L


2nd-Order Upwind dt=1
0.025
0.02
0.015
0.01
0.005
0
-0.005
η/A
0
0.2
0.4
0.6
0.8
1
x/L


2nd-Order Upwind dt=2
50
0
-50
-100
η/A
0
0.2
0.4
0.6
0.8
1
x/L



4th-Order Cen. Diff. dt=1
0.2
0.15
0.1
0.05
0
-0.05
-0.1
η/A
0
0.2
0.4
0.6
0.8
1
x/L
$\times 10^5$
4th-Order Cen. Diff. dt=2
1
0
-1
-2
-3
-4
η/A
0
0.2
0.4
0.6
0.8
1
x/L