2.993: Principles of Internet Computing
Homework #9 Solutions

1. *Hamming Bound* Let C be a given (n,k) binary codeword, i.e. n coded bits for every k data bits.
   a) Argue that M (defined below) is the total number of codewords (including C) that are at most t Hamming distance away from C. Hence conclude that a t-error correcting (n,k) binary code C must satisfy the following inequality:

$$2^{n-k} \geq M = \sum_{j=0}^{t} \binom{n}{j} \qquad \text{where } \binom{n}{j} = \frac{n!}{(n-j)!\,j!} \ .$$

The term $\binom{n}{j}$, also known as "n choose j", is simply the number of ways of choosing k from n objects without order. For example, 4 letters A, B, C and D can be chosen $6 = \binom{4}{2} = \frac{4!}{2!2!}$ different ways: AB, AC, AD, BC, BD and CD.

Ans: For a (n,k) code, there are 2n possible codewords, but only 2k are valid codewords since the encoding (mapping) is one-to-one. For decoding at the receiving end, every possible codeword must be mapped into a valid codeword (or its corresponding data bits). Thus, we partition 2n codewords into 2k disjoint sets, each containing 2^{n-k} codewords (LHS). In order to correct up to t bit errors, each set should include codewords that are up to t Hamming distance away from a valid codeword (RHS). Done!

   b) Compute the minimum required codelength n for k =1 to 12, t = 1.

Ans: (n,k)=(3,1), (5,2), (6,3), (7,4), (9,5), (10,6), (11,7), (12,8), (13,9), (14,10) (15,11), (17,12)

   c) When equality holds for any (n,k) code using the Hamming bound, it is said to be *perfect*. Among the codes in b), which are perfect?

Ans: (3,1), (7,4), (15,11)

2. *Hamming Code* A binary, single-error correcting, perfect code is known as a Hamming code. A classic Hamming code is the (7,4) code. This code can correct any single bit error among 7 coded bits.

   a) Obtain the original data bits from the following stream of coded bits by decoding 7 bits at a time.
   1001011 0100011 0010011 1110101

Ans: We assume the code is systematic, i.e. the first k bits are data bits followed by (n-k) parity-check bits.

1001	0100	0010 (5th bit in error)	1100 (3rd bit in error)

b) The (7,4) code can either detect 2 errors or correct a single error. Give an example of a block of 7 coded bits with 2 errors, that leads to a decoding error; i.e. by error correction, a 4-bit pattern is decoded incorrectly as another 4-bit pattern.

Ans: In part(a), 0010011 and 1110101 can result from 2-bit errors:

0010011 → 0100 (2nd and 3rd bits in error)
1110101 → 0110 (1st and 7th bits in error) OR
1010 (2nd and 5th bits in error)