22.02 Fall 2006 Problem Set #3 Solutions

1. Region II: ψ_{II} , Incident: ψ_{inc}

Condition associated with x: $|\psi_{II}|^2 = \frac{1}{4}|\psi_{inc}|^2$

$\psi_{II} = Ce^{-\kappa x}$, $\psi_{inc} = Ae^{ikx} \Rightarrow |Ce^{-\kappa x}|^2 = \frac{1}{4}|Ae^{ikx}|^2 \Rightarrow |C|^2 e^{-2\kappa x} = \frac{1}{4}|A|^2$

$$e^{-2\kappa x} = \frac{1}{4}\left|\frac{A}{C}\right|^2 \Rightarrow x = \frac{\ln\left(4\left|\frac{C}{A}\right|^2\right)}{2\kappa}$$

Region I: $\psi_I = Ae^{ikx} + Be^{-ikx}$

$\psi_I(x=0) = \psi_{II}(x=0) \Rightarrow Ae^{ik(0)} + Be^{-ik(0)} = Ce^{-\kappa(0)} \Rightarrow A+B=C$

$\psi_I'(x=0) = \psi_{II}'(x=0) \Rightarrow ikAe^{ik(0)} - ikBe^{-ik(0)} = -\kappa Ce^{-\kappa(0)} \Rightarrow$

$ikA - ikB = -\kappa C \Rightarrow 1 - \frac{B}{A} = \frac{i\kappa}{k}\left(\frac{C}{A}\right) \Rightarrow \frac{B}{A} = 1 - \frac{i\kappa}{k}\left(\frac{C}{A}\right)$

$A+B=C \Rightarrow \frac{B}{A} = \frac{C}{A} - 1 \Rightarrow \frac{C}{A} = \frac{2}{1+\frac{i\kappa}{k}} \Rightarrow \left|\frac{C}{A}\right|^2 = \frac{4}{1+\frac{\kappa^2}{k^2}}$

$k^2 = \frac{2mE}{\hbar^2}$, $\kappa^2 = \frac{2m(V_0-E)}{\hbar^2} = \frac{1}{4}\left(\frac{2mE}{\hbar^2}\right) \Rightarrow \frac{\kappa^2}{k^2} = \frac{1}{4}$

$\left|\frac{C}{A}\right|^2 = \frac{4}{1+\frac{1}{4}} = \frac{16}{5} \Rightarrow \kappa x = \frac{1}{2}\ln\left(4\left(\frac{16}{5}\right)\right) = 1.2747$

$x = \frac{1.2747}{\kappa} = 1.2747\sqrt{\frac{2\hbar^2}{mE}} = 1.2747\sqrt{\frac{2\hbar^2c^2}{m_pc^2\cdot E}}$

$\hbar c = 197$ MeV·F, $m_pc^2 = 938$ MeV, $E = 10$ MeV $\Rightarrow$

$\boxed{x = 3.67\text{ F}}$

2. Rough estimate: $P_{tun} \simeq \exp(-2\kappa L)$, $m_nc^2 \approx 1000$ MeV , $\hbar c \approx 200$ MeV·F

$V = 40$ MeV, $E = 20$ MeV, $L = 2a = 10$ F

$\kappa = \sqrt{\frac{2m_n(V-E)}{\hbar^2}} = \sqrt{\frac{2(m_nc^2)(V-E)}{(\hbar c)^2}} = \sqrt{\frac{2\cdot 1000\cdot 20}{200^2}} = 1\text{ F}^{-1}$

$P_{tun} \simeq \exp(-2\cdot 1\cdot 10) \Rightarrow \boxed{P_{tun} \simeq 2.06\times 10^{-9}}$

Estimate with proper values: $m_nc^2 = 939.6$ MeV , $\hbar c = 197$ MeV·F

$P_{tun} \simeq \exp(-2\cdot 0.9841\cdot 10) \Rightarrow \boxed{P_{tun} \simeq 2.83\times 10^{-9}}$

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Calculate using Liboff, eq. 7.147 : $T = \dfrac{1}{1+\frac{1}{4}\left[\frac{V^2}{E(V-E)}\right]\sinh^2(2\kappa a)}$

$2a = L$, $\kappa L >> 1$ (assumption) $\Rightarrow \sinh(2\kappa a) \to \frac{1}{2}\exp(\kappa L) \Rightarrow$

$$T = \frac{1}{1+\frac{1}{16}\left[\frac{V^2}{E(V-E)}\right]\exp(2\kappa L)} \to \frac{16}{\left[\frac{V^2}{E(V-E)}\right]\exp(2\kappa L)} = 4\exp(-2\kappa L)$$

Using rough numbers, $\kappa = 1\ F^{-1} \Rightarrow T = 4\exp(-20) \Rightarrow \boxed{T = 8.24\times10^{-9}}$

Using exact numbers, $\kappa = 0.9841\ F^{-1} \Rightarrow T = 4\exp(-19.7) \Rightarrow \boxed{T = 1.13\times10^{-8}}$

The estimate is $\frac{1}{4}$ the probability calculated using the Liboff equation.

3. $V_0 = 35$ MeV, $a = 1.76$ F, $mc^2 = 469$ MeV, $\hbar c = 197$ MeV·F

$\lambda = 4a$, $k = \dfrac{2\pi}{\lambda} = \dfrac{\pi}{2a} \Rightarrow k^2 = \dfrac{\pi^2}{4a^2}$

$E_{kin} = \dfrac{\hbar^2 k^2}{2m} = \dfrac{(\hbar c)^2 k^2}{2mc^2} = \dfrac{(197)^2}{2(469)}\left(\dfrac{\pi^2}{4(1.76)^2}\right) \Rightarrow E_{kin} = 32.96$ MeV

Well depth $= -V_0 = -35$ MeV

$E_0 = -V_0 + E_{kin} = -2.04$ MeV $\Rightarrow \boxed{E_0 = -2.04\text{ MeV}}$

$E_{binding} = -E_0 = 2.04$ MeV