22.02 Fall 2006 Problem Set #5 Solutions

1. $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi = E_n W_n(x)$, $\Psi(x,0) = a_2 \Psi_2(x) + a_5 \Psi_5(x)$

$\Psi(x,t) = C_n(t) W_n(x) \Rightarrow i\hbar \frac{\partial}{\partial t} C_n(t) = E_n C_n(t)$

$\Psi(x,t) = \sum_n C_n(t) W_n(x) \Rightarrow \Psi_n(x,t) = \sum_n C_n(0) \exp(-i\omega_n t) W_n(x)$

$P\Delta x = |\Psi|^2 \Delta x$ for $x = \frac{L}{2}$, $\Psi(x,0) = a_2 \sqrt{\frac{2}{L}} \sin(k_2 x) + a_5 \sqrt{\frac{2}{L}} \sin(k_5 x)$, $k_n = \frac{n\pi}{L}$

$\Psi(x,t) = a_2 \sqrt{\frac{2}{L}} \exp(-i\omega_2 t) \sin(k_2 x) + a_5 \sqrt{\frac{2}{L}} \exp(-i\omega_5 t) \sin(k_5 x)$

$|\Psi(x=\frac{L}{2},t)|^2 \Delta x = \frac{2}{L} \{ |a_2|^2 \cdot 0 + |a_5|^2 \cdot 1 + (a_2 a_5^* + a_2^* a_5) \cos[(\omega_2 - \omega_5)t] \cdot 0 \cdot 1 \} \Delta x$

$P(x=\frac{L}{2} \to x = \frac{L}{2} + \Delta x) = P(x=\frac{L}{2}) \Delta x = \boxed{\frac{2}{L} |a_5|^2 \Delta x}$

(a) $P(E=E_1) = 0$

(b) $P(E=E_2) = |a_2|^2$

(c) $P(E=E_3) = 0$

(d) $P(E=E_5) = |a_5|^2$

2. Prove $[\hat{L}_z, \hat{L}^2] = 0$

$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, $[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$, $[\hat{L}_z, \hat{L}_x] = i\hbar L_y$, $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$

$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

$[\hat{L}_z, \hat{L}^2] = [\hat{L}_z, \hat{L}_x^2] + [L_z, \hat{L}_y^2] + [\hat{L}_z, \hat{L}_z^2] = [\hat{L}_z, \hat{L}_x \hat{L}_x] + [\hat{L}_z, \hat{L}_y \hat{L}_y]$

$= (i\hbar \hat{L}_y)\hat{L}_x + \hat{L}_x(i\hbar \hat{L}_y) + (-i\hbar \hat{L}_x)\hat{L}_y + \hat{L}_y(-i\hbar \hat{L}_x)$

$= i\hbar(\hat{L}_y \hat{L}_x + \hat{L}_x \hat{L}_y - \hat{L}_x \hat{L}_y - \hat{L}_y \hat{L}_x) = 0$

Same relation will hold for x & y components because of the definition of $\hat{L}^2$ (all components are equivalent, not distinguishable)

3. Liboff 9.1: See solution in Liboff (page 357).

Liboff 9.3: $L = \hbar\sqrt{56} = \hbar\sqrt{\ell(\ell+1)} \Rightarrow \ell = 7$

$\cos^2\theta = \frac{m^2}{\ell(\ell+1)}$, $\theta_{min} = \cos^{-1}\left[\left(\frac{m_{max}^2}{\ell(\ell+1)}\right)^{1/2}\right]$, $m_{max} = \ell = 7 \Rightarrow$

$\theta_{min} = \cos^{-1}\left[\left(\frac{7}{8}\right)^{1/2}\right] = 21^\circ \Rightarrow \boxed{\theta_{min} = 21^\circ}$

4. $\phi_{5,4} \Rightarrow L_z = \hbar m = 4\hbar$

$L^2 = L_x^2 + L_y^2 + L_z^2 \Rightarrow L_x^2 = L^2 - L_y^2 - L_z^2 \Rightarrow L_x^2 \leq L^2 - L_z^2$

$L^2 = \hbar^2 \ell(\ell+1) = 30\hbar^2 \quad , \quad L_z^2 = 16\hbar^2$

$L_x^2 \leq 30\hbar^2 - 16\hbar^2 = 14\hbar^2 \Rightarrow |L_x| \leq \sqrt{14}\,\hbar$

$\boxed{L_x = \{-3\hbar, -2\hbar, -\hbar, 0, \hbar, 2\hbar, 3\hbar\}}$

5. Liboff 9.23:

(a) Ladder operators : $\hat{L}_+ = \hat{L}_x + i\hat{L}_y \, , \, \hat{L}_- = \hat{L}_x - i\hat{L}_y \, , \, \hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-), \hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)$

$\hat{L}_+ |\ell, m\rangle = |\ell, m+1\rangle \Rightarrow \langle \ell, m | L_+ | \ell, m\rangle = \langle \ell, m | \ell, m+1\rangle = 0$

$\hat{L}_- |\ell, m\rangle = |\ell, m-1\rangle \Rightarrow \langle \ell, m | L_- | \ell, m\rangle = \langle \ell, m | \ell, m-1\rangle = 0$

$\langle L_x \rangle = \frac{1}{2}(0+0) = 0 \, , \langle L_y \rangle = \frac{1}{2i}(0+0) = 0 \Rightarrow \langle L_x \rangle = \langle L_y \rangle = 0$

(b) $\langle L_x^2 \rangle = \langle L_y^2 \rangle = \underline{\hbar^2 \ell(\ell+1) - m^2\hbar^2}$
$\qquad\qquad 2$

$L_x^2 + L_y^2 = L^2 - L_z^2 \Rightarrow \langle L_x^2 \rangle + \langle L_y^2 \rangle = \langle L^2 \rangle - \langle L_z^2 \rangle$

$\langle L_x^2 \rangle = \langle L_y^2 \rangle$ by symmetry $\Rightarrow 2\langle L_x^2 \rangle = \langle L^2 \rangle - \langle L_z^2 \rangle = \hbar^2(\ell(\ell+1)) - (\hbar m)^2$

$\langle L_x^2 \rangle = \langle L_y^2 \rangle = \dfrac{\hbar^2 \ell(\ell+1) - m^2\hbar^2}{2}$

Liboff 9.25(a): $\psi(\theta, \phi, 0) = \frac{3}{\sqrt{26}} Y_1^1 + \frac{4}{\sqrt{26}} Y_7^3 + \frac{1}{\sqrt{26}} Y_7^1$

$(\ell, m) = (1,1) \Rightarrow L = \sqrt{\hbar^2 \ell(\ell+1)} = \hbar\sqrt{2} \, , \, L_z = \hbar m = \hbar \, , \, P = \left|\frac{3}{\sqrt{26}}\right|^2 = \frac{9}{26}$

$(\ell, m) = (7,3) \Rightarrow L = \hbar\sqrt{56} \, , \, L_z = 3\hbar \, , \, P = \frac{8}{13}$

$(\ell, m) = (7,1) \Rightarrow L = \hbar\sqrt{56} \, , \, L_z = \hbar \, , \, P = \frac{1}{26}$

6. Liboff 9.30:

$\hat{L}^2 |\ell, m\rangle = \hbar^2 \ell(\ell+1) \, , \, \hat{L}_z |\ell, m\rangle = \hbar m$

$(\hat{L}_1^2, \hat{L}_{1z}, \hat{L}_2^2, \hat{L}_{2z}) = (\hbar^2 \ell_1(\ell_1+1), \hbar m_1, \hbar^2 \ell_2(\ell_2+1), \hbar m_2)$

Liboff 9.34: $|l m l_1 l_2\rangle = |1,-1,1,1\rangle$ (coupled)

$m=-1$; $l_1 = 1 \Rightarrow m_1 = -1, 0, 1$; $l_2 = 1 \Rightarrow m_2 = -1, 0, 1$

$m = m_1 + m_2 \Rightarrow (m_1, m_2) = (-1, 0)$ or $(0, -1)$

Possible measurement values = $L_{1z} = -\hbar$ & $L_{1z} = 0$

$|l m l_1 l_2\rangle = |1,-1,1,1\rangle = C_{0-1}|1,0\rangle_1 |1,-1\rangle_2 + C_{-10}|1,-1\rangle_1 |1,0\rangle_2$

$\hat{L}_-|l,-1,1,1\rangle = 0 = \sqrt{2}\,(C_{0-1} + C_{-10})|1,-1\rangle_1 |1,-1\rangle_2 \Rightarrow C_{0-1} = -C_{-10} \Rightarrow |C_{0-1}|^2 = |C_{-10}|^2 = \frac{1}{2}$

$\boxed{P(-\hbar) = \frac{1}{2},\ P(0) = \frac{1}{2}}$

Liboff 9.35: $l_1 = 1 \Rightarrow m_1 = -1, 0, 1$, $l_2 = 1 \Rightarrow m_2 = -1, 0, 1$

$m = -2$, $m = m_1 + m_2 \Rightarrow m_1 = m_2 = -1$

$\hat{L}_{1z}|l,m\rangle = -\hbar$, $\hat{L}_{2z}|l,m\rangle = -\hbar$ (l & m = 1 & -1)

$\boxed{\text{Joint probability of finding } L_{1z} = L_{2z} = -\hbar \text{ is } 1}$