

3

Electric and Magnetic Forces

Electromagnetic forces determine all essential features of charged particle acceleration and transport. This chapter reviews basic properties of electromagnetic forces. Advanced topics, such as particle motion with time-varying forces, are introduced throughout the book as they are needed.

It is convenient to divide forces between charged particles into electric and magnetic components. The relativistic theory of electrodynamics shows that these are manifestations of a single force. The division into electric and magnetic interactions depends on the frame of reference in which particles are observed.

Section 3.1 introduces electromagnetic forces by considering the mutual interactions between pairs of stationary charges and current elements. Coulomb's law and the law of Biot and Savart describe the forces. Stationary charges interact through the electric force. Charges in motion constitute currents. When currents are present, magnetic forces also act.

Although electrodynamics is described completely by the summation of forces between individual particles, it is advantageous to adopt the concept of fields. Fields (Section 3.2) are mathematical constructs. They summarize the forces that could act on a test charge in a region with a specified distribution of other charges. Fields characterize the electrodynamic properties of the charge distribution. The Maxwell equations (Section 3.3) are direct relations between electric and magnetic fields. The equations determine how fields arise from distributed charge and current and specify how field components are related to each other.

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Electric and magnetic fields are often visualized as vector lines since they obey equations similar to those that describe the flow of a fluid. The field magnitude (or strength) determines the density of lines. In this interpretation, the Maxwell equations are fluidlike equations that describe the creation and flow of field lines. Although it is unnecessary to assume the physical existence of field lines, the concept is a powerful aid to intuit complex problems.

The Lorentz law (Section 3.2) describes electromagnetic forces on a particle as a function of fields and properties of the test particle (charge, position and velocity). The Lorentz force is the basis for all orbit calculations in this book. Two useful subsidiary functions of field quantities, the electrostatic and vector potentials, are discussed in Section 3.4. The electrostatic potential (a function of position) has a clear physical interpretation. If a particle moves in a static electric field, the change in kinetic energy is equal to its charge multiplied by the change in electrostatic potential. Motion between regions of different potential is the basis of electrostatic acceleration. The interpretation of the vector potential is not as straightforward. The vector potential will become more familiar through applications in subsequent chapters.

Section 3.6 describes an important electromagnetic force calculation, motion of a charged particle in a uniform magnetic field. Expressions for the relativistic equations of motion in cylindrical coordinates are derived in Section 3.5 to apply in this calculation.

3.1 FORCES BETWEEN CHARGES AND CURRENTS

The simplest example of electromagnetic forces, the mutual force between two stationary point charges, is illustrated in Figure 3.1a. The force is directed along the line joining the two particles, \mathbf{r} . In terms of \mathbf{u}_r (a vector of unit length aligned along \mathbf{r}), the force on particle 2 from particle 1 is

$$\mathbf{F}(1 \Rightarrow 2) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2 \mathbf{u}_r}{r^2} \text{ (newtons)}. \quad (3.1)$$

The value of ϵ_0 is

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ (A-s/V-m)}.$$

In Cartesian coordinates, $\mathbf{r} = (x_2 - x_1)\mathbf{u}_x + (y_2 - y_1)\mathbf{u}_y + (z_2 - z_1)\mathbf{u}_z$. Thus, $r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$. The force on particle 1 from particle 2 is equal and opposite to that of Eq. (3.1). Particles with the same polarity of charge repel one another. This fact affects high-current beams. The electrostatic repulsion of beam particles causes beam expansion in the absence of strong focusing.

Currents are charges in motion. Current is defined as the amount of charge in a certain cross section (such as a wire) passing a location in a unit of time. The mks unit of current is the ampere (coulombs per second). Particle beams may have charge and current. Sometimes, charge effects

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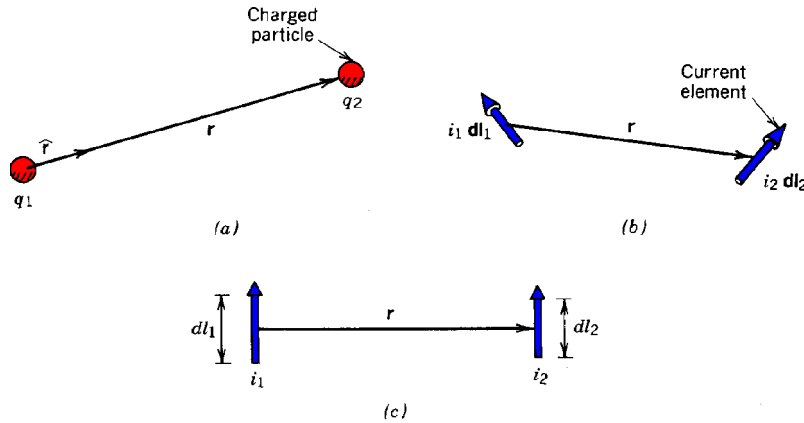


Figure 3.1 Forces between charged particles. (a) Electrostatic force. (b) Magnetostatic force. (c) Magnetostatic force between parallel current elements.

can be neutralized by adding particles of opposite-charge sign, leaving only the effects of current. This is true in a metal wire. Electrons move through a stationary distribution of positive metal ions. The force between currents is described by the law of Biot and Savart. If $i_1 d\mathbf{l}_1$ and $i_2 d\mathbf{l}_2$ are current elements (e.g., small sections of wires) oriented as in Figure 3.1b, the force on element 2 from element 1 is

$$d\mathbf{F} = \frac{\mu_o}{4\pi} \frac{i_2 d\mathbf{l}_2 \times (i_1 d\mathbf{l}_1 \times \mathbf{u}_r)}{r^2} . \quad (3.2)$$

where \mathbf{u}_r is a unit vector that points from 1 to 2 and

$$\mu_o = 4\pi \times 10^{-7} = 1.26 \times 10^{-6} \text{ (V-s/A-m)}.$$

Equation (3.2) is more complex than (3.1); the direction of the force is determined by vector cross products. Resolution of the cross products for the special case of parallel current elements is shown in Figure 3.1c. Equation (3.2) becomes

$$d\mathbf{F}(1 \Rightarrow 2) = -\frac{\mu_o}{4\pi} \frac{i_1 i_2 dl_1 dl_2}{r^2} \mathbf{u}_r.$$

Currents in the same direction attract one another. This effect is important in high-current relativistic electron beams. Since all electrons travel in the same direction, they constitute parallel current elements, and the magnetic force is attractive. If the electric charge is neutralized by ions, the magnetic force dominates and relativistic electron beams can be self-confined.

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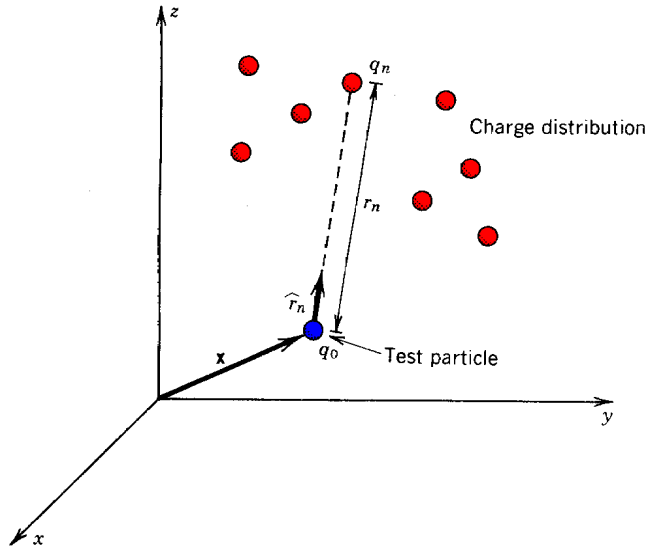


Figure 3.2 Electrical force exerted on a test particle by a distribution of charged particles.

3.2 THE FIELD DESCRIPTION AND THE LORENTZ FORCE

It is often necessary to calculate electromagnetic forces acting on a particle as it moves through space. Electric forces result from a specified distribution of charge. Consider, for instance, a low-current beam in an electrostatic accelerator. Charges on the surfaces of the metal electrodes provide acceleration and focusing. The electric force on beam particles at any position is given in terms of the specified charges by

$$\mathbf{F} = \sum_n \frac{1}{4\pi\epsilon_0} \frac{q_0 q_n \mathbf{u}_{rn}}{r_n^2},$$

where q_0 is the charge of a beam particle and the sum is taken over all the charges on the electrodes (Fig. 3.2).

In principle, particle orbits can be determined by performing the above calculation at each point of each orbit. A more organized approach proceeds from recognizing that (1) the potential force on a test particle at any position is a function of the distribution of external charges and (2) the net force is proportional to the charge of the test particle. The function $\mathbf{F}(\mathbf{x})/q_0$ characterizes the action of the electrode charges. It can be used in subsequent calculations to determine the orbit of any test particle. The function is called the *electric field* and is defined by

$$\mathbf{E}(\mathbf{x}) = \sum_n \frac{1}{4\pi\epsilon_0} \frac{q_n \mathbf{u}_{rn}}{r_n^2}. \quad (3.3)$$

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The sum is taken over all specified charges. It may include freely moving charges in conductors, bound charges in dielectric materials, and free charges in space (such as other beam particles). If the specified charges move, the electric field may also be a function of time-, in this case, the equations that determine fields are more complex than Eq. (3.3).

The electric field is usually taken as a smoothly varying function of position because of the $1/r^2$ factor in the sum of Eq. (3.3). The smooth approximation is satisfied if there is a large number of specified charges, and if the test charge is far from the electrodes compared to the distance between specified charges. As an example, small electrostatic deflection plates with an applied voltage of 100 V may have more than 10^{10} electrons on the surfaces. The average distance between electrons on the conductor surface is typically less than 1 μm .

When \mathbf{E} is known, the force on a test particle with charge q_0 as a function of position is

$$\mathbf{F}(\mathbf{x}) = q_0 \mathbf{E}(\mathbf{x}). \quad (3.4)$$

This relationship can be inverted for measurements of electric fields. A common nonperturbing technique is to direct a charged particle beam through a region and infer electric field by the acceleration or deflection of the beam.

A summation over current elements similar to Eq. (3.3) can be performed using the law of Biot and Savart to determine forces that can act on a differential test element of current. This function is called the magnetic field \mathbf{B} . (Note that in some texts, the term magnetic field is reserved for the quantity \mathbf{H} , and \mathbf{B} is called the magnetic induction.) In terms of the field, the magnetic force on $i\mathbf{dl}$ is

$$d\mathbf{F} = i\mathbf{dl} \times \mathbf{B}. \quad (3.5)$$

Equation (3.5) involves the vector cross product. The force is perpendicular to both the current element and magnetic field vector.

An expression for the total electric and magnetic forces on a single particle is required to treat beam dynamics. The differential current element, $i\mathbf{dl}$, must be related to the motion of a single charge. The correspondence is illustrated in Figure 3.3. The test particle has charge q and velocity \mathbf{v} . It moves a distance dl in a time $dt = |dl|/|v|$. The current (across an arbitrary cross section) represented by this motion is $q/(|dl|/|v|)$. A moving charged particle acts like a current element with

$$i\mathbf{dl} = \frac{q\mathbf{dl}}{|dl|/|v|} = q\mathbf{v}.$$

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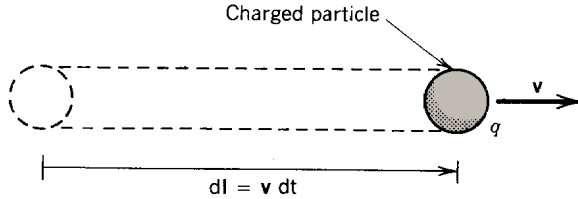


Figure 3.3 Representation of a moving charged particle as a current element.

The magnetic force on a charged particle is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \quad (3.6)$$

Equations (3.4) and (3.6) can be combined into a single expression (the Lorentz force law)

$$\mathbf{F}(\mathbf{x},t) = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (3.7)$$

Although we derived Equation (3.7) for static fields, it holds for time-dependent fields as well. The Lorentz force law contains all the information on the electromagnetic force necessary to treat charged particle acceleration. With given fields, charged particle orbits are calculated by combining the Lorentz force expression with appropriate equations of motion. In summary, the field description has the following advantages.

1. Fields provide an organized method to treat particle orbits in the presence of large numbers of other charges. The effects of external charges are summarized in a single, continuous function.
2. Fields are themselves described by equations (Maxwell equations). The field concept extends beyond the individual particle description. Chapter 4 will show that field lines obey geometric relationships. This makes it easier to visualize complex force distributions and to predict charged particle orbits.
3. Identification of boundary conditions on field quantities sometimes makes it possible to circumvent difficult calculations of charge distributions in dielectrics and on conducting boundaries.
4. It is easier to treat time-dependent electromagnetic forces through direct solution for field quantities.

The following example demonstrates the correspondence between fields and charged particle distributions. The parallel plate capacitor geometry is shown in Figure 3.4. Two infinite parallel metal plates are separated by a distance d . A battery charges the plates by transferring electrons from one plate to the other. The excess positive charge and negative electron charge spread uniformly on the inside surfaces. If this were not true, there would be electric fields inside the metal. The problem is equivalent to calculating the electric fields from two thin sheets of charge,

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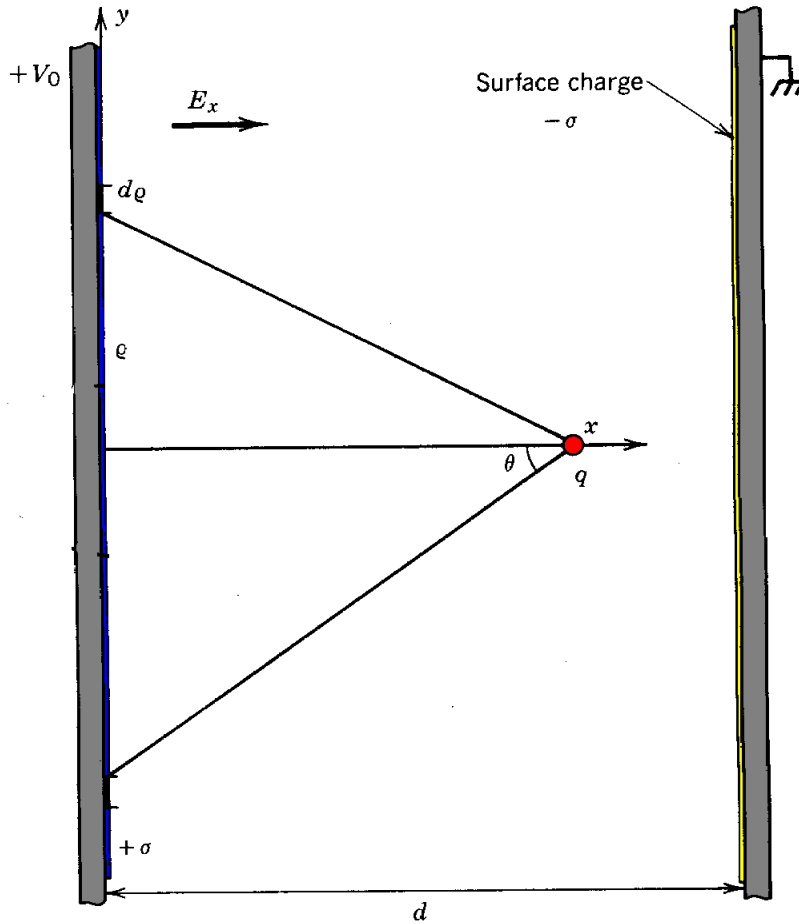


Figure 3.4 Calculation of force on a test particle between uniformly charged plates.

as shown in Figure 3.4. The surface charge densities, $\pm \sigma$ (in coulombs per square meter), are equal in magnitude and opposite in sign.

A test particle is located between the plates a distance x from the positive electrode. Figure 3.4 defines a convenient coordinate system. The force from charge in the differential annulus illustrated is repulsive. There is force only in the x direction; by symmetry transverse forces cancel. The annulus has charge $(2\pi\rho d\rho \sigma)$ and is a distance $(\rho^2 + x^2)^{1/2}$ from the test charge. The total force [from Eq. (3.1)] is multiplied by $\cos\theta$ to give the x component.

$$dF_x = \frac{2\pi\rho d\rho \sigma q_o \cos\theta}{4\pi\epsilon_o (\rho^2 + x^2)},$$

where $\cos\theta = x/(\rho^2 + x^2)^{1/2}$. Integrating the above expression over ρ from 0 to ∞ gives the net force

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$$F^+ = \int_0^{\infty} \frac{\rho \, d\rho \, \sigma q_o \, x}{2\epsilon_o (\rho^2 + x^2)^{3/2}} = \frac{q_o \sigma}{2\epsilon_o}. \quad (3.8)$$

A similar result is obtained for the force from the negative-charge layer. It is attractive and adds to the positive force. The electric field is found by adding the forces and dividing by the charge of the test particle

$$E_x(x) = (F^+ - F^-)/q = \sigma/\epsilon_o. \quad (3.9)$$

The electric field between parallel plates is perpendicular to the plates and has uniform magnitude at all positions. Approximations to the parallel plate geometry are used in electrostatic deflectors; particles receive the same impulse independent of their position between the plates.

3.3 THE MAXWELL EQUATIONS

The Maxwell equations describe how electric and magnetic fields arise from currents and charges. They are continuous differential equations and are most conveniently written if charges and currents are described by continuous functions rather than by discrete quantities. The source functions are the *charge density*, $\rho(x, y, z, t)$ and *current density* $\mathbf{j}(x, y, z, t)$.

The charge density has units of coulombs per cubic meters (in MKS units). Charges are carried by discrete particles, but a continuous density is a good approximation if there are large numbers of charged particles in a volume element that is small compared to the minimum scale length of interest. Discrete charges can be included in the Maxwell equation formulation by taking a charge density of the form $\rho = q\delta[\mathbf{x} - \mathbf{x}_o(t)]$. The delta function has the following properties:

$$\begin{aligned} \delta(\mathbf{x} - \mathbf{x}_o) &= 0, \quad \text{if } \mathbf{x} \neq \mathbf{x}_o, \\ \int dx \int dy \int dz \, \delta(\mathbf{x} - \mathbf{x}_o) &= 1. \end{aligned} \quad (3.10)$$

The integral is taken over all space.

The current density is a vector quantity with units amperes per square meter. It is defined as the differential flux of charge, or the charge crossing a small surface element per second divided by the area of the surface. Current density can be visualized by considering how it is measured (Fig. 3.5). A small current probe of known area is adjusted in orientation at a point in space until

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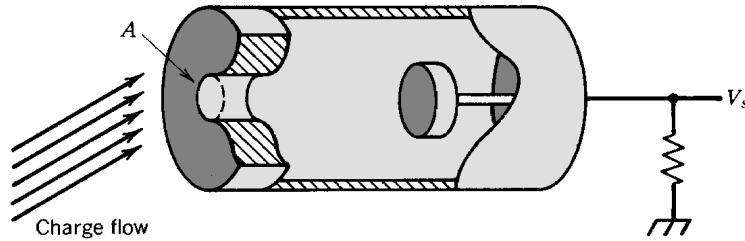


Figure 3.5 Probe to measure current density.

the current reading is maximized. The orientation of the probe gives the direction, and the current divided by the area gives the magnitude of the current density.

The general form of the Maxwell equations in MKS units is

$$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t, \quad (3.11)$$

$$\nabla \times \mathbf{B} = (1/c^2) \partial \mathbf{E} / \partial t + \mu \mathbf{j}, \quad (3.12)$$

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0, \quad (3.13)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (3.14)$$

Although these equations will not be derived, there will be many opportunities in succeeding chapters to discuss their physical implications. Developing an intuition and ability to visualize field distributions is essential for understanding accelerators. Characteristics of the Maxwell equations in the static limit and the concept of field lines will be treated in the next chapter.

No distinction has been made in Eqs. (3.11)-(3.14) between various classes of charges that may constitute the charge density and current density. The Maxwell equations are sometimes written in terms of vector quantities \mathbf{D} and \mathbf{H} . These are subsidiary quantities in which the contributions from charges and currents in linear dielectric or magnetic materials have been extracted. They will be discussed in Chapter 5.

3.4 ELECTROSTATIC AND VECTOR POTENTIALS

The electrostatic potential is a scalar function of the electric field. In other words, it is specified by a single value at every point in space. The physical meaning of the potential can be demonstrated by considering the motion of a charged particle between two parallel plates (Fig. 3.6). We want to find the change in energy of a particle that enters that space between the plates with kinetic energy

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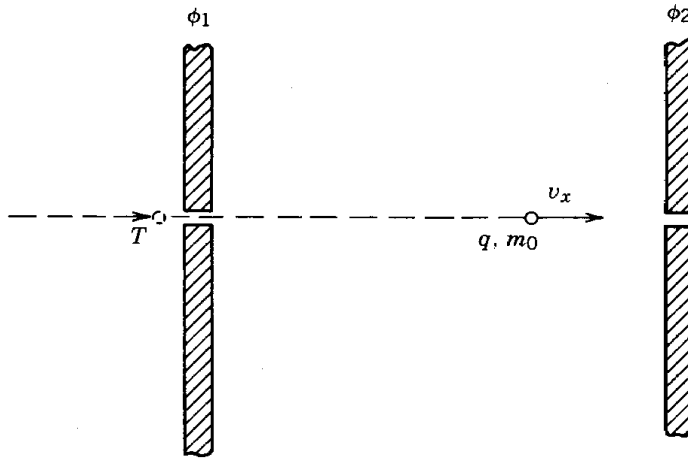


Figure 3.6 Acceleration of a particle between charged parallel plates.

T. Section 3.2 has shown that the electric field E_x , is uniform. The equation of motion is therefore

$$dp_x/dt = F_x = qE_x.$$

The derivative can be rewritten using the chain rule to give

$$(dp_x/dx)(dx/dt) = v_x dp_x/dx = qE_x.$$

The relativistic energy E of a particle is related to momentum by Eq. (2.37). Taking the derivative in x of both sides of Eq. (2.37) gives

$$c^2 p_x dp_x/dx = E dE/dx.$$

This can be rearranged to give

$$dE/dx = [c^2 p_x/E] dp_x/dx = v_x dp_x/dx. \quad (3.15)$$

The final form on the right-hand side results from substituting Eq. (2.38) for the term in brackets. The expression derived in Eq. (3.15) confirms the result quoted in Section 2.9. The right-hand side is dp_x/dt which is equal to the force F_x . Therefore, the relativistic form of the energy [Eq. (2.35)] is consistent with Eq. (2.6). The integral of Eq. (3.15) between the plates is

$$\Delta E = q \int dx E_x. \quad (3.16)$$

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The *electrostatic potential* ϕ is defined by

$$\phi = - \int \mathbf{E} \cdot d\mathbf{x}. \quad (3.17)$$

The change in potential along a path in a region of electric fields is equal to the integral of electric field tangent to the path times differential elements of pathlength. Thus, by analogy with the example of the parallel plates [Eq. (3.1 6)] $\Delta E = -q\Delta\phi$. If electric fields are static, the total energy of a particle can be written

$$E = m_o c^2 + T_o - q(\phi - \phi_o), \quad (3.18)$$

where T_o is the particle kinetic energy at the point where $\phi = \phi_o$.

The potential in Eq. (3.18) is not defined absolutely; a constant can be added without changing the electric field distribution. In treating electrostatic acceleration, we will adopt the convention that the zero point of potential is defined at the particle source (the location where particles have zero kinetic energy). The potential defined in this way is called the absolute potential (with respect to the source). In terms of the absolute potential, the total energy can be written

$$E = m_o c^2 - q\phi$$

or

$$\gamma = 1 - q\phi/m_o c^2. \quad (3.19)$$

Finally, the static electric field can be rewritten in the differential form,

$$\begin{aligned} \mathbf{E} &= -\nabla\phi = (\partial\phi/\partial x) \mathbf{u}_x + (\partial\phi/\partial y) \mathbf{u}_y + (\partial\phi/\partial z) \mathbf{u}_z \\ &= E_x \mathbf{u}_x + E_y \mathbf{u}_y + E_z \mathbf{u}_z. \end{aligned} \quad (3.20)$$

If the potential is known as a function of position, the three components of electric field can be found by taking spatial derivatives (the gradient operation). The defining equation for electrostatic fields [Eq. (3.3)] can be combined with Eq. (3.20) to give an expression to calculate potential directly from a specified distribution of charges

$$\phi(\mathbf{x}) = \sum_n \frac{q_n/4\pi\epsilon_o}{|\mathbf{x} - \mathbf{x}_n|}. \quad (3.21)$$

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The denominator is the magnitude of the distance from the test charge to the n th charge. The integral form of this equation in terms of charge density is

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \iiint d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} . \quad (3.22)$$

Although Eq. 3.22 can be used directly to find the potential, we will usually use differential equations derived from the Maxwell equations combined with boundary conditions for such calculations (Chapter 4). The *vector potential* \mathbf{A} is another subsidiary quantity that can be valuable for computing magnetic fields. It is a vector function related to the magnetic field through the vector curl operation

$$\mathbf{B} = \nabla \times \mathbf{A} . \quad (3.23)$$

This relationship is general, and holds for time-dependent fields. We will use \mathbf{A} only for static calculations. In this case, the vector potential can be written as a summation over source current density

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \iiint d^3x' \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} . \quad (3.24)$$

Compared to the electrostatic potential, the vector potential does not have a straightforward physical interpretation. Nonetheless, it is a useful computational device and it is helpful for the solution of particle orbits in systems with geometry symmetry. In cylindrical systems it is proportional to the number of magnetic field lines encompassed within particle orbits (Section 7.4).

3.5 INDUCTIVE VOLTAGE AND DISPLACEMENT CURRENT

The static concepts already introduced must be supplemented by two major properties of time-dependent fields for a complete and consistent theory of electrodynamics. The first is the fact that time-varying magnetic fields lead to electric fields. This is the process of magnetic induction. The relationship between inductively generated electric fields and changing magnetic flux is stated in Faraday's law. This effect is the basis of betatrons and linear induction accelerators. The second phenomenon, first codified by Maxwell, is that a time-varying electric field leads to a virtual current in space, the displacement current. We can verify that displacement currents "exist" by measuring the magnetic fields they generate. A current monitor such as a Rogowski loop enclosing an empty space with changing electric fields gives a current reading. The combination of inductive fields with the displacement current leads to predictions of electromagnetic oscillations.

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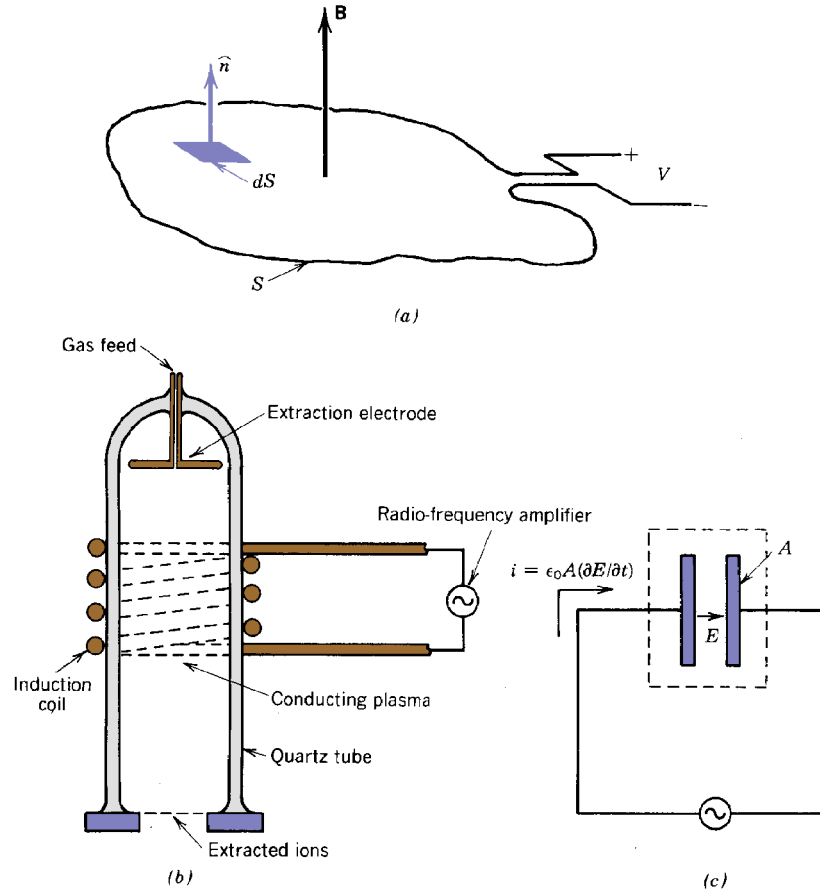


Figure 3.7 Inductive voltage and displacement currents. (a) Faraday's law. (b) Inductively coupled plasma source. (c) Alternating-current circuit with a parallel-plate capacitor.

Propagating and stationary electromagnetic waves are the bases for RF (radio-frequency) linear accelerators.

Faraday's law is illustrated in Figure 3.7a. A wire loop defines a surface S . The magnetic flux ψ passing through the loop is given by

$$\psi = \iint \mathbf{B} \cdot \mathbf{n} \, dS, \quad (3.25)$$

where \mathbf{n} is a unit vector normal to S and dS is a differential element of surface area. Faraday's law states that a voltage is induced around the loop when the magnetic flux changes according to

$$V = -d\psi/dt. \quad (3.26)$$

The time derivative of ψ is the total derivative. Changes in ψ can arise from a time-varying field at

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constant loop position, motion of the loop to regions of different field magnitude in a static field, or a combination of the two.

The term *induction* comes from induce, to produce an effect without a direct action. This is illustrated by the example of Figure 3.7b. an inductively coupled plasma source. (A plasma is a conducting medium of hot, ionized gas.) Such a device is often used as an ion source for accelerators. In this case, the plasma acts as the loop. Currents driven in the plasma by changing magnetic flux ionize and heat the gas through resistive effects. The magnetic flux is generated by windings outside the plasma driven by a high-frequency ac power supply. The power supply couples energy to the plasma through the intermediary of the magnetic fields. The advantage of inductive coupling is that currents can be generated without immersed electrodes that may introduce contaminants.

The sign convention of Faraday's law implies that the induced plasma currents flow in the direction opposite to those of the driving loop. Inductive voltages always drive reverse currents in conducting bodies immersed in the magnetic field; therefore, oscillating magnetic fields are reduced or canceled inside conductors. Materials with this property are called diamagnetic. Inductive effects appear in the Maxwell equations on the right-hand side of Eq. (3.11). Application of the Stokes theorem (Section 4.1) shows that Eqs. (3.11) and (3.26) are equivalent.

The concept of displacement current can be understood by reference to Figure 3.7c. An electric circuit consists of an ac power supply connected to parallel plates. According to Eq. 3.9, the power supply produces an electric field E between the plates by moving an amount of charge

$$Q = \epsilon_0 E_x A.$$

where A is the area of the plates. Taking the time derivative, the current through the power supply is related to the change in electric field by

$$i = \epsilon_0 A (\partial E_x / \partial t). \quad (3.27)$$

The partial derivative of Eq. (3.27) signifies that the variation results from the time variation of E_x with the plates at constant position. Suppose we considered the plate assembly as a black box without knowledge that charge was stored inside. In order to guarantee continuity of current around the circuit, we could postulate a virtual current density between the plates given by

$$j_d = \epsilon_0 (\partial E_x / \partial t). \quad (3.28)$$

This quantity, the displacement current density, is more than just an abstraction to account for a change in space charge inside the box. The experimentally observed fact is that there are magnetic fields around the plate assembly that identical to those that would be produced by a real wire connecting the plates and carrying the current specified by Eq. (3.27) (see Section 4.6). There is thus a parallelism of time-dependent effects in electromagnetism. Time-varying magnetic fields

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produce electric fields, and changing electric fields produce magnetic fields. The coupling and interchange of electric and magnetic field energy is the basis of electromagnetic oscillations. Displacement currents or, equivalently, the generation of magnetic fields by time-varying electric fields, enter the Maxwell equations on the right side of Eq. (3.12). Noting that

$$c = 1/\sqrt{\epsilon_0\mu_0} , \quad (3.29)$$

we see that the displacement current is added to any real current to determine the net magnetic field.

3.6 RELATIVISTIC PARTICLE MOTION IN CYLINDRICAL COORDINATES

Beams with cylindrical symmetry are encountered frequently in particle accelerators. For example, electron beams used in applications such as electron microscopes or cathode ray tubes have cylindrical cross sections. Section 3.7 will introduce an important application of the Lorentz force, circular motion in a uniform magnetic field. In order to facilitate this calculation and to derive useful formulas for subsequent chapters, the relativistic equations of motion for particles in cylindrical coordinates are derived in this section.

Cylindrical coordinates, denoted by (r, θ, z) , are based on curved coordinate lines. We recognize immediately that equations of the form $dp_r/dt = F_r$ are incorrect. This form implies that particles subjected to no radial force move in a circular orbit ($r = \text{constant}$, $dp_r/dt = 0$). This is not consistent with Newton's first law. A simple method to derive the proper equations is to express $d\mathbf{p}/dt = \mathbf{F}$ in Cartesian coordinates and make a coordinate transformation by direct substitution.

Reference to Figure 3.8 shows that the following equations relate Cartesian coordinates to cylindrical coordinates sharing a common origin and a common z axis, and with the line $(r, 0, 0)$ lying on the x axis:

$$x = r \cos\theta, \quad y = r \sin\theta, \quad z = z, \quad (3.30)$$

and

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x). \quad (3.31)$$

Motion along the z axis is described by the same equations in both frames, $dp_z/dt = F_z$. We will thus concentrate on equations in the (r, θ) plane. The Cartesian equation of motion in the x direction is

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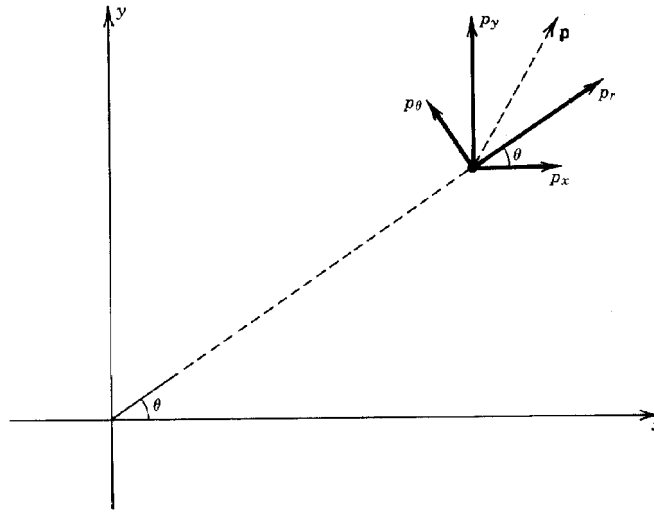


Figure 3.8 Relationship between cylindrical and Cartesian coordinates.

$$dp_x/dt = F_x. \quad (3.32)$$

Figure 3.8 shows that

$$p_x = p_r \cos\theta - p_\theta \sin\theta, \quad F_x = F_r \cos\theta - F_\theta \sin\theta.$$

Substituting in Eq. (3.32),

$$(dp_r/dt)\cos\theta - p_r \sin\theta(d\theta/dt) - (dp_\theta/dt)\sin\theta - p_\theta \cos\theta(d\theta/dt) = F_r \cos\theta - F_\theta \sin\theta.$$

The equation must hold at all positions, or at any value of θ . Thus, terms involving $\cos\theta$ and $\sin\theta$ must be separately equal. This yields the cylindrical equations of motion

$$dp_r/dt = F_r + [p_\theta d\theta/dt], \quad (3.33)$$

$$dp_\theta/dt = F_\theta - [p_r d\theta/dt]. \quad (3.34)$$

The quantities in brackets are correction terms for cylindrical coordinates. Equations (3.33) and (3.34) have the form of the Cartesian equations if the bracketed terms are considered as virtual forces. The extra term in the radial equation is called the centrifugal force, and can be rewritten

$$\text{Centrifugal force} = \gamma m_o v_\theta^2/r, \quad (3.35)$$

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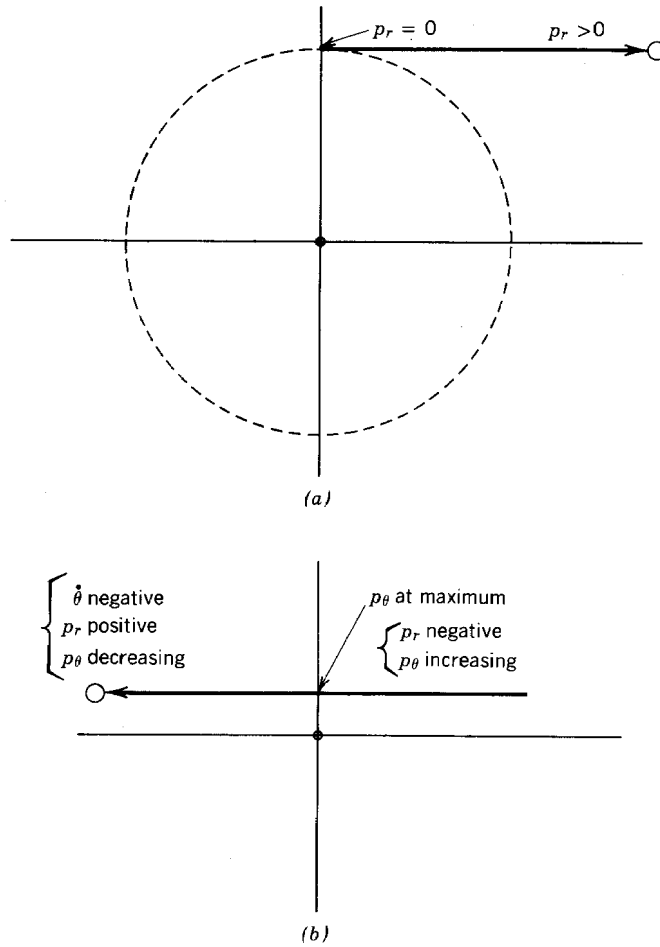


Figure 3.9 Virtual forces in a cylindrical coordinate system. (a) Centrifugal force: particle moves in straight line; p_r increases continually. (b) Coriolis force.

noting that $v_\theta = r d\theta/dt$. The bracketed term in the azimuthal equation is the Coriolis force, and can be written

$$\text{Coriolis force} = -\gamma m_o v_r v_\theta / r. \quad 3.36$$

Figure 3.9 illustrates the physical interpretation of the virtual forces. In the first example, a particle moves on a force-free, straight-line orbit. Viewed in the cylindrical coordinate system, the particle (with no initial v_r) appears to accelerate radially, propelled by the centrifugal force. At large radius, when v_θ approaches 0, the acceleration appears to stop, and the particle moves outward at constant velocity. The Coriolis force is demonstrated in the second example. A particle from large radius moves in a straight line past the origin with nonzero impact parameter.

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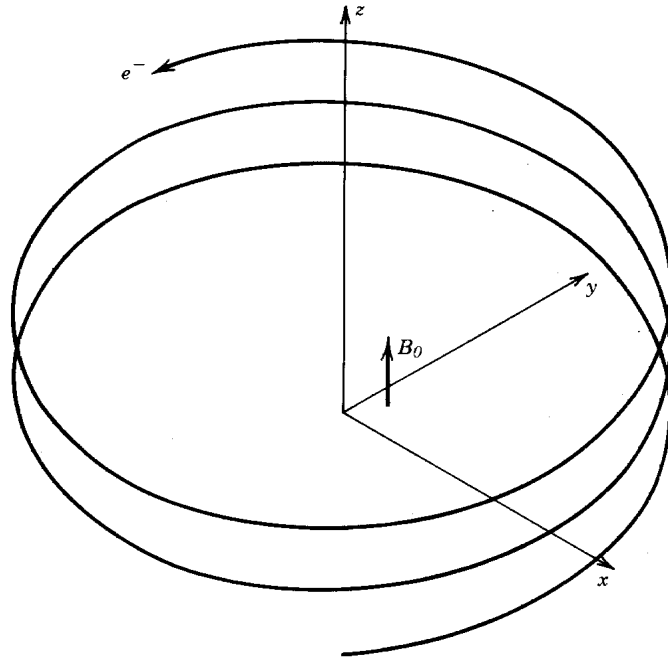


Figure 3.10 Helical orbit of a charged particle in a uniform magnetic field.

The azimuthal velocity, which was initially zero, increases as the particle moves inward with negative u , and decreases as the particle moves out. The observer in the cylindrical coordinate system notes a negative and then positive azimuthal acceleration.

Cylindrical coordinates appear extensively in accelerator theory. Care must be exercised to identify properly the orientation of the coordinates. For example, the z axis is sometimes aligned with the beam axis, while in other cases, the z axis may be along a symmetry axis of the accelerator. In this book, to avoid excessive notation, (r, θ, z) will be used for all cylindrical coordinate systems. Illustrations will clarify the geometry of each case as it is introduced.

3.7 MOTION OF CHARGED PARTICLES IN A UNIFORM MAGNETIC FIELD

Motion of a charged particle in a uniform magnetic field directed along the z axis, $\mathbf{B} = B_0 \mathbf{u}_z$, is illustrated in Figure 3.10. Only the magnetic component of the Lorentz force is included. The equation of motion is

$$d\mathbf{p}/dt = d(\gamma m_0 \mathbf{v})/dt = q \mathbf{v} \times \mathbf{B}. \quad (3.37)$$

By the nature of the cross product, the magnetic force is always perpendicular to the velocity of

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the particle. There is no force along a differential element of pathlength, $d\mathbf{x}$. Thus, $\int \mathbf{F} \cdot d\mathbf{x} = 0$. According to Eq. (2.6), magnetic fields perform no work and do not change the kinetic energy of the particle. In Eq. (3.37), γ is constant and can be removed from the time derivative.

Because the force is perpendicular to \mathbf{B} , there is no force along the z axis. Particles move in this direction with constant velocity. There is a force in the x - y plane. It is of constant magnitude (since the total particle velocity cannot change), and it is perpendicular to the particle motion. The projection of particle motion in the x - y plane is therefore a circle. The general three-dimensional particle orbit is a helix.

If we choose a cylindrical coordinate system with origin at the center of the circular orbit, then $dp_r/dt = 0$, and there is no azimuthal force. The azimuthal equation of motion [Eq. (3.34)] is satisfied trivially with these conditions. The radial equation [Eq. (3.33)] is satisfied when the magnetic force balances the centrifugal force, or

$$qv_{\theta}B_o = \gamma m_o v_{\theta}^2/r.$$

The particle orbit radius is thus

$$r_g = \gamma m_o v_{\theta} / |q| B_o. \quad (3.38)$$

This quantity is called the *gyroradius*. It is large for high-momentum particles; the gyroradius is reduced by applying stronger magnetic field. The point about which the particle revolves is called the *gyrocenter*. Another important quantity is the angular frequency of revolution of the particle, the *gyrofrequency*. This is given by $\omega_g = v_{\theta}/r$, or

$$\omega_g = |q| B_o / \gamma m_o. \quad (3.39)$$

The particle orbits in Cartesian coordinates are harmonic,

$$x(t) - x_o = r_g \cos(\omega_g t),$$

$$y(t) - y_o = r_g \sin(\omega_g t),$$

where x_o and y_o are the coordinates of the gyrocenter. The gyroradius and gyrofrequency arise in all calculations involving particle motion in magnetic fields. Magnetic confinement of particles in circular orbits forms the basis for recirculating high-energy accelerators, such as the cyclotron, synchrotron, microtron, and betatron.