3.11 Mechanics of Materials F03
Exam #2 Solutions : Tuesday 12/16/03

*(*show all of your work / calculations to get as much credit as possible : Be careful of units)*

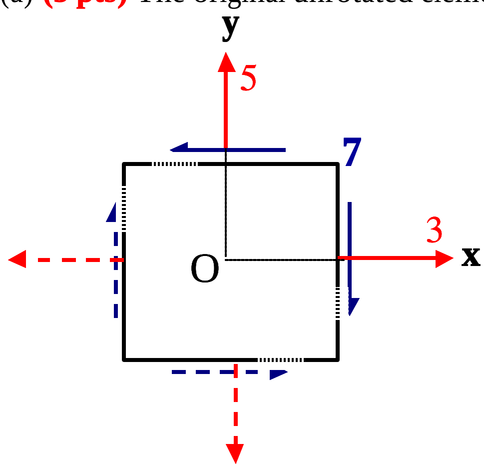
(58 pts total)

1. Mohr's Circle. A state of plane stress consists of a tensile stress of σ_x=3 MPa, σ_y=5 MPa, and τ_{xy}=-7 MPa: (17 pts total)

- **(a)** **Draw the original unrotated element and the corresponding 2-D Mohr's circle construction showing the x-face and y-face coordinates. (1 pt)**
- **(b)** **Calculate the principal stresses, σ_1 and σ_2 (2 pts) and their corresponding principal angles, θ_{p1}, θ_{p2} (2 pts) and show all of these on your Mohr's circle construction (2 pts) and a properly oriented stress element. (2 pts)**
- **(c)** **Calculate the maximum shear stresses, $\pm\tau_{MAX}$, (2 pt) and their corresponding angles of maximum shear stress, θ_{s1}, θ_{s2} (2 pts) and show all of these on your Mohr's circle construction (2 pts) and a properly oriented stress element. (2 pts)**

<u>ANSWER 1 :</u>

(a) **(3 pts)** The original unrotated element is shown below :

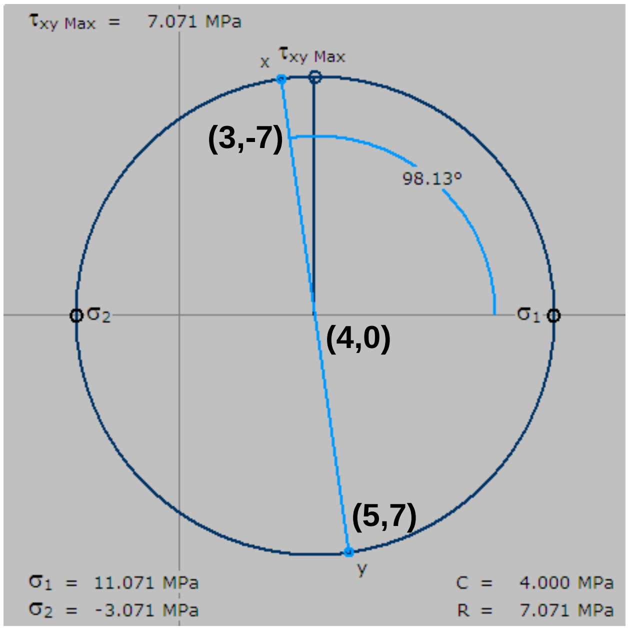


$$C = (\sigma_{ave}, 0)$$

$$\sigma_{ave} = \frac{(\sigma_x + \sigma_y)}{2} = \frac{(3+5)}{2} = 4$$

$$R = \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2} = \sqrt{\left[\frac{3-5}{2}\right]^2 + 7^2} = \sqrt{\left[\frac{-2}{2}\right]^2 + 7^2} = \sqrt{50} = 7.071$$

The Mohr's circle construction is shown below :



(b) **(4 pts)** The principal stresses, σ_1 and σ_2 and their corresponding principal angles, θ_{p1},θ_{p2} are calculated as follows :

Principal Stresses:

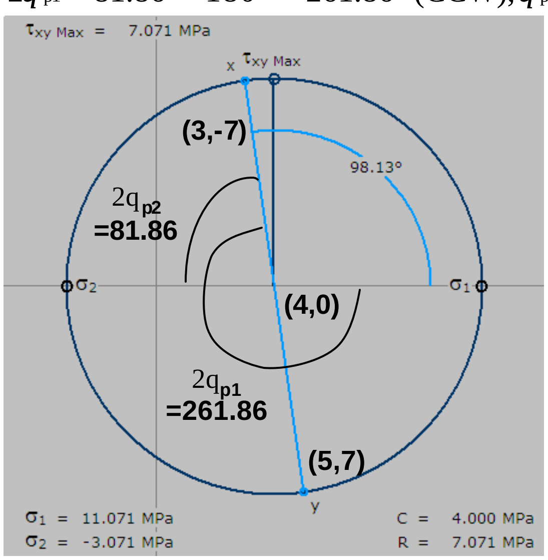
$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2} = 11.071 \text{ MPa.,-3.071 MPa}$$

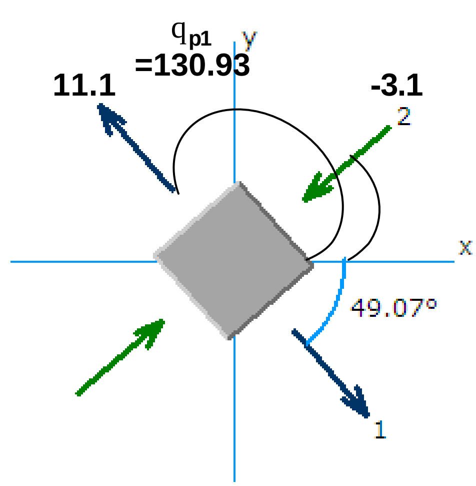
Principal Angles / Planes :

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-7)}{3-5} = 7$$

$$2\theta_{p2} = 81.86^\circ \text{ (CCW)}, \theta_{p2} = 40.93^\circ \text{ (CCW)}$$

$$2\theta_{p1} = 81.86^\circ + 180^\circ = 261.86^\circ \text{ (CCW)}, \theta_{p1} = 130.93^\circ \text{ (CCW)}$$





(c) **(4 pts)** The maximum shear stresses, $\pm\tau_{MAX}$, and their corresponding angles of maximum shear stress, θ_{s1},θ_{s2} are calculated as follows :

Maximum Shear Stresses:

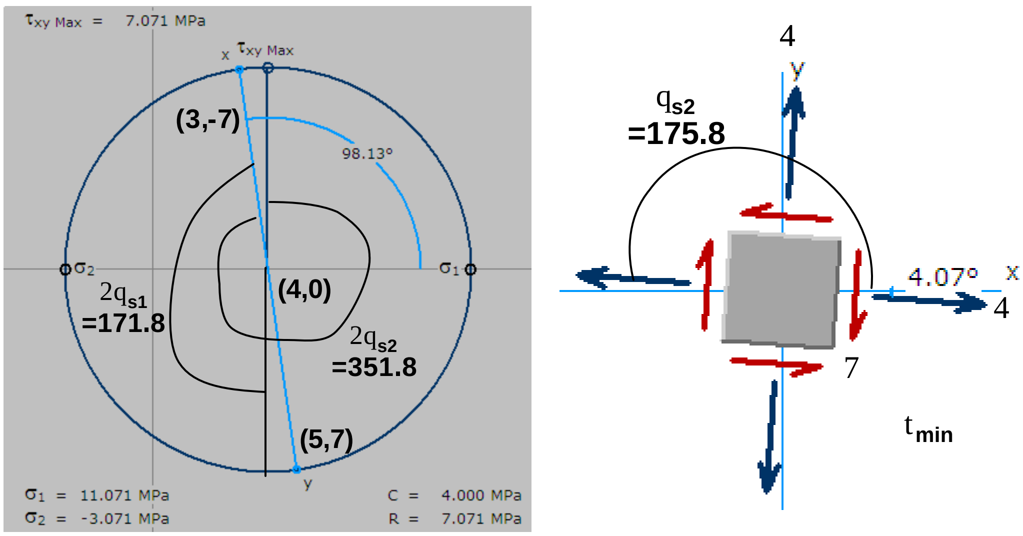
$$\tau_{\max,\min} = \pm\sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2} = \pm R = \pm 7.071\text{MPa}$$

Planes / Angles of Maximum Shear :

$$\tan(2\theta_s) = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}} = \frac{-(5-3)}{2(-7)} = -0.142$$

$$2\theta_{s2} = -8.08^o\,(\text{CW}) = 351.80^o\ (\text{CCW}), \theta_{s2} = 175.8^o\ (\text{CCW})$$

$$2\theta_{s1} = 351.80^o\ \text{-}180^o\ = 171.8\ (\text{CCW}), \theta_{s1} = 85.9^o\ (\text{CCW})$$



Need to calculate $\sigma_{x'}, \sigma_{y'}$ **corresponding to** $\tau_{\max,}\ 2\theta_{s1} = 171.8^o$ (CCW)

using stress transformation equations :

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{(\sigma_x - \sigma_y)\cos(2\theta)}{2} + \tau_{xy}\sin(2\theta) = 4 - \cos(171.8) - 7\sin(171.8) = 4$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{(\sigma_x - \sigma_y)\cos(2\theta)}{2} - \tau_{xy}\sin(2\theta) = 4$$

2. Rubber Elasticity. Three engineering stress versus strain curves in uniaxial tension are given below for NR=natural rubber (polyisoprene), NR+M2 and NR+M3=natural rubber composite filled with 10 wt % of 2 different organically modified clay particles (montmorillonites M2 and M3, data taken from *Chem. Mater.*, 14 (10), 4202-4208, 2002). The composite material is also initially macroscopically isotropic. (13 pts total)

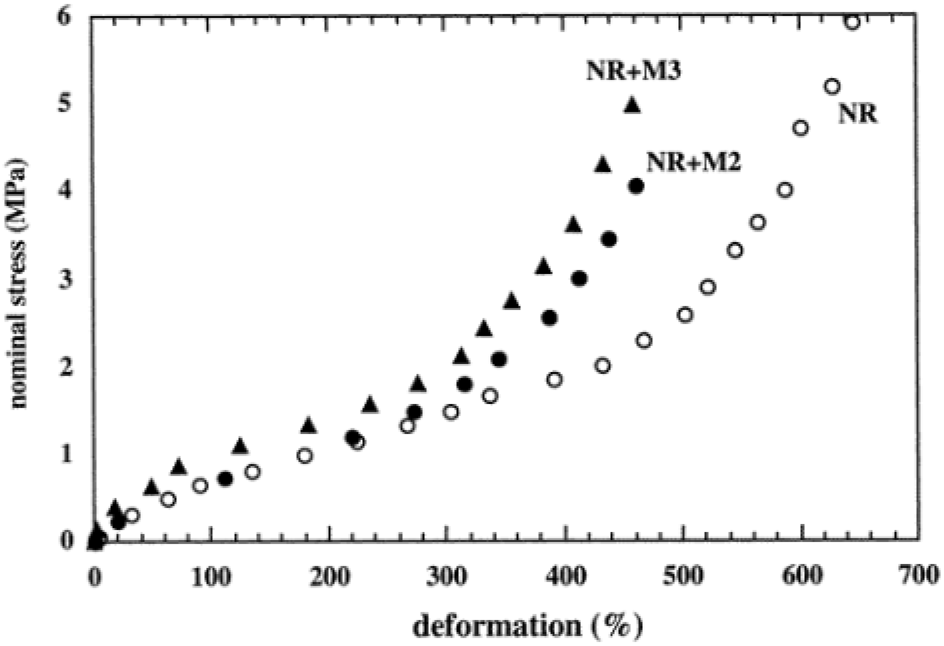
(a) Calculate the elastic modulus for each of the 3 samples. (3 pts)

(b) Using the Gaussian theory of rubber elasticity, calculate the network strand density n_x **(strands/m^3) for each of the 3 samples. (3 pts)**

(c) Explain the difference between the mechanical properties for the pure natural rubber and the composites for e>300%. **(2 pts)**

(d) Calculate the predicted stresses, s_x and s_y, from the Gaussian theory of rubber elasticity if this same block of natural rubber was deformed in biaxial tension to l_x=3 and l_y=4. **(2 pts)**

(e) Do any of these samples yield and if not, how come? **(2 pts)**



Answer. 2a.

The elastic modulus is the slope of the initial portion of the stress versus strain curve : note the strain needs to be converted from %.

$$E = \frac{\Delta s}{\Delta e}_{e \to 0}$$

For NR from ε=0-30% : $E = \frac{0.25}{0.3} = 0.83$ MPa

For NR+M2 from ε=0-20% : $E = \frac{0.2}{0.2} = 1$ MPa

For NR+M3 from ε=0-20%: $E = \frac{0.3}{0.2} = 1.5$ MPa

2b. For uniaxial tension of an elastomer as predicted by the Gaussian theory of rubber elasticity:

$$E = 3n_x k_B T \rightarrow n_x\left(\frac{\text{strands}}{m^3}\right) = \frac{E\ (Pa = N/m^2)}{(3k_B T = 1.23 \bullet 10^{-20}\ Nm)}$$

$$NR + M3 = \frac{1.5 \bullet 10^6}{1.23 \bullet 10^{-20}} = 1.21 \bullet 10^{26}\ \frac{\text{strands}}{m^3}$$

$$NR + M2 = \frac{1 \bullet 10^6}{1.23 \bullet 10^{-20}} = 8.1 \bullet 10^{25}\ \frac{\text{strands}}{m^3}$$

$$NR = \frac{0.83 \bullet 10^6}{1.23 \bullet 10^{-20}} = 6.7 \bullet 10^{25}\ \frac{\text{strands}}{m^3}$$

2c. The clay particles inhibit the entropic extension and orientation of random coil network strands, acting as crosslinks, requiring a higher stress for equivalent deformation.

2d.

The stress versus strain law for biaxial tension is :

$$s_x = E\left[l_x - \frac{1}{l_x^3 l_y^2}\right]$$

$$s_y = E\left[l_y - \frac{1}{l_x^2 l_y^3}\right]$$

Substituting in the given values one obtains:

$$s_x = 2.487 MPa$$

$$s_y = 3.3 MPa$$

2e. Rubbers do not yield because they are *entropically nonlinear elastic* based on the molecular entropic elasticity of an individual coil, they recover their deformation completely because the crosslinks prevent plastic deformation, permanent chain sliding.

3 a) For a brass alloy, the stress at which plastic deformation begins is 50,000 psi, and the modulus of elasticity, E, is 15 x 10^6 psi. (14 pts total)

i) What is the maximum load that may be applied to a specimen with a cross-sectional area of 0.2 in^2 without causing plastic deformation? (2 pts)

ii) If the original specimen length is 3.0 in., what is the maximum length to which it may be stretched without causing plastic deformation? (2 pts)

b) Consider a cylindrical rod, 20 in. (500 mm) long, having a diameter of 0.5 in. (12.7 mm). The rod is subject to a tensile load. If the rod is to experience neither plastic deformation nor an elongation of more than 0.05 in. (1.3 mm) when the applied load is 6500 lbs. (29,000 N), which of the four metals or alloys listed below are possible candidates? (4 pts)

Material	E (psi)	s_y (psi)	s_t (psi)

Aluminum	10 x 10^6	37,000	61,000
Brass	14.6 x 10^6	50,000	61,000
Copper	16 x 10^6	30,000	40,000
Steel	30 x 10^6	65,000	80,000

c) For aluminum under plane stress with s_x=15 ksi, s_y=45 ksi, and t_{xy}=-7 ksi: is the materials predicted to yield by the i) Tresca and ii) von Mises criterion? (6 pts)

3a. Answer

i) In order to prevent plastic deformation $\sigma<\sigma_Y$. Using this in conjunction with the definition of stress gives :

$$s_y = \frac{F_{max}}{A_0} \Rightarrow F_{max} = s_y \cdot A_0 = \left(50{,}000\, lbs/in^2\right)\left(0.2\, in^2\right) = 10{,}000\, lbs.$$

(ii) To find the maximum elongation without plastic deformation,use Hooke's Law as a approximation :

$$e_{max} = \frac{s_y}{E} = \frac{50{,}000\ psi}{15\,x10^6\ psi} = 3.333x10^{-3}\ in./in.$$

Since $e = \frac{\Delta L}{L_0} = \frac{d}{L_0} 3.333x10^{-3}\ in./in. \Rightarrow d = e \cdot L_0 = \left(3.33\,x10^{-3}\ in./in.\right)(3\,in.)$

$$\Rightarrow d = 0.01\,in.$$

The final length is: $L_f = L_0 + d = 3.0 + 0.01 = 3.01\,in.$

b. Two design criterion are given **:**

1) no plastic deformation: $\sigma<\sigma_Y$:

$$\sigma = \frac{F}{A_0} = \frac{6500\text{ lbs.}}{\pi\left(\frac{0.50\text{ in.}}{2}\right)^2} = \frac{6500}{0.196\text{ in}^2} = 33{,}100\text{ psi}$$

Thus, at this stage, only the aluminum, brass, and steel still qualify. But, we must still satisfy the elongation restriction, with $\Delta L \leq 0.05$ in. under the 6500 lb. load. This must mean that a particular upper limit on the strain is required, which must mean a lower limit on the Young's modulus:

$e = \frac{d_{MAX}}{L_0} = \frac{0.05in.}{20in.} = 0.0025in./in.$ With the stress already determined, we can take stress over strain and find a desired Young's modulus, and compare this desired modulus to the actual ones available on the list:

$E = \frac{\sigma}{\varepsilon} = \frac{33{,}100\text{ psi}}{0.0025} = 13.2 \times 10^6$ psi. Since the strain is a ***maximum***, the elasticity modulus, E, must be a minimum. By looking at the table, we see that of the three alloys left, only brass and steel satisfy both the criteria.

d.

Principal Stresses:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2} = \sigma_1 = 46.55 \text{ ksi}, \sigma_2 = 13.45 \text{ ksi}, \sigma_3 = 0 \text{ ksi}$$

$$\textit{Yielding takes place in 1D when: } \sigma \geq \sigma_Y$$

$$\textit{where: } \sigma_Y = \textit{yield stress in uniaxial tension}$$

$$\textit{Define "equivalent stress": } \overline{\sigma} = f(\sigma_{p1}, \sigma_{p2}, \sigma_{p3})$$

$$\textit{Yielding takes place in 3D when: } \overline{\sigma} \geq \sigma_Y$$

1. Tresca or Maxmimum Shear Stress Criterion:

$$\overline{\sigma}_{TRESCA} = 2\tau_{max} = \sigma_{p1} - \sigma_{p3} \geq \sigma_Y$$

$$\overline{\sigma}_{TRESCA} = 46.55 - 0 = 46.55 ksi > \sigma_Y(Al) = 37 ksi \therefore \textbf{the material is predicted to yield}$$

2. Von Mises Criterion:

$$\overline{\sigma}_{VM} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{p1} - \sigma_{p3})^2 + (\sigma_{p2} - \sigma_{p3})^2 + (\sigma_{p3} - \sigma_{p1})^2} \geq \sigma_Y$$

$$\overline{\sigma}_{VM}(2D) = 25 ksi < \sigma_Y(Al) = 37 ksi \therefore \textbf{the material is predicted not to yield}$$

4) Define the following words in one to two sentences using <u>no formulas</u>: (2 pts each=14 points total)

a) Lennard-Jones Potential : model of interatomic/intermolecular interactions, force vresus separation distance, which take into account both a long range attractive van der Waals component as well a s interatomic repulsion between overlapping electron clouds.

b) Critical Flaw Length : The length of the largest flaw in a sample that will cause the sample to fail under a given load, can be calculated from linear elastic fracture mechanics theory.

c) Strain Hardening : an increase in stress with increasing strain, could be nonlinear elastic as in rubbers or due to plastic deformation in metals/polymers.

d) Impact Energy : The energy absorbed when a material is deformed at high rates by an impactor. Depending on the magnitude will have contributions from elastic deformation, plastic deformation, and fracture.

e) Spherulite : A microstructural feature that occurs in semicrystalline polymers which consist of crystalline lamellae which expand radially outward from a nucleation site and contain amorphous regions between the lamellae.

f) Maxwell Model : Model for viscoelasticity which consists of a single linear elastic Hookean spring in series with a single Newtonian dashpot.

g) Craze : Highly localized region of plastic deformation in amorphous polymers which contains a fibrillar structure, is 50% voided, and is load bearing unlike a crack.