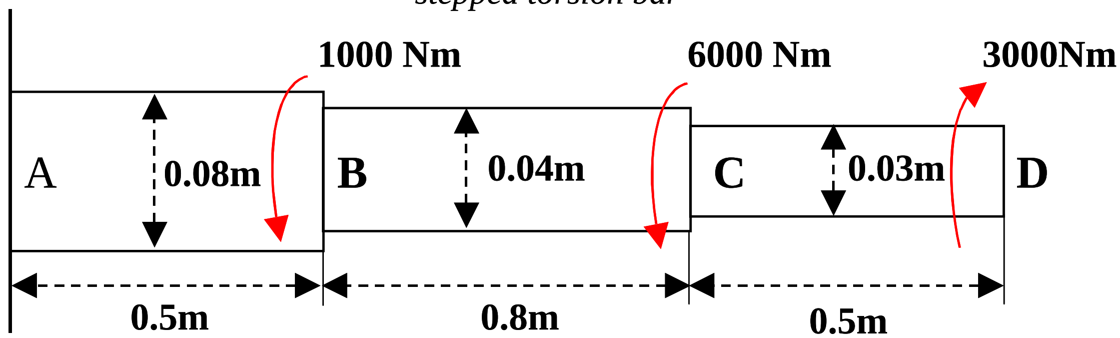
3.11 Mechanics of Materials F01
Exam #2 Solutions : Friday 11/08/01

*(*show all of your work / calculations to get as much credit as possible)*

1. A stepped steel (G=80 GPa) torsion bar ABCD consisting of solid circular cross-sections is subjected to three external torques, in the directions shown in the Figure below :

stepped torsion bar



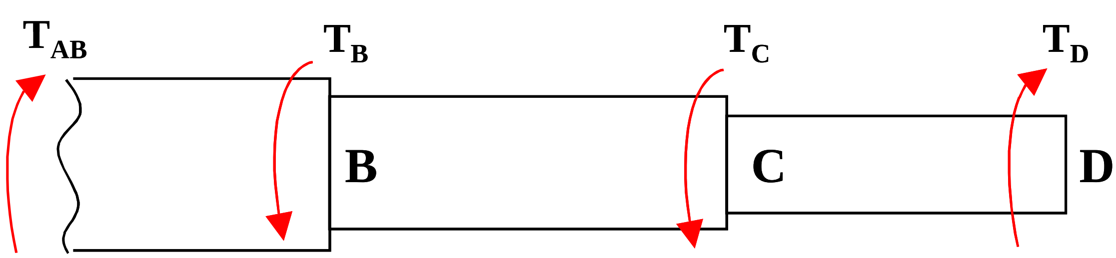
(a) calculate the maximum shear stress in the bar, τ_{MAX} (MPa)
(b) calculate the angle of twist at the end of the bar, ϕ_D (degrees)
USE THE FOLLOWING SIGN CONVENTIONS :
(CCW+) and (CW-) (10 pts total)

ANSWER 1:

To solve both (a) and (b) first one needs to calculate the **internal torques** in each of the three sections of the torsion bar which be uniform and constant in each of the sections. **(3 pts)** To find these internal torques, make cuts in each of the three sections, draw free body diagrams, and use the equations of static rotational equilibrium as follows: *(*note you can start on either side of the bar)*

In section : AB$\rightarrow$ the internal torque is $\mathbf{T_{AB}}$
BC$\rightarrow$ the internal torque is $\mathbf{T_{BC}}$
CD$\rightarrow$ the internal torque is $\mathbf{T_{CD}}$

To find T_{AB} :



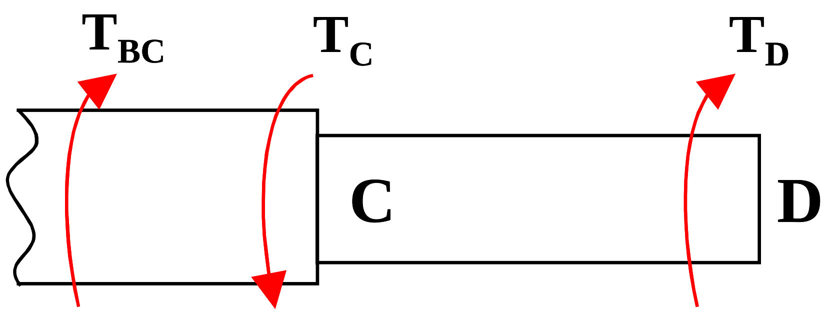
Using the sign conventions of CCW(+) and CW(-) :

$\Sigma T = \Sigma M = 0 = \mathbf{T_B + T_C - T_D - T_{AB}}$

$\mathbf{T_{AB}}$=(1000+6000-3000)Nm=+4000Nm (CW-)

*(*positive answer means assumed direction is correct)*

<u>Similarly for T_{BC} :</u>

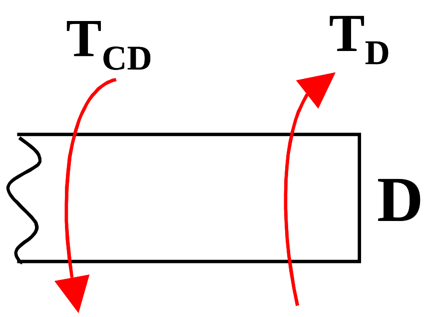


$\Sigma T = \Sigma M = 0 = \mathbf{T_C - T_D - T_{BC}}$

$\mathbf{T_{BC}}$=(6000-3000)Nm=+3000Nm (CW-)

*(*positive answer means assumed direction is correct)*

<u>Similarly for T_{CD} :</u>



$\Sigma T = \Sigma M = 0 = \mathbf{-T_3 + T_D}$

$\mathbf{T_{CD}}$=(3000)Nm=+3000Nm (CCW+)

*(*positive answer means assumed direction is correct)*

Find the polar moments of inertia, I_p or J, for each of the three different cross-sections **(1 pt)** :

$$I_p\,(\text{solid circular cross section}) = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$$

$$\left(I_p\right)_{AB} = \frac{\pi(0.08\text{m})^4}{32} = 4.0192 \bullet 10^{-6}\,\text{m}^4$$

$$\left(I_p\right)_{BC} = \frac{\pi(0.04\text{m})^4}{32} = 2.512 \bullet 10^{-7}\,\text{m}^4$$

$$\left(I_p\right)_{CD} = \frac{\pi(0.03\text{m})^4}{32} = 7.948 \bullet 10^{-8}\,\text{m}^4$$

(a) Calculate the maximum shear stress in the bar, τ_{MAX} (MPa) : (3 pts)

First, find shear stress in each section (where T is opposite sign to internal torques calculated above) :

$$\tau_{MAX} = \frac{Tr}{I_p}\ (\text{for solid circular cross-sections or tubes})\ \text{or}\ \tau_{MAX} = \frac{16T}{\pi d^3}(\text{for solid circular cross-sections})$$

$$\tau_{AB} = \frac{T_{AB} r_{AB}}{\left(I_p\right)_{AB}} = \frac{(+4000\text{ Nm})(0.04\text{ m})}{\left(4.0192 \bullet 10^{-6}\,\text{m}^4\right)} = +39808917\frac{\text{N}}{\text{m}^2} = +39.8\text{ MPa}$$

$$\tau_{BC} = \frac{T_{BC} r_{BC}}{\left(I_p\right)_{BC}} = \frac{(+3000\text{ Nm})(0.02\text{ m})}{\left(2.512 \bullet 10^{-7}\,\text{m}^4\right)} = +238853503\frac{\text{N}}{\text{m}^2} = +238.8\text{ MPa}$$

$$\tau_{CD} = \frac{T_{CD} r_{CD}}{\left(I_p\right)_{CD}} = \frac{(-3000\text{ Nm})(0.015\text{ m})}{\left(7.948 \bullet 10^{-8}\,\text{m}^4\right)} = -566180171\frac{\text{N}}{\text{m}^2} = -566.2\text{ MPa}$$

Hence, the maximum positive shear stress is $\tau_{MAX}=\tau_{AB}=+40$ MPa.
The maximum negative shear stress is $\tau_{MAX}=\tau_{CD}=-566$ MPa.

(b) Calculate the angle of twist at the end of the bar, ϕ_D (degrees) (3 pts)

$$\phi = \frac{TL}{GI_p}$$

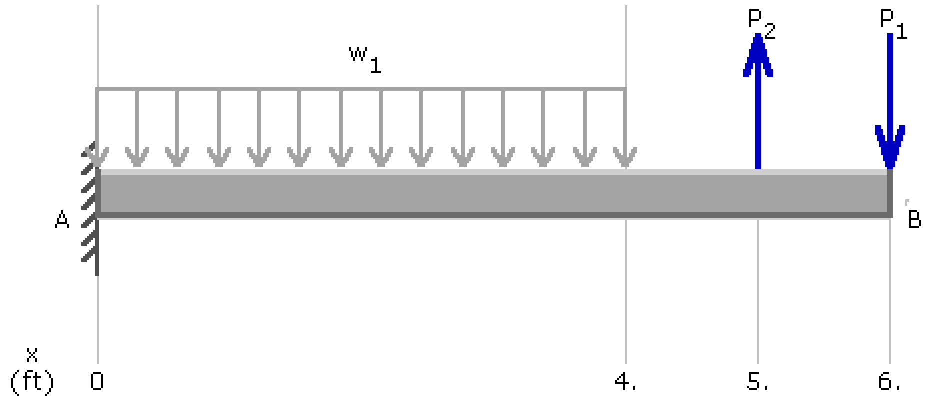
$$\phi_{AB} = \frac{T_{AB} L_{AB}}{G\left(I_p\right)_{AB}} = \frac{(+4000\text{ Nm})(0.5\text{ m})}{(80 \bullet 10^9\,\frac{\text{N}}{\text{m}^2})\left(4.0192 \bullet 10^{-6}\,\text{m}^4\right)} = +6.22 \bullet 10^{-3}\,\text{rad} = +0.00622\text{ rad}$$

$$\phi_{BC} = \frac{T_{BC} L_{BC}}{G\left(I_p\right)_{BC}} = \frac{(+3000\text{ Nm})(0.8\text{ m})}{(80 \bullet 10^9\,\frac{\text{N}}{\text{m}^2})\left(2.512 \bullet 10^{-7}\,\text{m}^4\right)} = +0.1194\text{ rad}$$

$$\phi_{CD} = \frac{T_{CD} L_{CD}}{G\left(I_p\right)_{CD}} = \frac{(-3000\text{ Nm})(0.5\text{ m})}{(80 \bullet 10^9\,\frac{\text{N}}{\text{m}^2})\left(7.948 \bullet 10^{-8}\,\text{m}^4\right)} = -0.2359\text{ rad}$$

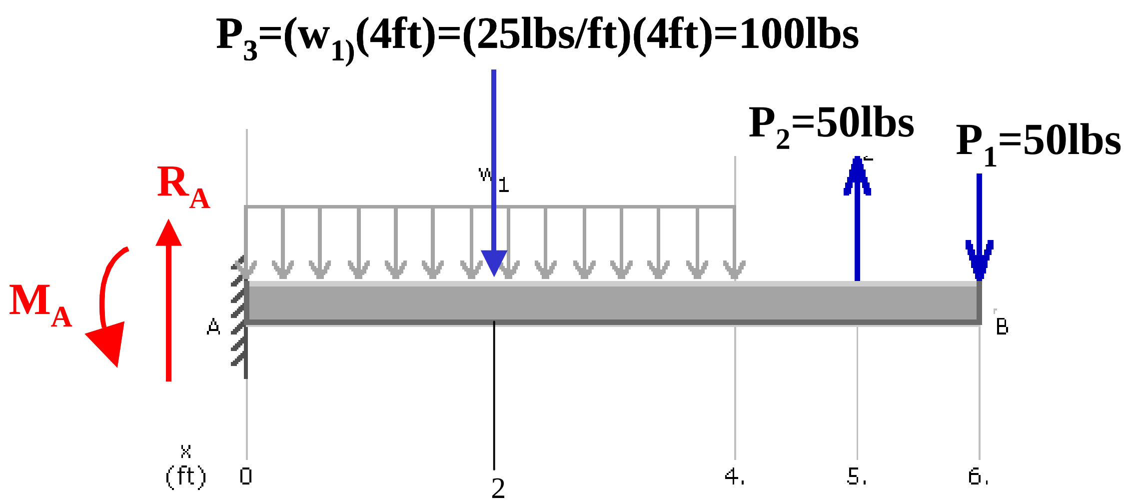
$$\phi_D = \phi_{AB} + \phi_{BC} + \phi_{CD} = \left(+0.00622 + 0.1194 - 0.2359\right)\text{rad} = -0.11\text{ rad} \bullet \frac{57.53\ \text{degrees}}{1\,\text{rad}} = \mathbf{-6.34^{o}\ CW}$$

2. A cantilever beam is used to support a uniformly distributed load of intensity w_1 = 25 lb/ft and two concentrated loads $P_1 = P_2$ = 50 lbs, as shown in the figure below.



(a) Draw the shear force diagram and label the location (along x-axis) and magnitude of the maximum shear force.
(b) Draw the bending moment diagram and label the location (along x-axis) and magnitude of the maximum bending moment.
(10 pts total)

<u>ANSWER 2:</u>
(a) **(2 pts)** Consider a free-body diagram of the entire beam and the equations of static equilibrium to determine the reaction shear force and bending moment at A. Take upwards as (+) and downwards as (-). The distributed load can be represented by a concentrated load through the centroid of the area it acts upon with a magnitude equal to the area.



$\Sigma F_Y=0=-P_1+P_2-P_3+R_A$

$R_A=P_1-P_2+P_3=(50-50+100)\text{lbs}=+100\text{lbs}$

*(*positive answer means assumed direction is correct)*

Taking (CCW+) and (CW-) :

$\Sigma M=0=M_A-P_3(2\text{ ft})+P_2(5\text{ ft})-P_1(6\text{ ft})$

$\Sigma M=0=M_A-(100\text{ lbs})(2\text{ ft})+(50\text{ lbs})(5\text{ ft})-(50\text{ lbs})(6\text{ ft})$

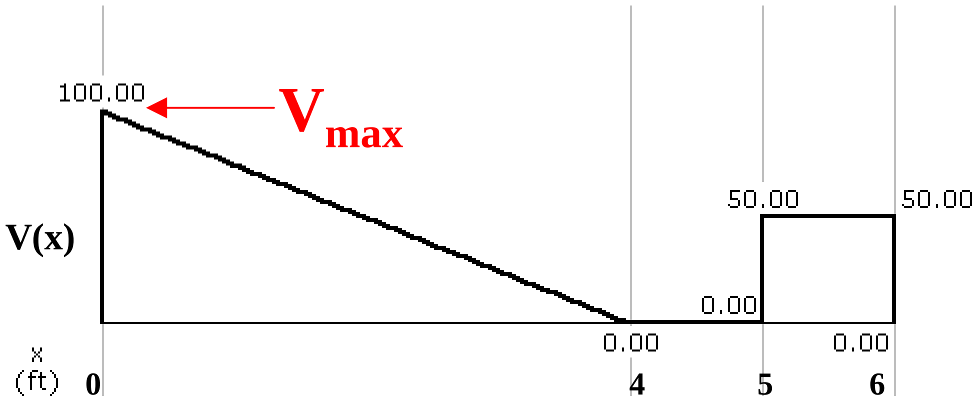
$\Sigma M=0=M_A-200\text{ lbs ft}+250\text{ lbs ft}-300\text{ lbs ft}$

$\Sigma M=0=M_A-250\text{ lbs ft}$

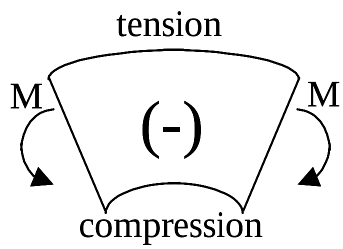
$M_A=250\text{ lbs ft}$

*(*positive answer means assumed direction is correct)*

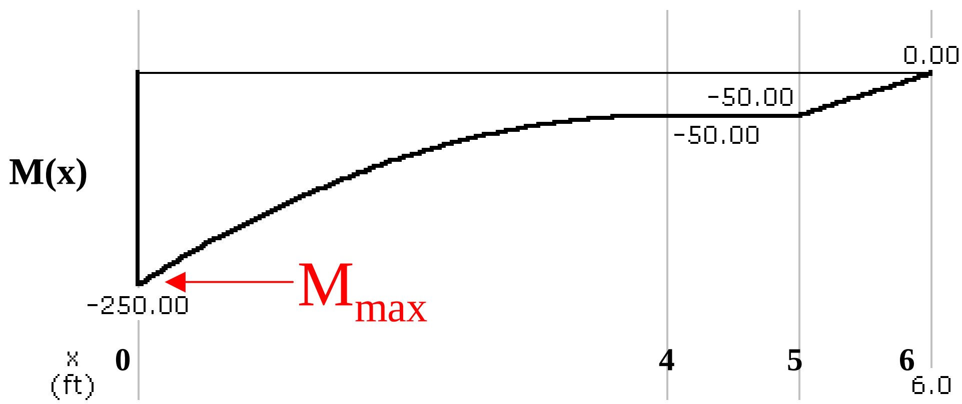
(4 pts) Starting from the right hand side of the beam everywhere there is a concentrated load, the shear will exhibit a discontinuous jump in the value of V(x) in the opposite direction. Hence, V(x) exhibits a positive jump of 50 lbs at x=6 ft and then a negative jump of 50 lbs at x=5ft. V(x) remains zero until it reaches the distributed load at x=4. The shear force is linear with distance for a distributed load since the slope, dV/dx=q=constant.= 25 lb ft, and we also know from the free body diagram above it has to be equal to 100 lbs at the left hand side of the beam. Hence, we can just connect the datapoints at x=4 (V=0) and x=0 (V=100 lbs) by a line.



(b) **(4 pts)** Intuitively, we can see that the bending moment is negative on the right hand side of the beam which will bend as shown below :



The bending moment, M(x), is the integral of the shear force diagram, V(x). We know that the bending moment is equal to -250 lbs ft at the left hand side of the beam and has to be quadratic with x (integral of linear function) up until x=4. Between x=4 and x=5 the moment stays constant with a slope of zero (V(x)=0). Between x=5 and x=6 the moment has to be linear with x and the slope is equal to the V(x)=50 lbs.



3. A state of plane stress consists of a tensile stress of σ_x=8 ksi, σ_y=-5 ksi, and τ_{xy}=-10 ksi:

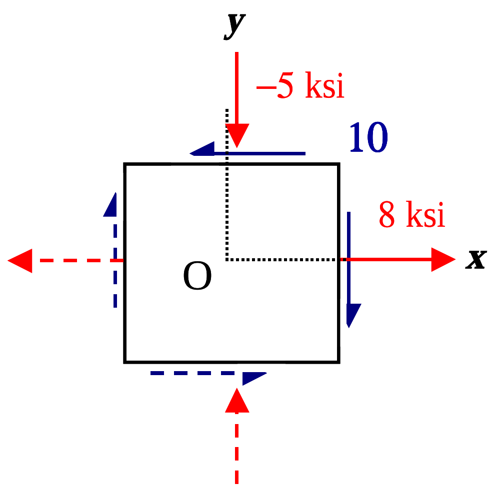
(a) **Draw the original unrotated element and the corresponding 2-D Mohr's circle construction showing the x-face and y-face coordinates.**

(b) **Calculate the principal stresses, σ_1 and σ_2 and their corresponding principal angles, θ_{p1},θ_{p2} and show all of these on your Mohr's circle construction.**

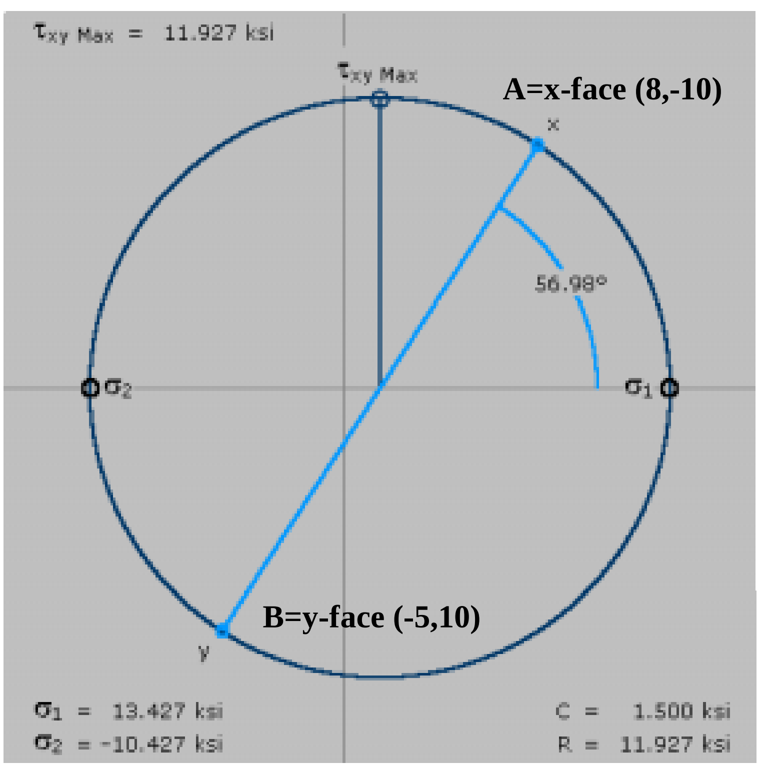
(c) **Calculate the maximum shear stresses, $\pm\tau_{MAX}$, and their corresponding angles of maximum shear stress, θ_{s1},θ_{s2} and show all of these on your Mohr's circle construction.**

ANSWER 3 :

(a) **(3 pts)** The original unrotated element is shown below :



The Mohr's circle construction is shown below :



(b) **(4 pts)** The principal stresses, σ_1 and σ_2 and their corresponding principal angles, θ_{p1},θ_{p2} are calculated as follows :

Principal Stresses:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2} = 13.427 \text{ ksi.}, -10.4 \text{ ksi}$$

Principal Angles / Planes :

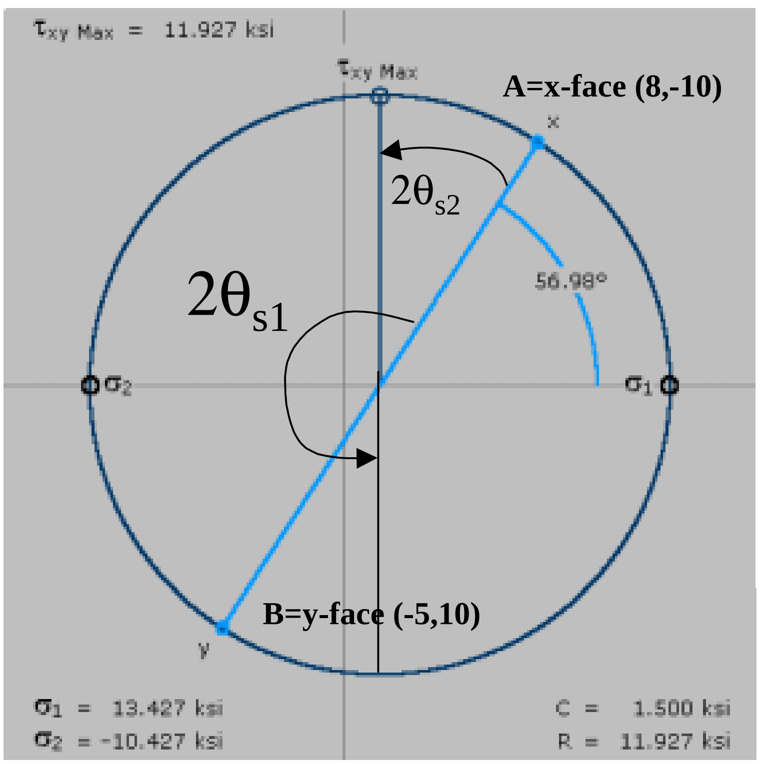
$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$2\theta_{p1} = -56.98^o$ (CW) = $360^o - 56.98^o = +303^o$ (CCW)

$2\theta_{p2} = 180^o - 56.98^o = +123^o$ (CCW) = -237^o (CW)

$\theta_{p1} = -28.49^o$ (CW) = $+152^o$ (CCW)

$\theta_{p2} = -118.5^o$ (CW) = $+61.5^o$ (CCW)



(c) **(4 pts)** The maximum shear stresses, $\pm\tau_{MAX}$, and their corresponding angles of maximum shear stress, θ_{s1}, θ_{s2} are calculated as follows :

Maximum Shear Stresses:

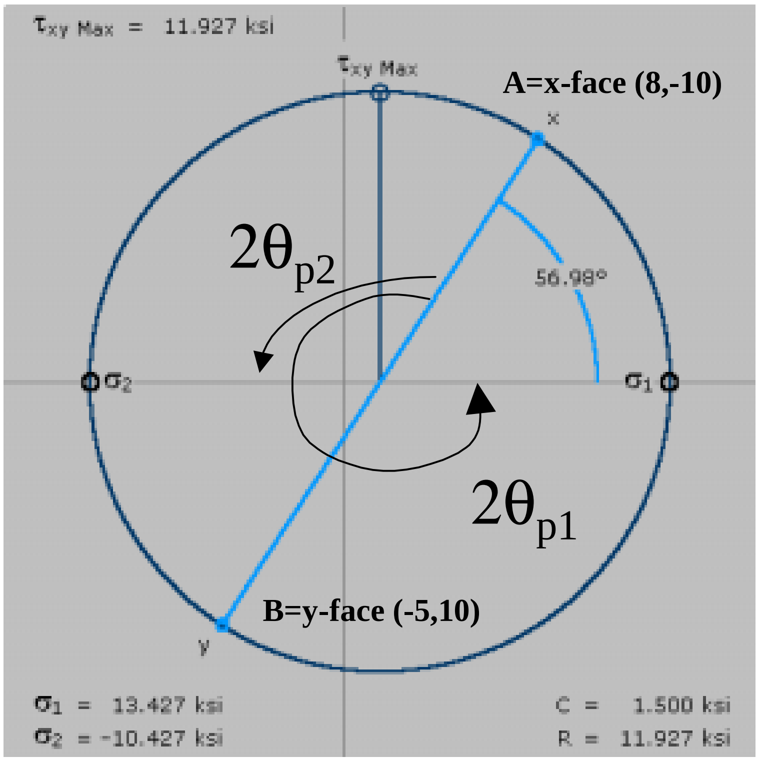
$$\tau_{\max,\min} = \pm\sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2} = \pm R = \pm 11.927\ \text{ksi}$$

Planes / Angles of Maximum Shear :

$$\tan(2\theta_s) = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$2\theta_{s2} = +33^o$ (CW) = -327^o (CCW), $\theta_{s2} = +16.5^o$ (CW) = -163.5^o (CCW)

$2\theta_{s1} = +33^o + 180^o = 213^o$ (CW) = -147^o (CCW), $\theta_{s1} = +106.5^o$ (CW) = -73.5^o (CCW)



θ_{p1}	θ_{p2}	θ_{s1}	θ_{s2}
152, -28.49	61.5, -118.5	106.5, -73.5	16.5, -163.5