3.11 Mechanics of Materials F00
Exam #3 SOLUTIONS

1. Q. (a) A sheet of rubber is 200 mm square and 100 mm thick. Its edges are aligned parallel to the x_1 and x_2 axis. The forces F_1 and F_2 are applied to the x_1 and x_2 directions respectively to stretch the sheet homogeneously and bring the x_1 dimension to 400 mm and the x_2 dimension to 300 mm. Calculate :

(1) F_1 and F_2 (kN)

(2) σ_{t1} and σ_{t2} (MPa) (the true stresses in the x_1 and x_2 directions)

assuming the rubber follow Gaussian rubber elasticity theory and has a Young's modulus of 1.5 MPa.

(b) It is typically observed that real elastomers, such as car tires, become softer (less stiff) as the temperature is increased. Explain in a few sentences this apparent discrepency with Gaussian rubber elasticity theory (HINT: start by explaining the equation for the elastic Young's modulus you derived on a recent problem set).

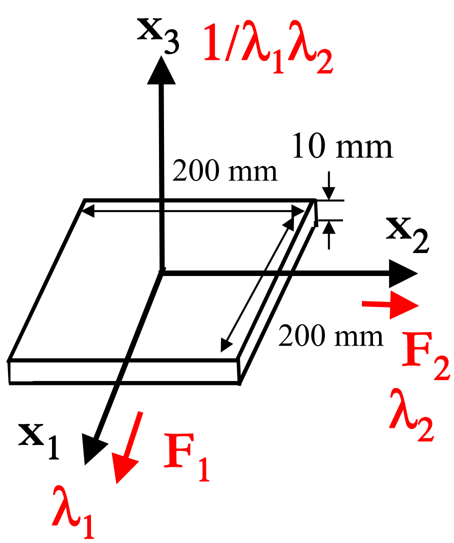
A. (a) Starting with the constant volume constraint :

$\lambda_1\lambda_2\lambda_3 = 1$

$\lambda_1 = \lambda_1 = L_{f1}/L_{o1} = 400mm/200mm = 2$ (extension)

$\lambda_2 = \lambda_2 = L_{f2}/L_{o2} = 300mm/200mm = 1.5$ (extension)

$\lambda_3 = 1/\lambda_1\lambda_2 = 1/(2 \bullet 1.5) = 0.33$ (contraction)



The nominal stress versus extension ratio equations for biaxial tension were derived in the homework :

$\sigma_1 = k_B T \nu_x [\lambda_1 - 1/\lambda_2^2\lambda_1^3]$ *(1)*

$\sigma_2 = k_B T \nu_x [\lambda_2 - 1/\lambda_1^2\lambda_2^3]$ *(2)*

$E = 3\nu_x k_B T$

$\nu_x k_B T = E/3 = 1.5\text{MPa}/3 = 0.5$ MPa

Substituting in values of $\nu_x k_B T = 0.5$ MPa, $\lambda_1 = 2$, and $\lambda_2 = 1.5$ into equations *(1)* and *(2)* we obtain :

$\sigma_1 = 0.5[2 - 1/1.5^2 2^3] = 0.97225$ MPa
$\sigma_2 = 0.5\ [1.5 - 1/2^2 1.5^3] = 0.7129$ MPa

$\sigma_1 = F_1/A_o$ (x_2/x_3 plane)$\Rightarrow F_1 = \sigma_1 A_o = 0.97225$ MPa (200 mm•10 mm)
$\sigma_2 = F_2/A_o$ (x_1/x_3 plane)$\Rightarrow F_2 = \sigma_2 A_o = 0.7129$ MPa (200 mm•10 mm)

σ_1=1.94 kN
σ_2=1.43 kN

$\sigma_{t1} = \sigma_1(\varepsilon_1 + 1)$
$\sigma_{t2} = \sigma_2(\varepsilon_2 + 1)$

where: $\lambda = 1 + \varepsilon$

$\sigma_{t1} = \sigma_1 \lambda_1 = 0.97225$ MPa • 2
$\sigma_{t2} = \sigma_2 \lambda_2 = 0.7129$ MPa • 1.5

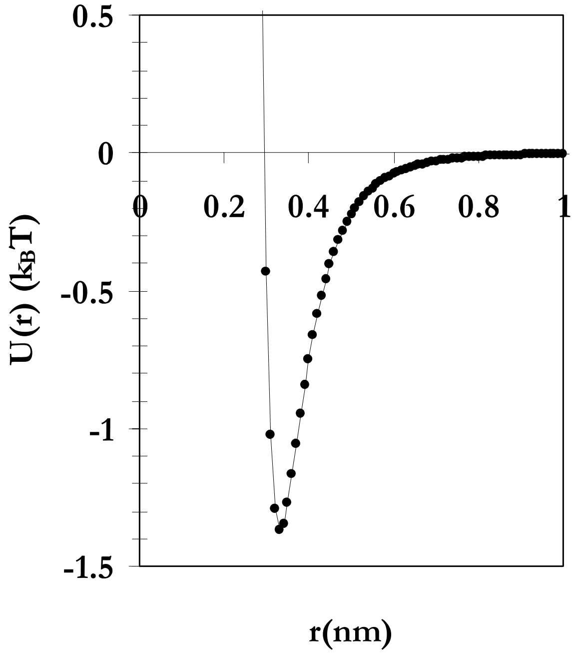
σ_{t1}=1.9445 MPa
σ_{t2}=1.06935 MPa

(b) The equation for the elastic Young's modulus in uniaxial tension derived from Gaussian rubber elasticity theory is :

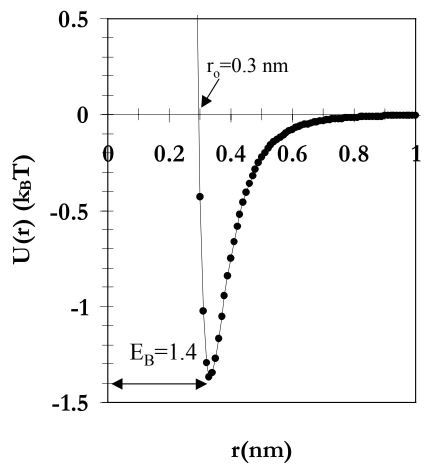
$E = 3\nu_x k_B T$

Hence, the modulus is predicted to increase linearly with temperature and with network strand density. In reality, the network strand density decreases rapidly with temperature due to covalent bond scission and/or disentanglement of physical crosslinks as the molecular mobility and thermal oscillations of the chain increase.

2. Q. Two atoms interact at T=0ºK via a van der Waals Lennard-Jones potential. The interaction energy versus separation distance plot is given in the following figure.
(a) Calculate the bond stiffness (N/m) (HINT : calculate A and B first).
(b) When is the bond stiffness proportional to ~ the Young's modulus, E? (e.g. thermodynamically, in what strain range, which types of materials)



A.(a) r_o can be read directly off the plot :



r_o=0.3 nm=$[B/A]^{1/6}$
$B/A=7.2899\bullet10^{-58}m^6$
$B=7.2899\bullet10^{-58}m^6A$ *(1)*

The binding energy E_B can be read directly off the plot :
$E_B=-1.4k_BT=-1.4\bullet 4.1\bullet 10^{-21}J=5.74\bullet 10^{-21}J$ *(2)*
$E_B=-[A^2/4B]$ *(3)*

Substitute equation *(1)* into *(3)*, set equal to *(2)*, and solve for A:
$-5.74\bullet 10^{-21}J=-[A^2/4\bullet 7.2899\bullet 10^{-58}m^6\bullet A]=-A/2.916\bullet 10^{-57}m^6$
$A=1.673784\bullet 10^{-77}J\ m^6$

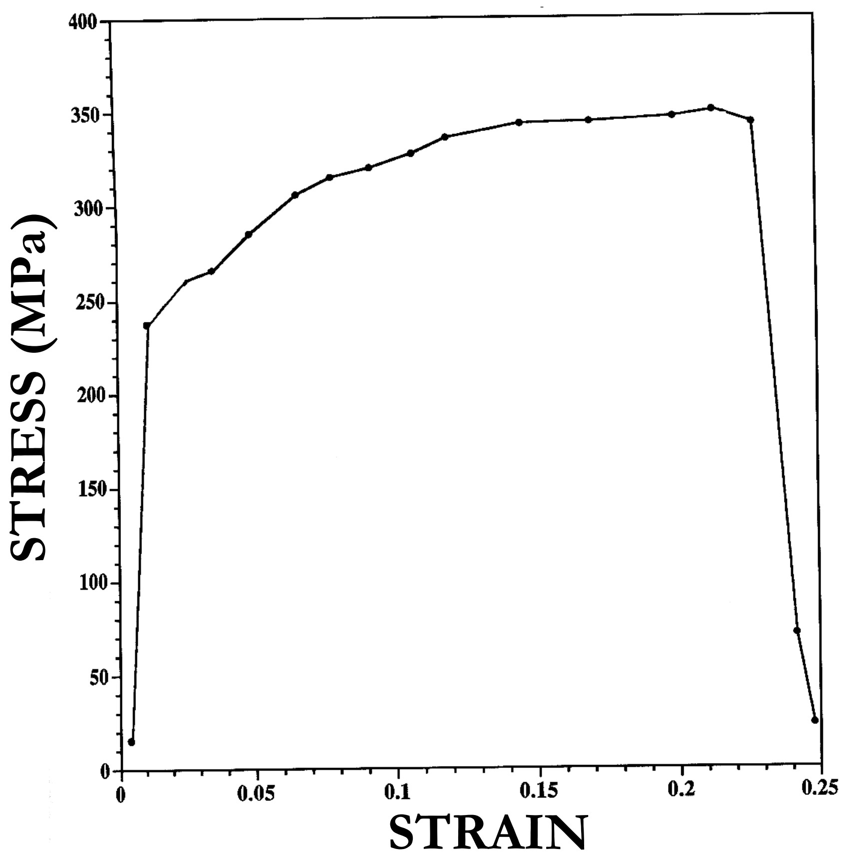
From equation *(1):*
$B=7.2899\bullet 10^{-58}m^6\bullet 1.673784\bullet 10^{-77}J\ m^6=1\bullet 10^{-135}Jm^{12}$

$r_e=1.12r_o=1.12\bullet 0.3\ nm=0.336\ nm$

$k_{bond}=42A/r_e^{\ 8}-156B/r_e^{\ 14}=42\bullet 1.7\bullet 10^{-77}Jm^6/(0.336\bullet 10^{-9})^8-1\bullet 10^{-135}\ Jm^{12}/(0.336\bullet 10^{-9})^{14}$
$k_{bond}=0.38\ N/m$

(b) The bond stiffness is ~ the Young's modulus, E, when the molecular origin of the elastic moduli is enthalpic or energetic in origin, i.e. when the resistance to deformation is provided by bond stretching / twisting / bending away from their equilibrium configuration. For most materials (e.g. metals, ceramics, crystalline materials) this occurs at small strains in the linear elastic, Hookean regime. For polymers this occurs at temperatures T<Tg.

3. Q. The following Figure is experimental data for the macroscopic uniaxial engineering stress versus strain curve I conducted as an undergrad in a lab course on iron, a ductile polycrystalline metal.

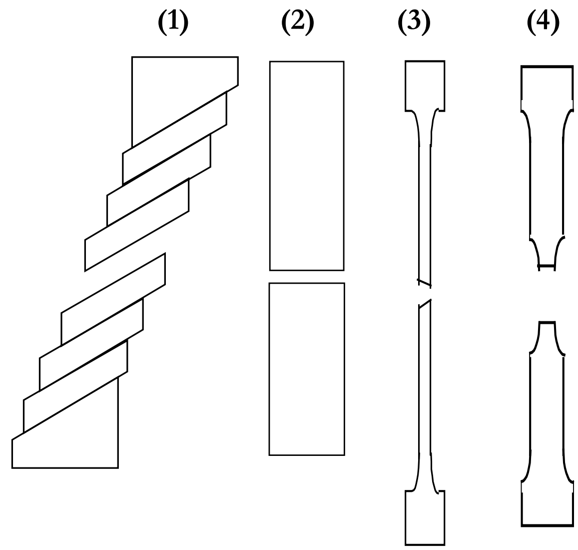


(a) From the above plot, approximate the following numerically and briefly (one sentence) explain how you obtained each value :

(1) Young's modulus, E (GPa)
(2) 0.2% yield stress (MPa)
(3) ultimate tensile strength (MPa)
(4) % strain at failure
(5) failure strength (MPa)
(6) tensile ductility (% strain)
(7) modulus of toughness (MPa)
(8) modulus of resilience (MPa)

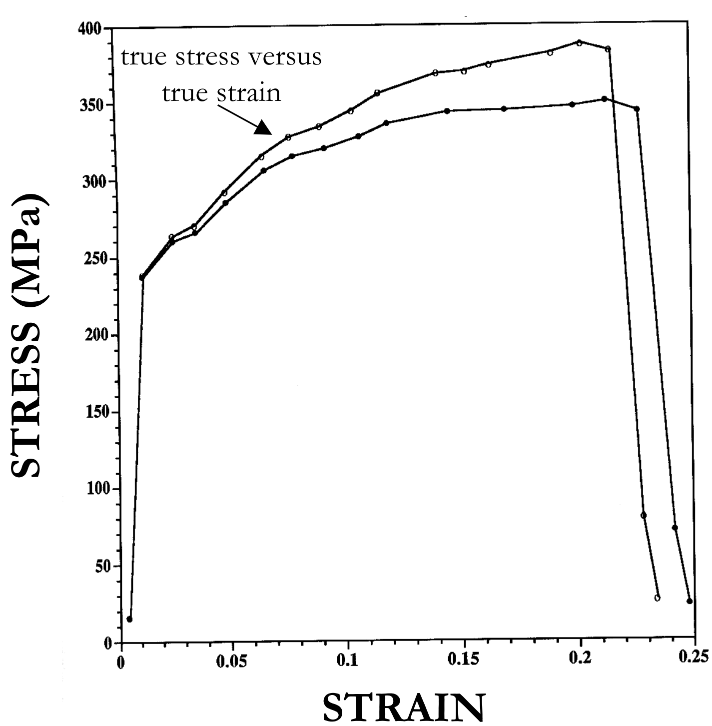
(b) On the same graph plot the true stress versus true strain curve.

(c) Below are four schematics of how this sample possibly fractured. Choose one which is most likely and explain why.



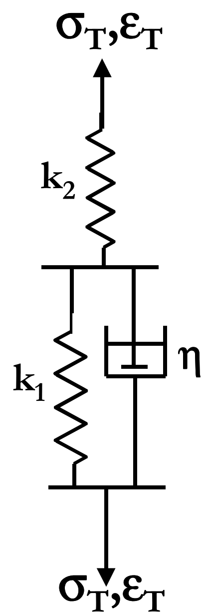
A. (a) (1) Young's modulus, E (GPa) : ≈20 GPa
(2) 0.2% yield stress (MPa) ≈240 MPa
(3) ultimate tensile strength (MPa) ≈ 340 MPa
(4) % strain at failure=24%
(5) failure strength (MPa) ≈340 MPa
(6) tensile ductility (% strain) ≈22%
(7) modulus of toughness (MPa) ≈ 75 MPa
(8) modulus of resilience (MPa) ≈ 1.2 MPa

(b)



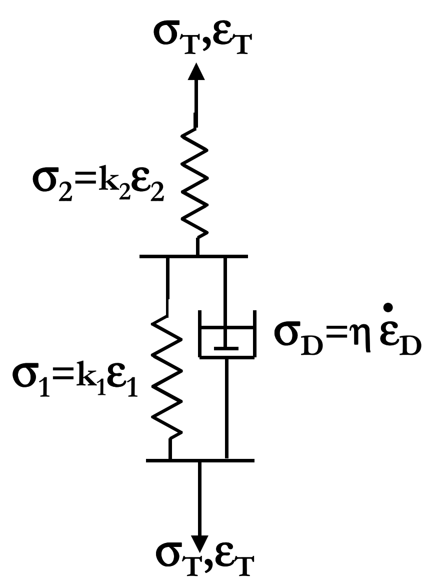
(c) (1) slip in a single crystal
(2) brittle fracture of for example, ceramics
(3) drawing in polymers
(4) ductile fracture due to multiple slip within grains just after formation of a neck **(ANSWER)**

4. Q. (a) Derive the governing differential equation (σ_T as a function of ε_T) for the following viscoelastic model in terms of k_1, k_2, η.



(b) Explain in a few sentences the term "Relaxation Modulus."

A. The governing equations for the individual elements are shown below :



In series $\Rightarrow$ the strains are additive, stresses equal
In parallel $\Rightarrow$ the strains are equal, stresses additive

$\varepsilon_T=\varepsilon_2+\varepsilon$ *(1)*
where : $\varepsilon=\varepsilon_1=\varepsilon_D$*(2)*

$\sigma_T=\sigma_2=\sigma_1+\sigma_D$ *(3)*

Substitute governing stress / strain laws for individual elements into equation *(3)* :
$\sigma_T=k_1\varepsilon_1+\eta d\varepsilon_D/dt$
$\sigma_T=k_1\varepsilon+\eta d\varepsilon/dt$ *(4)*

For equation *(4)* : we need ε and $d\varepsilon/dt$ in terms of ε_T and σ_T :

From *(1)* :
$\varepsilon_T=\sigma_2/k_2+\varepsilon=\sigma_T/k_2+\varepsilon$ *(5)*
rearrange equation *(5)* :
$\varepsilon=\varepsilon_T-\sigma_T/k_2$ *(6)*

Take time derivitive of equation *(1)* :
$d\varepsilon_T/dt=d\varepsilon_2/dt+d\varepsilon/dt$
$d\varepsilon_T/dt=(d\sigma_2/dt)/k_2+d\varepsilon/dt=(d\sigma_T/dt)/k_2+d\varepsilon/dt$
Solve for $d\varepsilon/dt$:
$d\varepsilon/dt=d\varepsilon_T/dt-(d\sigma_T/dt)/k_2$*(7)*

Substitute (6) and (7) into (4) :
$\sigma_T=k_1(\varepsilon_T-\sigma_T/k_2)+\eta(d\varepsilon_T/dt-(d\sigma_T/dt)/k_2)$
$\sigma_T=k_1\varepsilon_T-k_1\sigma_T/k_2+\eta(d\varepsilon_T/dt)-(d\sigma_T/dt)\eta/k_2$

rearrange :
$\boldsymbol{\sigma_T(1+k_1/k_2)+(d\sigma_T/dt)\eta/k_2=k_1\varepsilon_T+\eta(d\varepsilon_T/dt)}$