Matrix and Index Notation

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September 18, 2000

A *vector* can be described by listing its components along the xyz cartesian axes; for instance the displacement vector $\mathbf{u}$ can be denoted as u_x, u_y, u_z, using letter subscripts to indicate the individual components. The subscripts can employ numerical indices as well, with 1, 2, and 3 indicating the x, y, and z directions; the displacement vector can therefore be written equivalently as u_1, u_2, u_3.

A common and useful shorthand is simply to write the displacement vector as u_i, where the i subscript is an *index* that is assumed to range over 1,2,3 (or simply 1 and 2 if the problem is a two-dimensional one). This is called the *range convention* for index notation. Using the range convention, the vector equation $u_i = a$ implies three separate scalar equations:

$$u_1 = a$$

$$u_2 = a$$

$$u_3 = a$$

We will often find it convenient to denote a vector by listing its components in a vertical list enclosed in braces, and this form will help us keep track of matrix-vector multiplications a bit more easily. We therefore have the following equivalent forms of vector notation:

$$\mathbf{u} = u_i = \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\} = \left\{ \begin{array}{c} u_x \\ u_y \\ u_z \end{array} \right\}$$

Second-rank quantities such as stress, strain, moment of inertia, and curvature can be denoted as 3×3 matrix arrays; for instance the stress can be written using numerical indices as

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Here the first subscript index denotes the row and the second the column. The indices also have a physical meaning, for instance σ_{23} indicates the stress on the 2 face (the plane whose normal is in the 2, or y, direction) and acting in the 3, or z, direction. To help distinguish them, we'll use brackets for second-rank tensors and braces for vectors.

Using the range convention for index notation, the stress can also be written as σ_{ij}, where both the i and the j range from 1 to 3; this gives the nine components listed explicitly above.

(Since the stress matrix is symmetric, i.e. $\sigma_{ij} = \sigma_{ji}$, only six of these nine components are independent.)

A subscript that is repeated in a given term is understood to imply summation over the range of the repeated subscript; this is the *summation convention* for index notation. For instance, to indicate the sum of the diagonal elements of the stress matrix we can write:

$$\sigma_{kk} = \sum_{k=1}^{3} \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

The multiplication rule for matrices can be stated formally by taking $\mathbf{A} = (a_{ij})$ to be an $(M \times N)$ matrix and $\mathbf{B} = (b_{ij})$ to be an $(R \times P)$ matrix. The matrix product $\mathbf{AB}$ is defined only when $R = N$, and is the $(M \times P)$ matrix $\mathbf{C} = (c_{ij})$ given by

$$c_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{iN} b_{Nk}$$

Using the summation convention, this can be written simply

$$c_{ij} = a_{ik} b_{kj}$$

where the summation is understood to be over the repeated index k. In the case of a 3×3 matrix multiplying a 3×1 column vector we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \left\{ \begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \right\} = \left\{ \begin{array}{c} a_{11} b_1 + a_{12} b_2 + a_{13} b_3 \\ a_{21} b_1 + a_{22} b_2 + a_{23} b_3 \\ a_{31} b_1 + a_{32} b_2 + a_{33} b_3 \end{array} \right\} = a_{ij} b_j$$

The *comma convention* uses a subscript comma to imply differentiation with respect to the variable following, so $f_{,2} = \partial f / \partial y$ and $u_{i,j} = \partial u_i / \partial x_j$. For instance, the expression $\sigma_{ij,j} = 0$ uses all of the three previously defined index conventions: range on i, sum on j, and differentiate:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

The *Kroenecker delta* is a useful entity is defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

This is the index form of the *unit matrix* $\mathbf{I}$:

$$\delta_{ij} = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, for instance

$$\sigma_{kk}\delta_{ij} = \begin{bmatrix} \sigma_{kk} & 0 & 0 \\ 0 & \sigma_{kk} & 0 \\ 0 & 0 & \sigma_{kk} \end{bmatrix}$$

where $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$.