Recitation #6

October 14, 2003

TORSION OF CIRCULAR SECTIONS

Assumptions

1) This analysis can only be applied to solid or hollow circular sections

2) The material must be homogeneous

3) Torque is constant and transmitted along bar by each section trying to shear over its neighbour.

4) Transverse planes remain parallel to each other.

Examine the deformation of a length dx between two transverse planed of a beam with an applied torque T.

For this element, assume the left end is fixed and the right end rotates by d due to the applied torque.

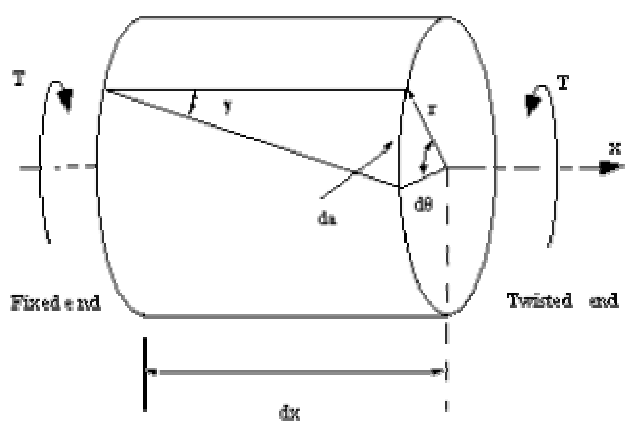


Figure 84: Small transverse element with applied torque T rotated by an amount dθ

The surface of radius 'r' rotates through angle , which is shear strain.

The arc is defined as length da, which is equal to :

$$da = r\,d\theta = \gamma\,dx$$

which gives that:

$$\gamma = r\frac{d\theta}{dx}$$

where :

$$\frac{d\theta}{dx} = \text{Rate of twist}$$

Equation 27 states that:

$$\tau = \gamma G$$

And by substituting for gamma gives that:

$$\tau = rG\frac{d\theta}{dx}$$

Which relates the shear stress linearly to the distance 'r' away from the centre of the section.

The shear stress distribution then looks like this.

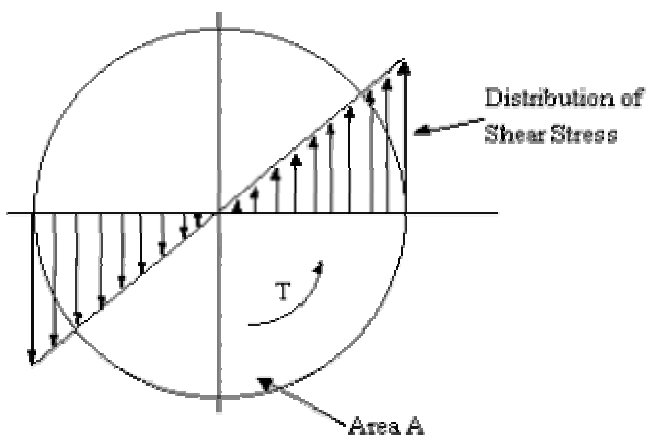


Figure 85: Shear stress distribution in circular section with applied torque T

We now equate the applied torque T to the torque generated in the section by the shear stress distribution. To do this look at a small circumferential section dA.

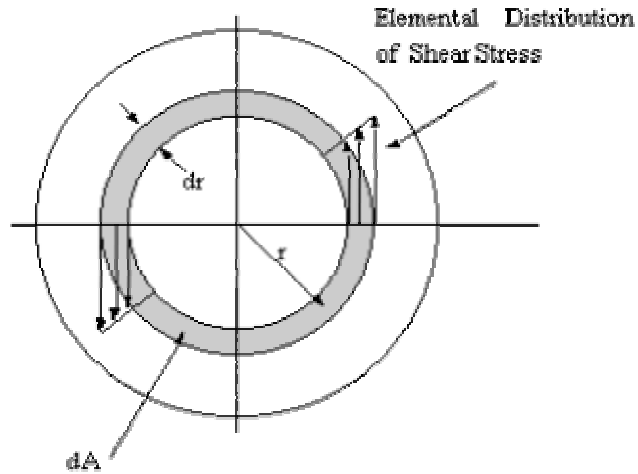


Figure 86: Shear stress distribution in elemental section dr of cross section

$$T = \int_A r\tau dA = \int_r r\tau 2\pi r dr$$

Substituting for shear stress:

$$T = \int_r 2\pi r\left(rG\frac{d\theta}{dx}\right)rdr$$

Since the rate of twist is constant through the section, it is not a function of r. If we assume a homogeneous material, G is also constant, so:

$$T = G\frac{d\theta}{dx}\int_r 2\pi r^3 dr$$

We represent the integral term as the geometric stiffness of the cross section in a similar way to I. We call this term the Polar Second Moment of Area, J.

$$\boxed{J = \int_r 2\pi r^3 dr} \quad \textbf{(29)}$$

and as for the second moment of area for beams under an applied bending moment, this term indicates the cross sectional properties to withstand the applied torque.

Which when equating with the shear stress term, gives that:

$$\boxed{\frac{T}{J} = G\frac{d\theta}{dx} = \frac{\tau}{r}} \quad \textbf{(30)}$$

r1">

called Engineer's Theory of Torsion (ETT).

Since this applies to circular bars, the standard terms for J are:

For solid Shaft of radius R, diameter D :

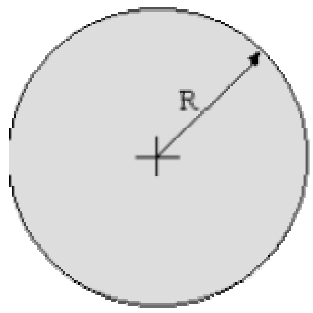


Figure 87: Solid shaft of radius R and diameter D

$$J = \int_0^R 2\pi r^3 dr = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}$$

For hollow shaft , ID = D_i, OD = D_o

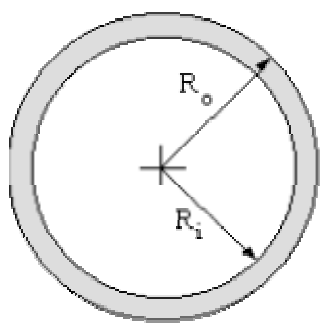


Figure 88: Hollow shaft of external radius R_o, and internal radius R_i

$$J = \frac{\pi}{32}\left(D_o^4 - D_i^4\right) = \frac{\pi}{2}\left(R_o^4 - R_i^4\right)$$

Approximation for this walled tube with t < R/10:

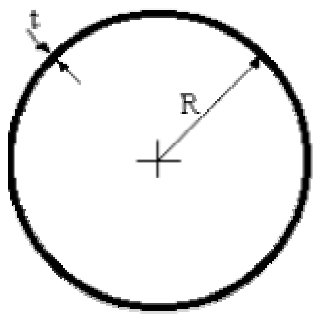


Figure 89: Thin walled tube of thickness t and radius R

$$J \approx 2\pi R^3 t$$

Non-Uniform TORSION OF CIRCULAR SECTIONS

(Section 3.4 Gere)

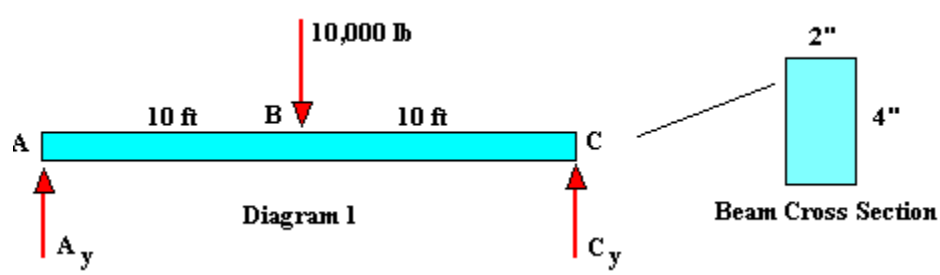
Can be analyzed by applying the formulas of pure torsion to finite segments of the bar and then adding the results , or by applying the formulas to differential elements of the bar and then integrating.

DO Example of Non-Uniform Torque → 3.4-1 From GERE

BEAM BENDING – Normal and Shear Stresses & Strains

Bending Stress - Example 1

In Diagram 1, we have shown a simply supported 20 ft. beam with a load of 10,000 lb. acting downward at the center of the beam. The beam used is a rectangular 2" by 4" steel beam. We would like to determine the maximum bending (axial) stress which develops in the beam due to the loading.



Step 1: Out first step in solving this problem is, of course, to apply static equilibrium conditions to determine the external support reactions. In this particular example, because of the symmetry of the problem, we will not go through the statics in detail, but point out that the two support forces will support the load at the center equally with forces of 5000 lb. each as shown in Diagram 2.

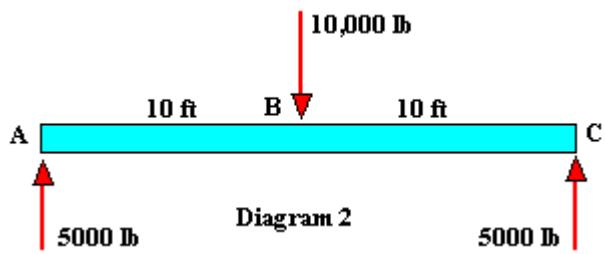


Diagram 2

Step 2: The second step is to draw the shear force and bending moment diagrams for the beam. We really don't need the shear force diagram at this point, except we will use it to make the bending moment diagram. We will normally be able to draw the shear force diagram by simply looking at the load forces and the support reactions. If necessary we will determine the shear force and bending moment expressions and make the shear force and bending moment diagrams from these expressions, as we did in the proceeding topic.

We first draw the shear force diagram. Due to the 5000 lb. support force, the shear force value begins at +5000 lb., and since there is no additional loads for the next 10 feet, the shear force remains constant at 5000 lb. between 0 and 10 feet. At 10 feet, the 10,000 lb. downward load drives the shear force down by that amount, from + 5000 lb. to a value of - 5000 lb. Then as there are no additional loads for the next 10 feet, the shear force will remain constant over the remainder of the beam. Graphing the shear force values produces the result in Diagram 3a.

Diagram 3a

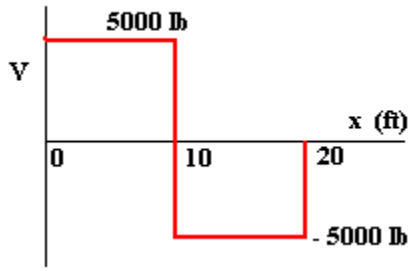
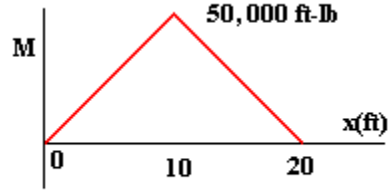


Diagram 3b



Then using the fact that for non-cantilevered beams the bending moment values are equal to the area under the shear force diagram, we develop the bending moment graph shown in Diagram 3b.

Step 3. We now apply the flexure formula: **Bending Stress = M y / I**
We wish to find the **maximum** bending stress, which occurs at the **outer edge of the beam** so:
M = maximum bending moment = 50,000 ft-lb. = 600,000 in-lb. (from bending moment diagram)
y = distance from the neutral axis of the cross section to outer edge of beam = 2 inches
I = moment of inertia of cross section; for rectangle I = (1/12) bd^3 = 1/12 (2" * $4"^3$) = 10.67 in^4.
Then, **Maximum Bending Stress = M y / I = (600,000 in-lb)*(2 in)/(10.67 in^4) = 112, 500 lb/in^2**

(if you needed the strain, You can plug in Hooke's Law to calc. for strain)

This is the correct value, but it is clearly excessive for normal steel. Thus if we tried to use a rectangular 2"x4" steel beam, it would fail under the load. We will have to use a stronger beam.
Notice that the **maximum bending moment** does not depend on the type of beam. The values of "y" and "I" in the flexure formula do depend on the beam used. Thus, if we had used a rectangular 2"x6" beam (instead of a 2" x 4" beam), the value of y would be: y = 3", and the value of I would be: I = (1/12)(2" * $6"^3$) = 36 in^4. Then the maximum bending stress for this beam would be:
Maximum Bending Stress = M y / I = (600,000 in-lb)*(3 in)/(36in^4) = 50,000 lb/in^2 . This value is in a more reasonable range for acceptable axial stresses in steel.

Beams -Horizontal Shear Stress

In addition to the bending (axial) stress which develops in a loaded beam, there is also a shear stress which develops, including both a **Vertical Shear Stress**, and a **Horizontal (longitudinal) Shear Stress.** It can be shown that **at any given point in the beam**. Diagram 1 shows a simply supported loaded beam.

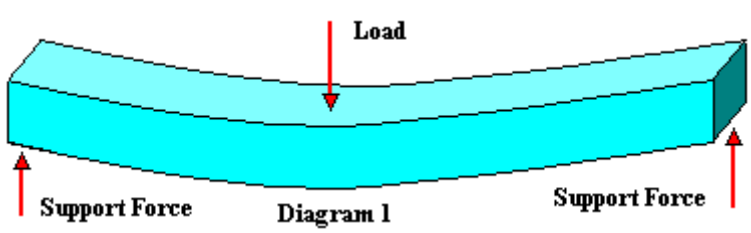


Diagram 1

In Diagram 2a, we have cut a section dx long out of the left end of the beam, and have shown the internal horizontal forces acting on the section.

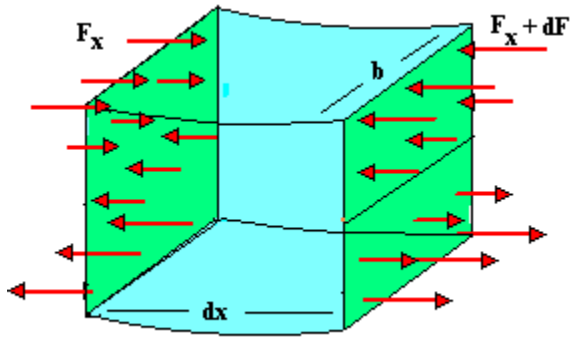


Diagram 2a

In Diagram 2b, we have shown a side view of section dx. Notice that the bending moment is larger on the right hand face of the section by an amount dM. (This is clear if we make the bending moment diagram for the beam, in which we see the bending moment increases from a value of zero at the left end to a maximum at the center of the beam.)

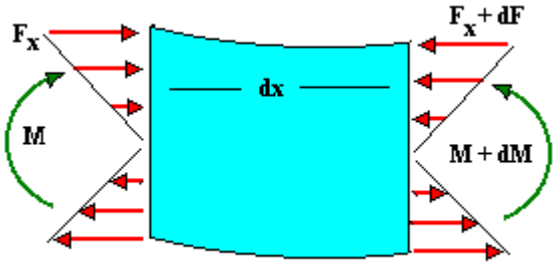


Diagram 2b

top is max stress in compression, bottom is max stress in tension. (In Notes Lecture 15)

τ = [(dM/dx)/Ib] Σ y dA, however dM/dx is equal to the shear force V (as discussed in the previous topic), and Σ y dA is the first moment of the area of the section, and may be written as A y', where A is the area of the section and y' is the distance from the centroid of the area A to the neutral axis of the beam cross section. Rewriting in a final form we have:

Horizontal Shear Stress: τ = VAy'/Ib, where for rectangle = -V/2I (h^2/4-y^2). At the max, Tau = -3V/2A

V = Shear force at location along the beam where we wish to find from the horizontal shear stress
A = cross sectional area, from point where we wish to find the shear stress at, to an outer edge of the beam cross section (top or bottom)
y' = distance from neutral axis to the centroid of the area A.
I = moment of inertia for the beam cross section.

b = width of the beam at the point we wish to determine the shear stress. (In some texts, the product Ay' is given the symbol Q and used in the shear stress equation)

If we consider our shear relationship a little, we observe that the **Horizontal Shear Stress is zero at the outer edge of the beam** - since the area A is zero there. **The Horizontal Shear Stress is (normally) a maximum at the neutral axis of the beam.** This is the opposite of the behavior of the Bending Stress which is maximum at the other edge of the beam, and zero at the neutral axis.

EXAMPLE: Horizontal Shear STress

We will now apply the **Horizontal Shear Stress formula**: Tau = **Shear Stress = Vay'/Ib**
We wish to find the **maximum shear stress**, which occurs at the **neutral axis of the beam:**

V = maximum shear force = 5,000 ft-lb. (from the shear force diagram)
I = moment of inertia of cross section; for rectangle
I = (1/12) bd^3 = 1/12 (2" * $4"^3$) = 10.67 in^4.
b = width of beam section where we wish to find shear stress at; b= 2 in.
a = area from point we wish to find shear stress at (neutral axis) to an outer edge of beam
a= (2" x 2")= 4 in^2.
y' = distance from neutral axis to the centroid of the area "a" which we used; y'= 1 in.
(See Diagram 4)

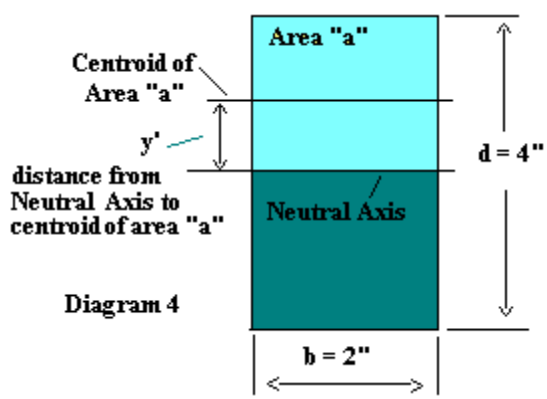


Diagram 4

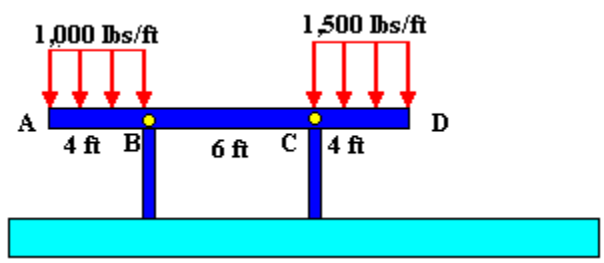
Placing the values into the equation, we find:
Maximum Horizontal Shear Stress = Vay'/Ib = (5000 lb)*(4 in^2)*(1 in)/ (10.67 in^4)(2 in)= 937 lb/in^2

This is the correct value; we notice it is not very large. The beam is clearly able to carry a heavy load without failing in shear.

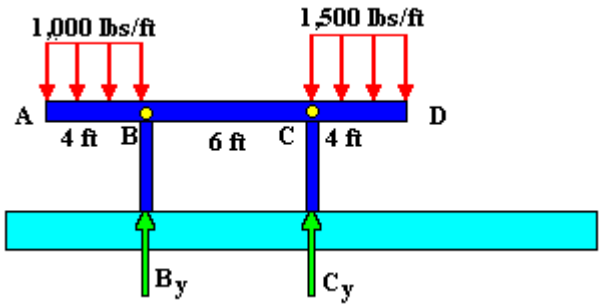
Horizontal Shear Stress

STATICS & STRENGTH OF MATERIALS - Example

A simply supported rectangular 2 x 12 beam is loaded as shown below. For this beam:
A. Determine the maximum bending stress 8 feet from the left end of the beam.
B. Determine the horizontal shear stress at a point 3 inches above the bottom of the beam cross section and 8 feet from the left end of the beam.
Unless otherwise indicated, all joints and support points are assumed to be pinned or hinged joints.



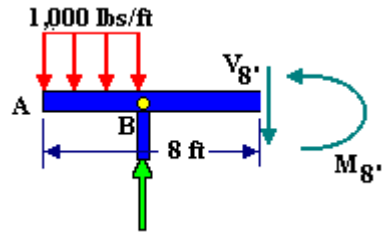
Solution:
Part A:
STEP 1: Determine the external support reactions:



1.) FBD of structure (See Diagram)
2.) Resolve all forces into x/y components
3.) Apply equilibrium conditions:

Sum F_x = 0 none
Sum F_y = B_y + C_y - 1,000 lbs/ft (4 ft) - 1,500 lbs/ft (4 ft) = 0
Sum T_B = 1,000 lbs/ft (4 ft) (2 ft) - 1,500 lbs/ft (4 ft) (8 ft) + C_y(6 ft) = 0
Solving: B_y = 3,330 lbs; C_y = 6,670 lbs

STEP 2: Determine the shear force and bending moment at x=8 ft.



1.) Cut beam at 8 ft. Draw the FBD of left end of beam, showing and labeling all external forces.
2.) Resolve all forces into x/y directions.
3.) Apply equilibrium conditions:

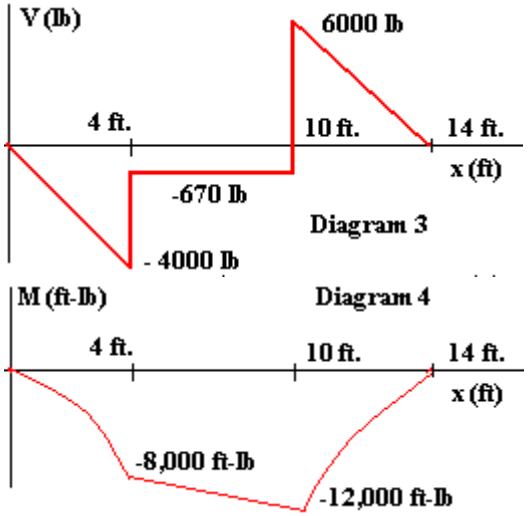
Sum F_x = 0 none
Sum F_y = -1,000 lbs/ft (4 ft) + 3,330 lbs - V_8 = 0
Sum T_A = -1,000 lbs/ft (4 ft) (2 ft) + 3,330 lbs (4 ft) + M_8 = 0
Solving: **V_8 = 667 lbs; M_8 = 10,670 ft-lbs**

STEP 3: Apply the Flexure Formula to determine the Maximum Bending Stress (MBS) at 8'.

MBS = My/I (Where $M_{8'}$ is the bending moment at 8 ft, and S is the section modulus for the beam. The section modulus is available from the Beam Tables. This is a 2 x 12 beam.)
MBS = -10,670 ft-lbs(12 in/ft)(6 in) /[(1/12)(2 in)(12 in)^3] = 2,667 lbs/in^2

Part B:
STEP 4: To determine the Horizontal Shear Stress (HSS) at 8 ft from the end of the beam and 3 inches above the bottom of the beam, apply the horizontal shear stress formula.
The form we will use is: **HSS = Vay'/Ib**
Where:
V = Shear force 8 ft from the end of the beam
a = cross sectional area from 3 in above the bottom of the beam to bottom of beam
y' = distance from neutral axis to the centroid of area a
I = moment of inertia of the beam (288 in^4 for 2 x 12 beam)
b = width of beam a 3 in above the bottom of the beam

HSS = [(667 lbs)(6 in^2)(4.5 in)]/[(288 in^4)(2 in)] = 31.3 psi



Transformations of Stress & Strain—probably next recitation. ☹