3.11 RECITATION #8

October 28, 2003

What is Mohr's Circle??

Mohr's Circle was the leading tool used to visualize relationships between normal and shear stresses, and to estimate the maximum stresses, before hand-held calculators became popular. Even today, Mohr's Circle is still widely used by engineers all over the world.

DERIVATION OF MOHR's CIRCLE

To establish Mohr's Circle, we first recall the stress transformation formulas for plane stress at a given location,

$$\begin{cases} \sigma_{x'} - \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \end{cases}$$

Using a basic trigonometric relation ($\cos^2 2\theta + \sin^2 2\theta = 1$) to combine the two above equations we have,

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^{\,2} = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}$$

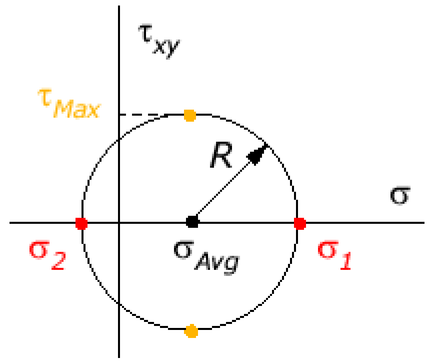
This is the equation of a circle, plotted on a graph where the abscissa is the normal stress and the ordinate is the shear stress. This is easier to see if we interpret σ_x and σ_y as being the two principal stresses, and τ_{xy} as being the maximum shear stress. Then we can define the average stress, σ_{avg}, and a "radius" R (which is just equal to the maximum shear stress),

$$\sigma_{\mathrm{Avg}} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}}$$

The circle equation above now takes on a more familiar form,

$$\left(\sigma_{x'} - \sigma_{\mathrm{Avg}}\right)^2 + \tau_{x'y'}^{\,2} = R^2$$

The circle is centered at the average stress value, and has a radius R equal to the maximum shear stress, as shown in the figure below,



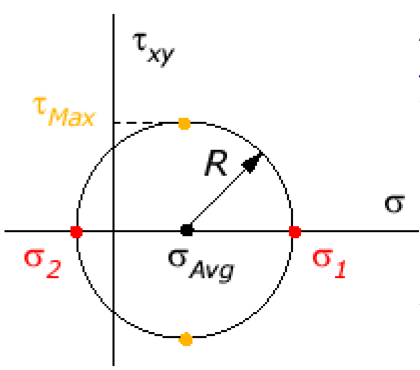
AS A REMINDER:

Basic Relations for Trigonometric Functions

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\sin^2\theta + \cos^2\theta = 1$
$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	$\sec^2\theta - \tan^2\theta = 1$
$\sec\theta = \dfrac{1}{\cos\theta}$	$\csc^2\theta - \cot^2\theta = 1$
$\csc\theta = \dfrac{1}{\sin\theta}$	$1 + \tan^2\theta = \dfrac{1}{\cos^2\theta}$
$\tan\theta\cot\theta = 1$	$1 + \cot^2\theta = \dfrac{1}{\sin^2\theta}$

TO DRAW MOHR'S CIRCLE

Principal Stresses from Mohr's Circle



A chief benefit of Mohr's circle is that the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max} are obtained immediately after drawing the circle,

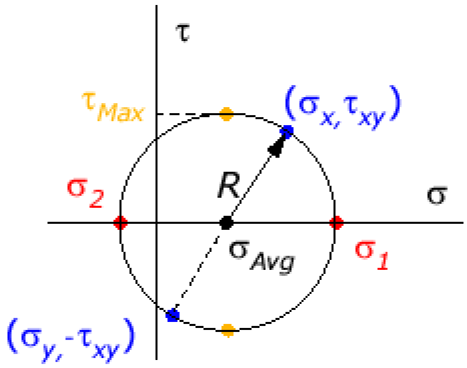
$$\begin{cases} \sigma_{1,2} = \sigma_{\mathrm{Avg}} \pm R \\ \tau_{\mathrm{Max}} = R \end{cases}$$

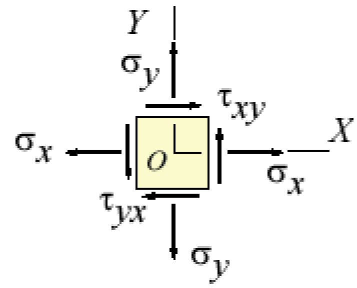
where,

$$\sigma_{\mathrm{Avg}} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Principal Directions from Mohr's Circle

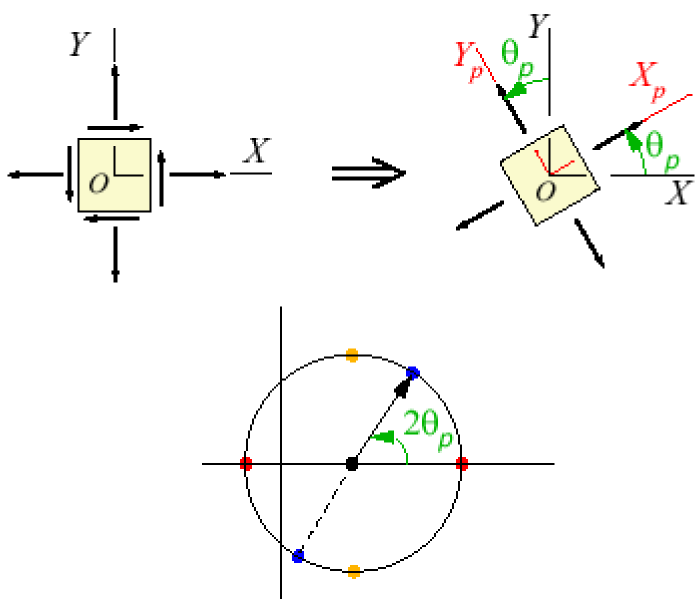
Mohr's Circle can be used to find the directions of the principal axes. To show this, first suppose that the normal and shear stresses, σ_x, σ_y, and τ_{xy}, are obtained at a given point *O* in the body. They are expressed relative to the coordinates *XY*, as shown in the stress element at right below.





The Mohr's Circle for this general stress state is shown at left above. Note that it's centered at σ_{avg} and has a radius *R*, and that the two points $\{\sigma_x, \tau_{xy}\}$ and $\{\sigma_y, -\tau_{xy}\}$ lie on opposites sides of the circle. The line connecting σ_x and σ_y will be defined as L_{xy}.

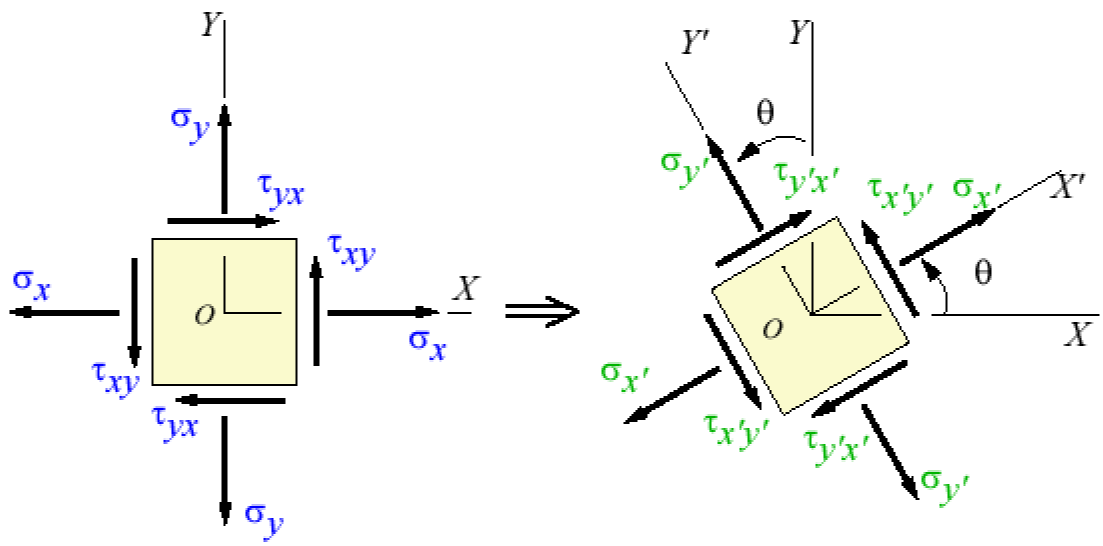
The **angle** between the current axes (*X* and *Y*) and the **principal axes** is defined as θ_p, and is equal to one half the angle between the line L_{xy} and the σ-axis as shown in the schematic below,



Stress Transform by Mohr's Circle

Mohr's Circle can be used to transform stresses from one coordinate set to another.

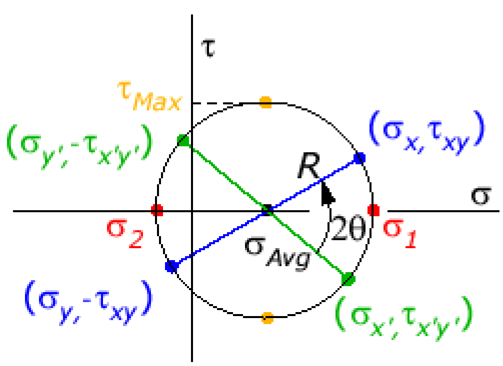
Suppose that the normal and shear stresses, σ_x, σ_y, and τ_{xy}, are obtained at a point *O* in the body, expressed with respect to the coordinates *XY*. We wish to find the stresses expressed in the new coordinate set *X'Y'*, rotated an angle θ from *XY*, as shown below:



Stresses at given coordinate system   Stresses transformed to another coordinate

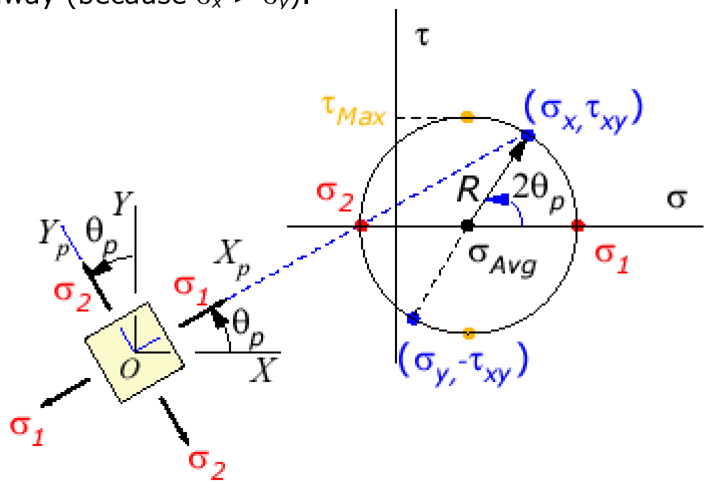
To do this we proceed as follows:

- Draw Mohr's circle for the **given stress state** (σ_x, σ_y, and τ_{xy}; shown below).
- Draw the line L_{xy} across the circle from (σ_x, τ_{xy}) to (σ_y, -τ_{xy}).
- Rotate the line L_{xy} by 2*θ (twice as much as the angle between *XY* and *X'Y'*) and in the *opposite* direction of θ.
- The **stresses in the new coordinates** ($\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$) are then read off the circle.



Problem #2 from Pset 7 that will be posted Friday and due Nov. 7
$\tau_{xy} > 0$ and $\sigma_x > \sigma_y$

The principal axes are counterclockwise to the current axes (because $\tau_{xy} > 0$) and no more than 45º away (because $\sigma_x > \sigma_y$).



Lennard-Jones Potential

Review equations covered in class by Prof. Ortiz.

Example 1. The interaction potential between two atoms is found to be:

$$U(r) = -\frac{10^{-76}\,Jm^6}{r^6} + \frac{10^{-133}\,Jm^{12}}{r^{12}}$$

a) Find the equilibrium distance r_e between the atoms and the binding energy of the atoms, in k_BT units. ($1\ k_BT = 4.1 * 10^{-21}$ J) *You may use Maple, Matlab or Mathematica <u>only to solve for r_e</u>, but you must set up the equation to be solved first*. Also, show the answer clearly.
b) What is the k_BT unit? What does it mean when the binding energy is greater or less than $1\ k_BT$?

Example 2. The interaction potential between two atoms is found to be:

$$U(r) = -\frac{A}{r^5} + \frac{B}{r^{11}}$$

In an attempt to determine A and B, physicists have measured the equilibrium distance between the atoms (r_e) to be 0.167 nm and the distance between the atoms where the potential is zero (r_0) to be 0.147 nm. They also measured the binding energy between the atoms (E_B) to be $0.101*10^{12}\ k_BT$. Determine A and B. Be very careful about your units.

EXAMPLE 1 Answer:

Example 1: Part (a) and (b) if time.

a) (0.5 pts each answer, 0.5 for method)
r_e is the distance at which F(r) = dU(r)/dr = 0

$$\frac{dU(r)}{dr} = \frac{6*10^{-76}\,Jm^6}{r^7} - \frac{12*10^{-133}\,Jm^{12}}{r^{13}} = 0$$

$$\frac{6*10^{-76}\,Jm^6}{r^7} = \frac{12*10^{-133}\,Jm^{12}}{r^{13}}$$

$$\frac{r_e^{\ 13}}{r_e^{\ 7}} = r_e^{\ 6} = \frac{2*10^{-133}\,Jm^{12}}{10^{-76}\,Jm^6} = 2*10^{-57}\,m^6$$

$$r_e = \sqrt[6]{2*10^{-57}\,m^6} = 0.355*10^{-9}\,m = \mathbf{0.355\ nm}$$

The binding energy is the depth of the potential well at r_e, so

$$U(r = 0.355*10^{-9}\,m) = -\frac{10^{-76}\,Jm^6}{(0.355*10^{-9}\,m)^6} + \frac{10^{-133}\,Jm^{12}}{(0.355*10^{-9}\,m)^{12}} = -2.50*10^{-20}\,J$$

Thus, the depth of the well is $2.50*10^{-20}$ J, or **6.1 k_BT**.

b) PLEASE NOTE. ANSWER DIFFERS SOMEWHAT FROM THE QUESTION THAT WAS ASKED.
(0.5 pts) The k_BT unit is an energy unit, corresponding to the **thermal energy at room temperature**. It is obtained by multiplying Boltzmann's constant (k_B) by the room temperature (298 K).
(0.5 pts) When the binding energy between two atoms is <u>much greater</u> than 1 k_BT, the energy provided by the thermal bath at room temperature is not enough to break the bond between them, so **the atoms will remain bonded, as a solid**. For example, covalent bonds have strengths that are 100-300 k_BT, hence you will need additional energy apart from room temperature to break the bond.
(0.5 pts) If the binding energy is <u>of the order of</u> 1 k_BT, the energy provided by the thermal bath at room temperature is of the order of the bond strength, and the bond between the atoms can break and reform easily without much external energy applied.

Example 2 ANSWER

Example 2: Answer → you get these equations using information given.

$$\begin{cases} U(r_e) = -E_B = -0.101*10^{12} k_B T = -\dfrac{A}{(0.167*10^{-9}m)^5} + \dfrac{B}{(0.167*10^{-9}m)^{11}} \\ U(r_0) = 0 = -\dfrac{A}{(0.147*10^{-9}m)^5} + \dfrac{B}{(0.147*10^{-9}m)^{11}} \end{cases} = -4.14*10^{-10} J$$

Solving for A and B will yield:

A = 10^{-58} J-m^5, B = 10^{-117} J-m^{11}

Note: using the derivative of U(r) = 0 instead of the first equation does not work, because that equation is not independent from the second equation. However, if you replace the second equation with $dU(r_e)/dr = 0$, then the system works.