3.11 Problem Set 2 Solution

1.
(a) $\tau_{aver} = V / A_{//}$

In the case of the upper or the lower plate, $V = P/2$ and $A_{//}$ = length * width
In the case of the middle plate, $V = P$ and $A_{//}$ = 2 * (length * width) (double shear)

$\tau_{aver} = V / A_{//} = 15{,}000 \text{ N} / (2 * 200 \text{ mm} * 150 \text{ mm}) = 0.25 \text{ MPa}$
$\tau_{aver} = G * \gamma_{aver} \Rightarrow \gamma_{aver} = \tau_{aver} / G = 0.25 \text{ MPa} / 0.83\text{MPa} =$ **0.301 (radians)**

(b) $\tan \gamma_{aver} = \sim \gamma_{aver}$ (for small angles) $= \delta / t$
$\Rightarrow \delta = \gamma_{aver} * t = 0.301 * 12 \text{ mm} =$ **3.6 mm**

2.

$$S_{ij} = \begin{bmatrix} 1/E & -\upsilon/E & -\upsilon/E & 0 & 0 & 0 \\ -\upsilon/E & 1/E & -\upsilon/E & 0 & 0 & 0 \\ -\upsilon/E & -\upsilon/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix}$$

$$Det.S_{ij} = \Delta = \begin{vmatrix} 1/E & -\upsilon/E & -\upsilon/E & 0 & 0 & 0 \\ -\upsilon/E & 1/E & -\upsilon/E & 0 & 0 & 0 \\ -\upsilon/E & -\upsilon/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{vmatrix} = 1/G \begin{vmatrix} 1/E & -\upsilon/E & -\upsilon/E & 0 & 0 \\ -\upsilon/E & 1/E & -\upsilon/E & 0 & 0 \\ -\upsilon/E & -\upsilon/E & 1/E & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 1/G \end{vmatrix}$$

$$= \frac{1}{G^2}\begin{vmatrix} \frac{1}{E} & -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & 0 \\ -\frac{\upsilon}{E} & \frac{1}{E} & -\frac{\upsilon}{E} & 0 \\ -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & \frac{1}{E} & 0 \\ 0 & 0 & 0 & \frac{1}{G} \end{vmatrix} = \frac{1}{G^3}\begin{vmatrix} \frac{1}{E} & -\frac{\upsilon}{E} & -\frac{\upsilon}{E} \\ -\frac{\upsilon}{E} & \frac{1}{E} & -\frac{\upsilon}{E} \\ -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & \frac{1}{E} \end{vmatrix}$$

$$= \frac{1}{G^3}\left[\begin{array}{l} (-1)^{1+1}\times\left(\frac{1}{E}\right)\times\left(\frac{1}{E^2}-\frac{\upsilon^2}{E^2}\right)+(-1)^{1+2}\times\left(-\frac{\upsilon}{E}\right)\times\left(-\frac{\upsilon}{E^2}-\frac{\upsilon^2}{E^2}\right) \\ +(-1)^{1+3}\times\left(-\frac{\upsilon}{E}\right)\times\left(\frac{\upsilon^2}{E^2}+\frac{\upsilon}{E^2}\right) \end{array}\right]$$

$$= \frac{1}{G^3}\left[\frac{1}{E^3}-\frac{\upsilon^2}{E^3}-\frac{\upsilon^2}{E^3}-\frac{\upsilon^3}{E^3}-\frac{\upsilon^3}{E^3}-\frac{\upsilon^2}{E^3}\right] = \frac{1-3\upsilon^2-2\upsilon^3}{G^3E^3}$$

$$= \frac{\mathbf{(1-2\upsilon)(1+\upsilon)^2}}{\mathbf{G^3E^3}}$$

$$S'_{11} = (-1)^{1+1}\frac{\begin{vmatrix} \frac{1}{E} & -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & 0 & 0 & 0 \\ -\frac{\upsilon}{E} & \frac{1}{E} & -\frac{\upsilon}{E} & 0 & 0 & 0 \\ -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{vmatrix}}{\Delta} = \frac{\frac{1}{G^3}\begin{vmatrix} \frac{1}{E} & -\frac{\upsilon}{E} \\ -\frac{\upsilon}{E} & \frac{1}{E} \end{vmatrix}}{\Delta}$$

$$= \frac{\left(\frac{1-\upsilon^2}{G^3E^2}\right)}{\Delta} = \frac{\left(\frac{(1-\upsilon)(1+\upsilon)}{G^3E^2}\right)}{\left(\frac{(1-2\upsilon)(1+\upsilon)^2}{G^3E^3}\right)} = \frac{\mathbf{E(1-\upsilon)}}{\mathbf{(1-2\upsilon)(1+\upsilon)}}$$

$$S'_{12} = (-1)^{1+2}\frac{\begin{vmatrix} \frac{1}{E} & -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & 0 & 0 & 0 \\ -\frac{\upsilon}{E} & \frac{1}{E} & -\frac{\upsilon}{E} & 0 & 0 & 0 \\ -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{vmatrix}}{\Delta} = -\frac{\frac{1}{G^3}\begin{vmatrix} -\frac{\upsilon}{E} & -\frac{\upsilon}{E} \\ -\frac{\upsilon}{E} & \frac{1}{E} \end{vmatrix}}{\Delta}$$

$$= -\frac{\left(\dfrac{-\upsilon-\upsilon^2}{G^3E^2}\right)}{\Delta} = \frac{\left(\dfrac{\upsilon(1+\upsilon)}{G^3E^2}\right)}{\left(\dfrac{(1-2\upsilon)(1+\upsilon)^2}{G^3E^3}\right)} = \frac{\mathbf{E\upsilon}}{\mathbf{(1-2\upsilon)(1+\upsilon)}}$$

$$S'_{44} = (-1)^{4+4}\frac{\begin{vmatrix} 1/E & -\upsilon/E & -\upsilon/E & 0 & 0 & 0 \\ -\upsilon/E & 1/E & -\upsilon/E & 0 & 0 & 0 \\ -\upsilon/E & -\upsilon/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{vmatrix}}{\Delta} = \frac{1/G^2\begin{vmatrix} 1/E & -\upsilon/E & -\upsilon/E \\ -\upsilon/E & 1/E & -\upsilon/E \\ -\upsilon/E & -\upsilon/E & 1/E \end{vmatrix}}{\Delta}$$

$$= \frac{\left(\dfrac{(1-2\upsilon)(1+\upsilon)^2}{G^2E^3}\right)}{\left(\dfrac{(1-2\upsilon)(1+\upsilon)^2}{G^3E^3}\right)} = \mathbf{G}$$

Doing the same thing for the other terms in the matrix finds that

$S'_{11} = S'_{22} = S'_{33} = \dfrac{\mathbf{E(1-\upsilon)}}{\mathbf{(1-2\upsilon)(1+\upsilon)}}$

$S'_{12} = S'_{13} = S'_{23} = S'_{21} = S'_{31} = S'_{32} = \dfrac{\mathbf{E\upsilon}}{\mathbf{(1-2\upsilon)(1+\upsilon)}}$

$S'_{44} = S'_{55} = S'_{66} = G = \dfrac{\mathbf{E}}{\mathbf{2(1+\upsilon)}}$

$$S'_{ij} = \begin{bmatrix} \frac{E(1-\upsilon)}{(1-2\upsilon)(1+\upsilon)} & \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & 0 & 0 & 0 \\ \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & \frac{E(1-\upsilon)}{(1-2\upsilon)(1+\upsilon)} & \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & 0 & 0 & 0 \\ \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & \frac{E(1-\upsilon)}{(1-2\upsilon)(1+\upsilon)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{E}{2(1+\upsilon)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{E}{2(1+\upsilon)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{E}{2(1+\upsilon)} \end{bmatrix}$$

$D_{ij} = (S'_{ij})^T = (S'_{ij})$ because the matrix is symmetric.

$$D_{ij} = \begin{bmatrix} \frac{E(1-\upsilon)}{(1-2\upsilon)(1+\upsilon)} & \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & 0 & 0 & 0 \\ \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & \frac{E(1-\upsilon)}{(1-2\upsilon)(1+\upsilon)} & \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & 0 & 0 & 0 \\ \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & \frac{E\upsilon}{(1-2\upsilon)(1+\upsilon)} & \frac{E(1-\upsilon)}{(1-2\upsilon)(1+\upsilon)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{E}{2(1+\upsilon)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{E}{2(1+\upsilon)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{E}{2(1+\upsilon)} \end{bmatrix}$$

$$= \frac{E}{(1-2\upsilon)(1+\upsilon)} \begin{bmatrix} 1-\upsilon & \upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & 1-\upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & \upsilon & 1-\upsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\upsilon)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\upsilon)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\upsilon)}{2} \end{bmatrix}$$

3. In this problem, there are no shear strains involved, so we can use just the upper-left quadrant of the compliance matrix.
plane stress problem => $\sigma_z = 0$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} = \begin{bmatrix} 1/E & -\upsilon/E & -\upsilon/E \\ -\upsilon/E & 1/E & -\upsilon/E \\ -\upsilon/E & -\upsilon/E & 1/E \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ 0 \end{Bmatrix}$$

$\varepsilon_x = \sigma_x/E - \sigma_y (\nu/E) \Rightarrow \sigma_x = E * \varepsilon_x + \sigma_y * \nu$ (1)
$\varepsilon_y = \sigma_x (-\nu/E) + \sigma_y/E$ (2)
$\varepsilon_z = (-\nu/E) (\sigma_x + \sigma_y)$ (3)

(1) into (2) $\Rightarrow \varepsilon_y = -\nu * \varepsilon_x - \sigma_y * \nu^2/E + \sigma_y/E$
$\varepsilon_y + \nu * \varepsilon_x = \sigma_y * (1-\nu^2)/E$
$\sigma_y = E * (\varepsilon_y + \nu * \varepsilon_x) / (1-\nu^2)$
$= 200$ GPa $* (85 + 0.3 * 350) * 10^{-6} / (1 - 0.3^2)$
$= 0.0418$ GPa = **41.8 MPa**

$\sigma_x = E * \varepsilon_x + \sigma_y * \nu$
$= 200$ GPa $* 350 * 10^{-6} + 0.0418$ GPa $* 0.30$
$= 0.0825$ GPa = **82.5 MPa**

(3) $\rightarrow \varepsilon_z = (-0.3/200{,}000 \text{ MPa}) (82.5 \text{ MPa} + 41.8 \text{ MPa}) = \Delta t/t$
$\Delta t/t = - 1.86 \times 10^{-4}$
$\Delta t = 10 \text{ mm} * (-1.86 \times 10^{-4}) =$ **$- 1.86 \times 10^{-3}$ mm**

4. Since this is a case where the cube is immersed in water, we have a hydrostatic stress, where the normal stresses are the same, in compression, and the shear stresses are zero. From Appendix H-2 (p.899), E = 70 GPa and $\nu = 0.33$.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} = \begin{bmatrix} 1/E & -\upsilon/E & -\upsilon/E \\ -\upsilon/E & 1/E & -\upsilon/E \\ -\upsilon/E & -\upsilon/E & 1/E \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix}$$

$\sigma_x = \sigma_y = \sigma_z = \sigma = - 40$ MPa
$\Rightarrow \varepsilon_x = \varepsilon_y = \varepsilon_z = \sigma * (1 - 2\nu)/E = - 40 \text{ MPa} (1 - 2*0.33) / 70{,}000 \text{ MPa} = -1.943 \times 10^{-4}$

$V = (L + L * \varepsilon)^3 = (1 \text{ m} - 1.943 \times 10^{-4} \text{ m})^3 =$ **0.9994 m^3**