Solutions Problem Set #9
Due Friday, November 21th

Spring-Dashpot

1. Describe the difference between a Creep Test and a Stress Relaxation Test. Use graphs of each to explain your answer. (You can state what part dominates)

Answer: Discussed in recitation. Creep test has constant stress and relaxation test has constant strain.

2.

a) The simpliest spring-dashpot models are the Maxwell and Voigt elements discussed in class. A better model is the "standard linear solid" which is shown below. Derive a constitutive equation (i.e. one that takes into account the parallel and series properties) for the standard linear solid, which shows how the overall stress, stress rate, strain and strain rate are related by the three parameters E_1, E_2 and η

b) For the standard linear solid discussed in part a), determine the overall $\varepsilon(t)$ in terms of E_1, E_2 and η_1 when in a state of constant stress.

c) The retardation time, τ, defined as $\tau = \eta/E_1$ is often used to replace viscosity and Young's modulus(in general, this equation is seen as $\tau = \eta/k$) . If immediately after applying stress, the strain is 0.002, after 1000 seconds the strain is 0.004 and after a very long time the strain tends to be 0.006, what is the retardation time τ?

Solution:

$$\sigma = \sigma_a = \sigma_b$$

$$\sigma = E_1\varepsilon_1 = E_2\varepsilon_2 + \eta\dot{\varepsilon}_2$$

$$\dot{\varepsilon} = \dot{\varepsilon}_2 + \dot{\varepsilon}_1$$

$$= \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta} - \frac{E_2\varepsilon_2}{\eta}$$

$$\varepsilon_2 = \varepsilon - \varepsilon_1 = \varepsilon - \frac{\sigma}{E_1}$$

$$\varepsilon = \frac{\sigma}{E_1} + \frac{\sigma}{\eta} - \frac{E_2}{\eta}\left(\varepsilon - \frac{\sigma}{E_1}\right)$$

$$\dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}\left(1 + \frac{E_2}{E_1}\right)$$

$$\dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}\left(\frac{E_1 + E_2}{E_1}\right)$$

b) For the standard linear solid discussed in part a), determine the overall ε(t) in terms of E_1, E_2, and η_1 when in a state of constant stress.

$$\boldsymbol{\eta}\dot{\boldsymbol{\varepsilon}} + \mathbf{E}_2\boldsymbol{\varepsilon} = \left(1 + \frac{\mathbf{E}_2}{\mathbf{E}_1}\right)\boldsymbol{\sigma}$$

Need to solve the differential equation of type a*y'+b*y=c

Using a differential equation solver:

$$\varepsilon(t) = K \bullet \exp\left(\frac{-E_2 t}{\eta}\right) + \frac{\left(1 + \frac{E_2}{E_1}\right)}{E_1}\sigma$$

$$\varepsilon(t) = K \bullet \exp\left(\frac{-E_2 t}{\eta}\right) + \frac{(E_1 + E_2)}{E_2 E_1}\sigma$$

Where K is an integration constant. We know that immediately after applying the stress, the strain will be entirely from the lone spring (ε1=0) and so:

$$\boldsymbol{\varepsilon}(\mathbf{t} = 0) = \frac{\boldsymbol{\sigma}}{\mathbf{E}_1}$$

$$\frac{\boldsymbol{\sigma}}{\mathbf{E}_2} = \mathbf{K}\frac{(\mathbf{E}_2 + \mathbf{E}_1)}{\mathbf{E}_1\mathbf{E}_2}\boldsymbol{\sigma}$$

$$\mathbf{K} = \frac{\boldsymbol{\sigma}}{\mathbf{E}_1}\left(1 - \frac{\mathbf{E}_2 + \mathbf{E}_1}{\mathbf{E}_2}\right) = \frac{\boldsymbol{\sigma}}{\mathbf{E}_1}\left(1 - \frac{\mathbf{E}_1}{\mathbf{E}_2} - 1\right) = \frac{-\boldsymbol{\sigma}}{\mathbf{E}_2}$$

$$\boldsymbol{\varepsilon}(\mathbf{t}) = \frac{-\boldsymbol{\sigma}}{\mathbf{E}_2} \bullet \exp\left(\frac{-\mathbf{E}_2\mathbf{t}}{\boldsymbol{\eta}}\right) + \frac{(\mathbf{E}_2 + \mathbf{E}_1)}{\mathbf{E}_1\mathbf{E}_2}\boldsymbol{\sigma}$$

$$\boldsymbol{\varepsilon}(\mathbf{t}) = \frac{\boldsymbol{\sigma}}{\mathbf{E}_2} \bullet \left(1 + \frac{\mathbf{E}_2}{\mathbf{E}_1} - \exp\left(\frac{-\mathbf{E}_2\mathbf{t}}{\boldsymbol{\eta}}\right)\right)$$

c) The retardation time, τ, defined as $\tau = \frac{\eta}{E_2}$ is often used to replace viscosity and Young's modulus. If immediately after applying stress, the strain is 0.002, after 1000 seconds the strain is 0.004 and after a very long time the strain tends to be 0.006, what is the retardation time τ?

Solution:

Defining $\tau \equiv \frac{\eta}{E_2}$

$$\varepsilon(0) = \frac{\sigma}{E_2}\left(1 + \frac{E_2}{E_1} - 1\right) = \frac{\sigma}{E_1} = 0.002$$

$$\varepsilon(\infty) = \frac{\sigma}{E_2}\left(1 + \frac{E_2}{E_1} - 0\right) = \frac{\sigma}{E_1} + \frac{\sigma}{E_1} = \frac{\sigma}{E_1} + 0.002 = 0.006 \rightarrow \frac{\sigma}{E_2} = 0.004$$

$$\varepsilon(1000) = \frac{\sigma}{E_2}\left(1 + \frac{E_2}{E_1} - \exp\left(\frac{-1000}{\tau}\right)\right) = \frac{\sigma}{E_2} + \frac{\sigma}{E_1} - \frac{\sigma}{E_2}\left(\exp\left(\frac{-1000}{\tau}\right)\right)$$

$$\varepsilon(1000) = 0.004 + 0.002 - 0.004\exp\left(\frac{-1000}{\tau}\right)$$

$$\varepsilon(1000) = 0.006 - 0.004\exp\left(\frac{-1000}{\tau}\right) = 0.004$$

$$\Rightarrow 0.004\left(\exp\left(\frac{-1000}{\tau}\right)\right) = 0.002$$

$$\exp\left(\frac{-1000}{\tau}\right) = 0.5$$

$$\frac{-1000}{\tau} = -0.693$$

$$\tau = \frac{1000}{0.693}$$

$$\tau = 1443\,\text{sec}$$

3. In a Kelvin-Voigt model, the creep response of a material is modeled by the following expression:

$$\varepsilon(t) = \frac{\sigma_0}{k}\left(1 - \exp\left(-t/\tau_c\right)\right)$$

where σ_0 is the constant stress applied to the material, k is the spring modulus and τ_c, the retardation time, is defined as $\tau_c = \eta/k$, where η is the viscosity of the dashpot.

In a creep test, a material with k = 600 MPa is initially loaded with a stress σ_0. Half an hour after the initial loading, the strain in the material is measured to be 0.111, and after another hour, it is found to be 0.264. What will be the strain in the material 3 hours after the initial loading? How much time did it take for the strain to reach a value of 0.001?

$$0.111 = \frac{\sigma_0}{600MPa}\left(1-\exp\left(-0.5hrs/\tau_c\right)\right) \qquad (1)$$

and

$$0.264 = \frac{\sigma_0}{600MPa}\left(1-\exp\left(-1.5hrs/\tau_c\right)\right) \qquad (2)$$

Dividing (1) by (2), we get:

$$\frac{0.111}{0.264} = \frac{\left(1-\exp\left(-0.5hrs/\tau_c\right)\right)}{\left(1-\exp\left(-1.5hrs/\tau_c\right)\right)}$$

Solving for τ_c, we find that τ_c = 1.97 hrs

Plugging this value back in (1) or (2), we can solve for σ_0 and find that
σ_0 = 0.111 * 600 MPa / exp(-0.5hrs / 1.97hrs)
σ_0 = 297.2 MPa

Thus, we finally have an expression for the creep response of the material:

$$\varepsilon(t) = \frac{297.2MPa}{600MPa}\left(1-\exp\left(-t/1.97hrs\right)\right)$$

The strain after 3 hours is

$$\varepsilon(t = 3hrs) = \frac{297.2MPa}{600MPa}\left(1-\exp\left(-3hrs/1.97hrs\right)\right)$$

= **0.39**

When the strain is 0.001, the time t is found to be

$$\varepsilon(t) = 0.001 = \frac{297.2MPa}{600MPa}\left(1-\exp\left(-t/1.97hrs\right)\right)$$

$$\Leftrightarrow t = -1.97hrs * \ln\left(1-0.001*\frac{600MPa}{297.2MPa}\right)$$

t = 0.00398 hrs = 14.3 s

4. A review of the rubber elasticity that wasn't covered—

a) starting with the expression for the change in Helmholtz free energy in an ideal rubber, derive the rubber elasticity equations of stress versus extension ratio for a biaxial stress (i.e., find $\sigma_x(\lambda_x,\lambda_y)$ and $\sigma_y(\lambda_x,\lambda_y)$.)

b) If E (the elastic modulus in uniaxial deformation) is 10.0 MPa for a rubber, what are the stresses σ_x and σ_y required to deform the rubber to $\lambda_x = 2$ and $\lambda_y = 1/2$? Is σ_x an overestimate or an underestimate, compared to experimental data?

(a)

$$\Delta A = \frac{1}{2}\upsilon_x k_B T\left(\lambda_x^2 + \lambda_y^2 + \lambda_z^2 - 3\right) \quad (1)$$

The stress in a particular direction in a rubber under deformation is found by taking the partial derivative of the change in Helmholtz free energy with respect to the extension ratio in the direction of interest, i.e., $\sigma_i = \frac{\partial \Delta A}{\partial \lambda_i}$, where i is x, y or z. Also, deformation in rubber happens at constant volume, which means that $\lambda_x\lambda_y\lambda_z = 1$. Since we are looking for the stresses as a function of λ_x and λ_y, we need to eliminate λ_z from the Helmholtz free energy change expression.

$\lambda_x\lambda_y\lambda_z = 1 \Leftrightarrow \lambda_z = \frac{1}{\lambda_x\lambda_y}$. Substitute this into Eqn (1), then differentiate with respect to λ_x to find the expression for σ_x, and with respect to λ_x to find the expression for σ_y.

$$\Delta A = \frac{1}{2}\upsilon_x k_B T\left(\lambda_x^2 + \lambda_y^2 + \frac{1}{\lambda_x^2\lambda_y^2} - 3\right)$$

$$\sigma_x = \frac{\partial \Delta A}{\partial \lambda_x} = \frac{1}{2}\upsilon_x k_B T\left(2\lambda_x - \frac{2}{\lambda_x^3\lambda_y^2}\right)$$

$$\boldsymbol{\sigma_x = \upsilon_x k_B T\left(\lambda_x - \frac{1}{\lambda_x^3\lambda_y^2}\right)}$$

and

$$\sigma_y = \frac{\partial \Delta A}{\partial \lambda_y} = \frac{1}{2}\upsilon_x k_B T\left(2\lambda_y - \frac{2}{\lambda_y^3\lambda_x^2}\right)$$

$$\boldsymbol{\sigma_y = \upsilon_x k_B T\left(\lambda_y - \frac{1}{\lambda_y^3\lambda_x^2}\right)}$$

b) (1.5 pts, 0.5 each) E = 3 ν_x k_B T = 10.0 MPa ➔ ν_x k_B T = 3.33 MPa

$$\sigma_x = \upsilon_x k_B T\left(\lambda_x - \frac{1}{\lambda_x^3\lambda_y^2}\right) = 3.33MPa\left(2 - \frac{1}{2^3(1/2)^2}\right) = \mathbf{5.0\ MPa}$$

$$\sigma_y = \upsilon_x k_B T\left(\lambda_y - \frac{1}{\lambda_y^3\lambda_x^2}\right) = 3.33MPa\left((1/2) - \frac{1}{2^2(1/2)^3}\right) = \mathbf{-5.0\ MPa}$$

From the comparison between theory and experiment in lecture 21, when $1.5 < \lambda_x < 6$, the theory **overestimates** the stress undergone by the rubber.