7 Filtering for Markov Chains

Let $E = \{1, \ldots, \mathbb{K}\}$ be the state space of a finite state Markov Chain $(X_n)_{n\geq 0}$, with stationary transition probabilities $p(i,j)\Big|_{i,j=1}^{\mathbb{K}}$. Let $P = \Big(p(i,j)\Big)_{i,j=1}^{\mathbb{K}}$ denote the transition matrix of the Markov Chain. Let $\mu = (\mu_1, \ldots, \mu_{\mathbb{K}})$ be the distribution of X_0. The state of the Markov Chain cannot be observed but instead we observe

$$Y_n = h_n(X_n) \tag{7.1}$$

where Y_n takes its values in the set $G = \{1, 2, \ldots, \mathbb{M}\}$, and

$$P\Big(Y_n = j \Big| X_n = k\Big) = q^k(j) \quad , \quad k \in E \quad , \quad j \in G \quad , \quad n = 0, 1, 2, \ldots \tag{7.2}$$

We assume that the Y_n's are conditionally independent given X_n, X_{n-1}. Let

$$\Pi_n^k = P\Big(X_n = k \Big| Y_0, \ldots, Y_n\Big) \tag{7.3}$$

and let $\coprod_n = (\Pi_n^1, \ldots, \Pi_n^{\mathbb{K}})$.

Filtering. We would like to obtain a recursion

$$\Pi_{n+1}^k = F_n(\Pi_{n+1}^K, Y_{k+1}) \quad , \quad k = 1, 2, \ldots, \mathbb{K} \tag{7.4}$$

for some appropriate function F_n. Now

$$P\Bigg(X_{n+1}=j \,\Bigg|\, Y_0=y_0,\dots Y_n=y_n, Y_{n+1}=y\Bigg)$$

$$= \frac{P\Big(X_{n+1}=j, Y_{n+1}=y \,\Big|\, Y_0=y_0,\dots Y_n=y_n\Big)}{P\Big(Y_{n+1}=y \,\Big|\, Y_0=y_0,\dots Y_n=y_n\Big)} \tag{7.5}$$

$$= \frac{\sum_{k\in S} P\Big(X_{n+1}=j, X_n=k, Y_{n+1}=y \,\Big|\, Y_0=y_0,\dots Y_n=y_n\Big)}{\sum_{l,m\in S} P\Big(X_{n+1}=\ell, X_n=m, Y_{n+1}=y \,\Big|\, Y_0=y_0,\dots Y_n=y_n\Big)}$$

$$\forall j\in E \quad , \quad \forall y_0,\dots,y_n,y\in G \quad .$$

Now,

$$P\Bigg(X_{n+1}=\ell, X_n=m, Y_{n+1}=y \,\Bigg|\, Y_0=y_0,\dots Y_n=y_n\Bigg)$$

$$= P\Bigg(X_{n+1}=\ell, X_n=m, h_{n+1}(\ell)=y \,\Bigg|\, Y_0=y_0,\dots Y_n=y_n\Bigg) \tag{7.6}$$

Now $h_{n+1}(\ell)$ is *independent* of $Y_0,\dots,Y_n$ given X_{n+1} and X_n. Hence

$$P\Bigg(X_{n+1}=\ell, X_n=m, h_{n+1}(\ell) \,\Bigg|\, Y_0=y_0,\dots Y_n=y_n\Bigg)$$

$$= q^\ell(y)\cdot P\Bigg(X_{n+1}=\ell, X_n=m \,\Bigg|\, Y_0=y_0,\dots Y_n=y_n\Bigg)$$

$$= q^\ell(y)\cdot P\Bigg(X_{n+1}=\ell \,\Bigg|\, X_n=m\Bigg)\cdot P\Bigg(X_n=m \,\Bigg|\, Y_0=y_0,\dots Y_n=y_n\Bigg)$$

$$= q^\ell(y)p(\ell,m)\Pi_n^m \tag{7.7}$$

Hence from (7.5) and (7.3), we get

$$\Pi^j_{n+1} = \frac{q^j(y)\sum_{k\in E} p(k,j)\Pi^k_n}{\sum_{i\in E} q^i(y)\sum_{k\in E} p(k,i)\Pi^k_n} \tag{7.8}$$

$$\Pi^j_0 = \frac{\mu_j q^j(z_0)}{\sum_k \mu_k q^k(z_0)} \tag{7.9}$$

Let $\sum = (\nu_1, \ldots, \nu_k)$ be the set of all probability vectors and let

$$\begin{aligned} F \quad &: \quad \Sigma \times G \to \Sigma \ \text{ be the map} \\ &: \quad (p,y) \mapsto \Big(F_1(p,y), \ldots, F_k(p,y)\Big) \end{aligned}$$

where

$$F_j(p,y) = \frac{q^j(y)\sum_{k\in S} p(k,j)\nu_k}{\sum_{i\in E} q^i(y)\sum_{k\in E} p(k,i)\nu_k} \tag{7.10}$$

We then have

$$\begin{aligned} \coprod_{n+1} &= F(\coprod_n, y) \quad , \quad n = 0,1,2,\ldots \\ \coprod_0 &= (\Pi^1_0, \ldots, \Pi^k_0) \qquad (7.11) \\ \Pi^j_0 &= \frac{\mu_j q^j(z_0)}{\sum_k \mu_k q^k(z_0)} \quad . \end{aligned}$$

We may view $(\coprod_n)_{n\geq 0}$ as a Markov Chain whose state space is the space of probability vectors $\nu = (\nu_1, \ldots, \nu_{\mathbb{K}}$. Let ψ be a bounded Borel function $\Sigma \to \mathbb{R}$. Then

$$\begin{aligned} &\mathbb{E}\left(\psi(\coprod_{n+1})\,\middle|\, Y_0 = y_0, \ldots Y_n = y_n\right) \\ = \;&\mathbb{E}\left(\psi[F(\coprod_n, Y_{n+1})]\,\middle|\, Y_0 = y_0, \ldots Y_n = y_n\right) \end{aligned} \tag{7.12}$$

But Π_n is a random variable measurable w.r.to $\sigma(Y_0, \ldots, Y_n)$. Hence

$$= \sum_{y\in G} \psi\Big[F(\boldsymbol{\Pi}_n, y)\Big]\mathbb{P}\Big(Y_{n+1} = y \,\Big|\, Y_0 = y_0, \ldots, Y_n = y_n\Big)$$

$$= \sum_{y\in G} \psi\Big[F(\boldsymbol{\Pi}_n, y)\Big] \sum_{i\in E} q^i(y) \sum_{k\in E} p(k,i)\Pi_n^k \tag{7.13}$$

(from (7.6) and (7.7)).

Now define the operator $\mathcal{P} : \Sigma \to \Sigma$, by

$$(\mathcal{P}\psi)(\nu) = \sum_{y\in G} \psi\Big[F(\nu, y) \sum_{i,k\in E} q^i(y)\nu_k p(k,i) \tag{7.14}$$

Then from (7.13) and (7.14) we get

$$\mathbb{E}\Big(\psi(\boldsymbol{\Pi}_{n+1}) \,\Big|\, Y_0 = y_0, \ldots, Y_n = y_n\Big) = \mathcal{P}\psi(\boldsymbol{\Pi}_n) \quad ,$$

showing that $\mathcal{P}$ is the transition operator of a Markov Chain evolving on Σ, the space of probability vectors $\nu = (\nu_1, \ldots, \nu_k)$.