Steady State Kalman Filter
&
the Wiener Filter

①

$$\dot{X}(t) = FX(t) + Gu(t) \quad X(t_0) = X_0$$

$$Z(t) = Hx(t).$$

Real field

$$Y(t) = Z(t) + V(t).$$

$$\mathrm{Cov}(X_0) = \Pi_0 \quad u(t) \perp X_0 \quad t > t.$$

$$\mathrm{Cov}(u(t), u(s)) = Q\delta(t-s)$$

$$\mathrm{Cov}(V(t), V(s)) = R\,\delta(t-s$$

$$u(t) \perp V(s)$$

F stability matrix.

$$R_X(t,s) = E(X(t)X'(s)) = \begin{cases} e^{F(t-s)}\Pi(s) & t \geq s \\ \Pi(t)\, e^{F'(s-t)} & t \leq s. \end{cases}$$

$$\Pi(t) = \mathrm{Cov}(X(t))$$

$$= e^{F(t-t_0)}\Pi_0 e^{F'(t-t_0)} + \int_{t_0}^{t} e^{F(t-\tau)} GQG' e^{F'(t-\tau)} d\tau$$

$$\dot{\Pi} = F\Pi + \Pi F' + GQG' \quad \Pi(0) = \Pi_0$$

$$R_Z(t,s) = H R_X(t,s) H'$$

$$t_0 \longrightarrow -\infty \quad \Pi(t) \longrightarrow \bar{\Pi}$$

$$\Pi(t) \longrightarrow \bar{\Pi} > 0 \quad (\text{Controllability } (F, GQ^{1/2})$$

$t_0 \longrightarrow -\infty$

$$R_z(t,s) = He^{F(t-s)}\overline{\Pi}H' \quad t \geq s.$$

$$S_z(j\omega) = H(j\omega I F)^{-4}\overline{\Pi}H'$$
$$+ H\overline{\Pi}(-j\omega I - F')^{-1}H'$$
$$= H(j\omega I - F)^{-1}GQG'(-j\omega I - F')^{-1}H'$$

Set $\Pi_0 = \overline{\Pi}$

then $\dot{\Pi} = 0 \ \forall t$

$\Rightarrow \quad \Pi(t) = \overline{\Pi} \quad \forall\, t \geq t_0$

<u>K-F</u>

$$\dot{\hat{X}}(t) = F\hat{X}(t) + K(t)\nu(t) \quad \hat{X}(0) = 0$$

$$K(t) = P(t)H'(t)$$

$$\dot{P}(t) = FP(t) + P(t)F' - K(t)K'(t) + GQG'$$

$$= (F - K(t)H)P(t) + P(t)(F - K(t)H)' + K(t)K'(t) + GQG'$$

Add & subtract

$P(t) \leq \Pi(t)$

If $\Pi(0) = \overline{\Pi}$

$\dot{P}(t) \leq 0$ follows from

$$\dot{P} = \Phi(t,0)\dot{P}(0)\phi'(t,0)$$
$$= \Phi(t,0)GG'\phi'(t,0)$$
$$\dot{P}(0) = -\overline{\Pi}H'H\overline{\Pi}$$

$P(t) \downarrow P$ as $t \to \infty$.

$\dot{P}(t) \longrightarrow 0$.

$\overline{P}$ satisfies ARE

Steady state Kalman Filter.

$y(\cdot) \longrightarrow \hat{z}(\cdot) \quad : \quad \mathcal{H}_K(s) = H\left(sI - F + \overline{K}H\right)\overline{K}$

Look at Scalar y

Wiener Filter: $H_W(s) = 1 - \left[S_y^+(s)\right]^{-1}$

$S_y(s) = \underset{\substack{\uparrow \\ \text{Stable \& Causal}}}{H}(s)\, H(-s)$

$$S_y(s) = 1 + S_z(s)$$

$$= \cancel{1} + H(sI-F)^{-1} GQG' (-sI-F')^{-1} H'.$$

<u>Show:</u> $S_y^+(s) = 1 + H(sI-F)^{-1}\overline{K}$

Innovation Repres. K-F.

$$y(\cdot) \longrightarrow \nu(\cdot) \longrightarrow \hat{z}(0).$$

$$\mathcal{H}_K(s) = H(sI-F)^{-1}\,\overline{K}\cdot\left[S_y^+(s)\right]^{-1}.$$

$$= \left[S_y^+(s) - 1\right]\left[S_y^+(s)\right]^{-1}.$$

$$= 1 - \left[S_y^+(s)\right]^{-1}.$$

$$= \mathcal{H}_W(s)$$

<u>Matrix Identity</u>

$$\left[1 + H(sI-F)^{-1}\overline{K}\right]^{-1} = 1 - H(sI-F+\overline{K}H)^{-1}\overline{K}.$$

Spectral ~~#~~a (5)

$$H = (1 \quad 0 \cdots) \quad G' = [\beta_1 \cdots \beta_N].$$

$$F = \begin{pmatrix} -\alpha_1 & 1 & & 0 \\ \vdots & & \ddots & \\ -\alpha_n & 0 & & 1 \end{pmatrix}.$$

$$S_y(s) = 1 + H(sI-F)^{-1} G Q G' (-sI-F')^{-1} H'$$

$$H(sI-F)^{-1} G = \frac{\beta(s)}{\cancel{\alpha}(s)}.$$

$$\gamma(s) = \det (sI-F)^{-1} = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n.$$

$$\beta(s) = \beta_1 s^{n-1} + \cdots + \beta_n.$$

$$H(sI-F)^{-1} = [s^{n-1} \; s^{n-2} \cdots] \frac{1}{\alpha(s)} \qquad (*)$$

$$S_y(s) = 1 + \frac{\beta(s)\beta(-s)}{\alpha(s)\alpha(-s)}.$$

$$S_y^+(s) = \left[\frac{\alpha(s)\alpha(-s) + \beta(s)\beta(-s)}{\alpha(s)\alpha(-s)}\right]^+ = \frac{[\quad]^+}{\alpha(s)}.$$

$$= \frac{\gamma(s)}{\alpha(s)}$$

(b)

$$\gamma(s) = s^n + \gamma_1 s^{n-1} \cdots + \gamma_n.$$

But $S_y^+(s) = 1 + H(sI-F)^{-1}\bar{P}H'$

$$\Rightarrow H(sI-F)^{-1}\bar{P}H' = \frac{\gamma(s)-\alpha(s)}{\alpha(s)}$$

$\Rightarrow$ From (*)

$$\bar{p}_{11}s^{n-1} + \cdots + \bar{p}_{n1} = \sum_{i=1}^{n}(\gamma_i - \alpha_i)s^{n-i}$$

or $\bar{p}_{11} = \gamma_1 - \alpha_1$. etc.

$\Rightarrow$ First col. of $\bar{P}$ completely determined by Spectral Factorization

$\bar{K} = \bar{P}H'$ only 1st coln of $\bar{P}$ is required

$\Rightarrow$ KF completely determined by Spectral Factor