Levinson-Durbin Algorithm

Consider the following system of linear equations with $(a_{k,i})_{i=1}^{k}$ and σ_k as unknowns:

$$\left.\begin{aligned} R_k \theta_k &= -r_k \\ \rho_0 + \theta_k^T r_k &= \sigma_k \end{aligned}\right\} k = 1, \dots n \qquad (1)$$

where

$$R_k = \begin{pmatrix} \rho_0 & \rho_1 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_0 & & \\ & & & \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_0 \end{pmatrix} \quad r_k = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{pmatrix} \qquad (2)$$

$\theta_k = (a_{k,1}, \dots, a_{k,k})^T$, and where

$(\rho_i)_{i=0}^{k}$ are given.

The matrices R_k are assumed to be non-singular.

Such a system of equations arises via the Yule-Walker equations when fitting AR-models of order $k = 1, \dots n$ given covariances $(\rho_i)_{i=0}^{k}$ of the data (perhaps estimated).

Note equation (1) can be written as:

$$\underbrace{\begin{pmatrix} 1 & a_{m,1} & \cdots & a_{m,m} \\ & \ddots & & \vdots \\ 0 & & 1 & a_{1,1} \\ & & & 1 \end{pmatrix}}_{U^{T}_{m+1}} \underbrace{\begin{pmatrix} \rho_0 & \cdots & & \rho_m \\ \rho_1 & \rho_0 & \cdots & \rho_{m-1} \\ \vdots & & & \\ \rho_m & \rho_{m-1} & \cdots & \rho_0 \end{pmatrix}}_{R_{m+1}} \underbrace{\begin{pmatrix} 1 & & & \\ a_{m,1} & 1 & & \\ & & \ddots & \\ a_{m,m} & \cdots & a_{1,1} & 1 \end{pmatrix}}_{U_{m+1}} = \underbrace{\begin{pmatrix} \sigma_m & & 0 \\ & \ddots & \\ 0 & & \rho_0 \end{pmatrix}}_{D_{m+1}} \qquad (3)$$

To show this note: for $m=1$.

$$\begin{pmatrix} 1 & a_{1,1} \\ & 1 \end{pmatrix}\begin{pmatrix} \rho_0 & \rho_1 \\ \rho_1 & \rho_0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_{1,1} & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1 & \\ & \rho_0 \end{pmatrix}$$

Now we show that if (3) holds for $m+1=k$. i.e.

$$U^{T}_{k} R_{k} U_{k} = D_{k}$$ then it holds

for $m+k = k+1$, ie..

$$U^{T}_{k+1} R_{k+1} U_{k+1} = \begin{pmatrix} 1 & \theta_k^{T} \\ 0 & U_k^{T} \end{pmatrix}\begin{pmatrix} \rho_0 & r_k^{T} \\ r_k & R_k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \theta_k & U_k \end{pmatrix}$$

$$= \begin{pmatrix} \rho_0 + \theta_k^{T} r_k & r_k^{T}\theta_k^{T} R_k \\ U_k^{T} r_k & U_k^{T} R_k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \theta_k & U_k \end{pmatrix}.$$

$$= \begin{pmatrix} \sigma_K & 0 \\ U_K^T r_K & U_K^T R_K \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \theta_K & U_K \end{pmatrix} \qquad \text{from } 2^{\text{nd}} \text{ eqn. in (1)}$$

$$= \begin{pmatrix} \sigma_K & 0 \\ U_K^T (r_K + R_K \theta_K) & U_K^T R_K U_K \end{pmatrix}.$$

$$= \begin{pmatrix} \sigma_K & \\ & D_K \end{pmatrix} = \begin{pmatrix} D_{K+1} \end{pmatrix}. \qquad \text{from } 2^{\text{nd}} \text{ eqn. in (1)}.$$

Now from (3)

$$R_{m+1}^{-1} = U_{m+1} D_{m+1}^{-1} U_{m+1}^T \qquad (4)$$

which shows that $U_{m+1} D_{m+1}^{-1/2}$ is the lower triangular Choleski factor of the Inverse Covariance matrix R_{m+1}^{-1}.

Now we wish to have a recursion for the estimates $(a_{k,i})_{i=1}^{k}$ and $(\sigma_k)_{k=1}^{n}$.

We exploit the Toeplitz structure of (R_k).

For a vector $x = (x_1 \cdots x_n)^T$ we consider the reverse vector $x^R = (x_n, \cdots x_1)^T$.

We can write (1) in the form:

$$\theta_{k+1}^R = - R_{k+1}^{-1} r_{k+1}^R \qquad (5)$$

Hence from (4)

$$\theta_{k+1}^R = - \begin{pmatrix} 1 & 0 \\ \theta_k & U_k \end{pmatrix} \begin{pmatrix} \sigma_k^{-1} & 0 \\ 0 & D_k^{-1} \end{pmatrix} \begin{pmatrix} 1 & \theta_k^T \\ 0 & U_k^T \end{pmatrix} \begin{pmatrix} \rho_{k+1} \\ r_k^R \end{pmatrix}.$$

$$= - \begin{pmatrix} \dfrac{1}{\sigma_k} & \theta_k^T/\sigma_k \\ \dfrac{\theta_k}{\sigma_k} & R_k^{-1} + \theta_k \theta_k^T/\sigma_k \end{pmatrix} \begin{pmatrix} \rho_{k+1} \\ r_k^R \end{pmatrix}.$$

$$= \begin{pmatrix} -(\rho_{k+1} + \theta_k^T r_k^R)/\sigma_k \\ -\theta_k \left(\rho_{k+1} + \dfrac{\theta_k r_k^R}{\sigma_k} + \theta_k^R\right) \end{pmatrix}. \qquad (6)$$

Now from (1)

$$\sigma_{k+1} = \rho_0 + \theta_{k+1}^T r_{k+1} = \rho_0 + \left(\theta_{k+1}^R\right)^T r_{k+1}^R$$

$$= \rho_0 + \left(a_{k+1,k+1}\left(\theta_k^R\right)^T + a_{k+1,k+1}\,\theta_k^T\right)\begin{pmatrix}\rho_{k+1}\\ r_k^R\end{pmatrix}.$$

$$= \rho_0 + \left(\theta_k^R\right)^T r_k^R + a_{k+1,k+1}\left(\rho_{k+1} + \theta_k^T r_k^R\right). \qquad (7)$$

$$= \sigma_k - a_{k+1,k+1}^2\,\sigma_k$$

Hence from (6) and (7), we get the recursion:

$k = 1 \dots n-1$

$$a_{k+1,k+1} = -\left(\rho_{k+1} + a_{k,1}\rho_k \cdots + a_{k,k}\rho_1\right)/\sigma_k$$

$$a_{k+1,i} = a_{k,i} + a_{k+1,k+1}\,a_{k,k+1-i} \qquad i = 1, \dots k.$$

$$\sigma_{k+1} = \sigma_k\left(1 - a_{k+1,k+1}^2\right).$$

with initial values:

$$a_{1,1} = \frac{-\rho_1}{\rho_0}$$

$$\sigma_1 = \rho_0\left(1 - a_{1,1}^2\right).$$

<u>Stopping criterion</u> if for some k, $a_{k+1,k+1} = 1$.

<u>Operation Count</u> : Requires $O(n^2)$ operations for determining $(\theta_k, \sigma_k)_{k=1}^{n}$, exploiting Toeplitz structure.

The Levinson-Durbin algorithm is useful for computing the UDU^T factorization of the inverse of some symmetric Toeplitz matrix

Solving Yule Walker equations for AR-fitting