Theory of Polarization Shift Keying Modulation

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Abstract—**A rigorous analysis of digital coherent optical modulation schemes using the state of polarization as the modulating parameter is presented, which permits to obtain the exact performance of all the polarization-based modulation schemes proposed in the literature so far, including a differential demodulation scheme, named DPOLSK, which does not require either electro-optic or electronic polarization tracking. Preliminary results involving multilevel transmission schemes based on the state of polarization are introduced. A spectral analysis of POLSK signals is also proposed.**

I. Introduction

As an alternative to the application of the standard coherent modulation techniques like ASK, FSK, PSK and DPSK to coherent optical communications, modulation methods exploiting the vector characteristics of the propagating light radiation have been recently proposed and/or experimentally demonstrated in laboratory [3]–[6].

They use the state of polarization (SOP) of a fully polarized lightwave as the information-bearing parameter, exploiting the two orthogonal channels available in free space as well as in a single-mode fiber propagation. In free space, two orthogonal input signals maintain their state of polarization while propagating. In the case of single-mode fibers fed by a monochromatic light source, orthogonal SOP pairs at the input lead to orthogonal output SOP pairs, although the input state of polarization is not maintained in general. Moreover, careful measurements reported in [13], [12], and [14] have shown that depolarization phenomena or polarization dependent losses are of little importance even after relatively long fiber spans. These properties are crucial to digital modulation based on SOP, called POLarization Shift Keying (POLSK).

Demodulation and detection is accomplished through the analysis of the SOP. A SOP is fully described by the knowledge of the *Stokes parameters* [1, ch. 10]. We will call *Stokes receiver* a coherent heterodyne receiver extracting the Stokes parameters from the IF signals (see Fig. 1).

All so far proposed POLSK systems make use of a binary modulation scheme (2-POLSK), i.e., information is sent by switching the polarization of the transmitted lightwave between two linear orthogonal SOP's. In the three-dimensional space defined by the Stokes parameters two orthogonal SOP's map onto opposite points with respect to the origin. Thus, detection of binary modulation schemes is simply accomplished by looking at the sign of the scalar product of the received SOP vector in the Stokes space with a reference vector representing one of the received SOP's in the absence of noise. This reference vector depends on the fiber induced changes on the transmitted SOP's and for this reason all 2-POLSK systems must somehow keep track of it.

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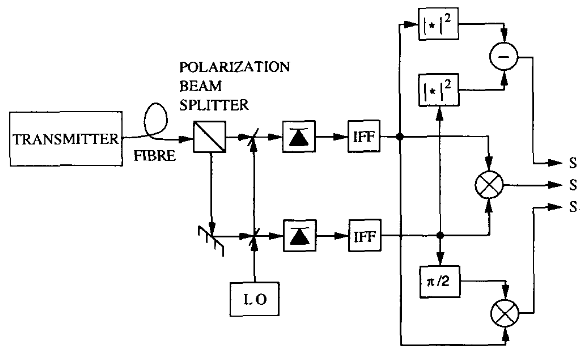

Fig. 1. Block diagram of the receiver front-end extracting the Stokes parameters from the received signal.

It is straightforward to see that the system proposed in [3] makes use only of the S_2 channel of Fig. 1, while the system outsketched in [4] exploits the S_1 path. A complete Stokes receiver is used by the system proposed in [5]. A fourth interesting scheme[1] (JMPSK) has been presented in [6], which uses a rather different signal processing. All these schemes can be viewed as POLSK systems. The above-mentioned reference recovery is accomplished electro-optically in [3],[4]. This implies that they can make use of only part of the Stokes receiver because it is supposed that active electro-optic controls force the received SOP's to align with a specific axis of the receiver Stokes space reference. The third and fourth systems, which will be shown to be equivalent, perform an electronic reference recovery by means of long-term averages over the received signals.

In terms of performance, approximate results based on signal-to-noise ratios [4] or to additive Gaussian hypothesis applied to the noise perturbing the Stokes parameters [5] have been presented. In [8] the exact performance of the system proposed in [3] has been inferred. In [32] the performances of

[1] For this system the authors suggest the use of an unbalanced power splitting between the polarization "channel" bearing only the reference carrier and the orthogonal one carrying the modulated signal. As a result the transmitted SOP's are no longer orthogonal and the signal tends to become phase modulated instead of polarization modulated. Therefore this version of the scheme gets out of the scope of the present work. Throughout this paper we shall refer strictly to the balanced-power 2-POLSK modulated version of the system.

the system [3] as a function of the depolarization effect taking place in the fiber are evaluated.

All the known results place POLSK modulation, in terms of required power, between ASK and DPSK, with a theoretical degradation of 3 dB with respect to DPSK in the absence of laser phase noise.

As to the effect of the laser phase noise, a general analytical treatment in coherent optical heterodyne receivers is not available. However, numerical analyses and experimental results have proved the considerable insensitivity to phase noise of receivers based on non-linear memoryless processing of the signal, provided that the IF filter bandwidth is large enough to avoid phase-to-amplitude noise conversion [9]–[11]. POLSK schemes can be thought of belonging to this class of systems, also called PNCHR (phase-noise-canceling heterodyne receivers) [32].

In this paper, we present a comprehensive description of the properties of a full Stokes receiver. The statistical characteristics of the polar coordinates of the received noisy SOP in the Stokes space will be obtained and we will prove that they are independent of the modifications of the SOP induced by the fiber. As a consequence, all binary POLSK modulation schemes will be shown to have the same shot-noise performance, which will be exactly calculated.

In addition, the shot-noise performance of a differential demodulation system, named DPOLSK [7], which does not need polarization tracking, will be exactly analyzed as well.

First approximate results are presented regarding the performance of multilevel POLSK modulation schemes, showing that the shot noise tolerance becomes closer to that of PSK and DPSK when the number of signal points is increased.

Finally, the power spectral density of POLSK signals is evaluated and some system considerations comparing different modulation schemes are presented.

We would like also to remark that POLSK is an abstract scheme of modulation and its applicability is not confined to fiber optic communications. Whenever the transmission medium is a transversally bi-dimensional non-depolarizing one, POLSK modulation can be used and the results presented here are fully valid. This is the case for instance of outer space, so that POLSK could be used in microwave or lightwave inter-satellite communications. However, in this paper, all the comments and system comparisons will be based on fiber optic communications.

II. Signals and Noise in Stokes Space

The state of polarization (SOP) of a fully polarized lightwave can be described through the Stokes parameters [1, pp. 554–556]. Given a reference plane $\hat{x}$, $\hat{y}$ normal to the $\hat{z}$ propagation axis of an electromagnetic field, the expression of which is

$$\begin{aligned} E_x &= a_x(t)e^{j(\omega t+\phi_x(t))} \\ E_y &= a_y(t)e^{j(\omega t+\phi_y(t))} \\ \vec{E} &= E_x\hat{x} + E_y\hat{y} \end{aligned} \tag{1}$$

the Stokes parameters can be calculated as follows:

$$\begin{aligned} S_1 &= a_x^2 - a_y^2 \\ S_2 &= 2a_x a_y \cos(\delta) \\ S_3 &= 2a_x a_y \sin(\delta) \end{aligned} \tag{2}$$

with

$$\delta = \phi_x - \phi_y.$$

In (2) we have omitted the dependence on time of the parameters for notational simplicity. This will be used throughout the paper in all unambiguous cases.

In classical optics an average is generally taken of the quantities appearing in the right hand side of (2). For our use we assume (2) to define "instantaneous" values for these parameters. A fourth parameter

$$S_0 = a_x^2 + a_y^2 \tag{3}$$

represents the total electromagnetic power density traveling in the $\hat{z}$ direction. The following also holds

$$S_0^2 = S_1^2 + S_2^2 + S_3^2. \tag{4}$$

The S_i can be represented in a three-dimensional space with unit vectors $\hat{s}_1$, $\hat{s}_2$, $\hat{s}_3$. We will call this space "Stokes space." For waves having the same power density, $\vec{S}$ lies on a sphere of radius S_0, called "Poincaré sphere."

A fundamental feature of this representation is that SOP's orthogonal according to the hermitian scalar product:

$$\left\langle \vec{E}' \cdot \vec{E}''^* \right\rangle = (\hat{v}' \cdot \hat{v}''^*)a'a''e^{j(\omega t+\phi')}e^{-j(\omega t+\phi'')} \tag{5}$$

map onto points which are antipodal on the Poincaré sphere. Moreover all linear polarizations lie on the (S_1, S_2) plane, while the points $\pm S_0\hat{s}_3$ represent the two possible circular SOP's. The other points represent elliptic SOP's.

A complete SOP analysis of the received field can be performed using the Stokes receiver configuration of Fig. 1, which implements the set of relationships (2).

We will prove in the following that the performance of every receiver operating on the Stokes parameters obtained from a received signal perturbed by additive white Gaussian noise (AWGN) are invariant with respect to the choice of the reference system $\hat{s}_1$, $\hat{s}_2$, $\hat{s}_3$ in the receiver, or, equivalently, with respect to the modifications induced by the fiber onto the SOP.

We assume that the power of the local oscillator is equally split between the x and y channels. Shot noise processes appearing on the x and y IF channels after photodetection can be written as [21]:

$$\begin{aligned} x &= x_c\cos(\omega_{IF}t) - x_s\sin(\omega_{IF}t) \\ y &= y_c\cos(\omega_{IF}t) - y_s\sin(\omega_{IF}t) \end{aligned} \tag{6}$$

where $x_{c,s}$, $y_{c,s}$ are independent white Gaussian random processes with power spectral density N_0.

In optical systems the received field is generally affected by phase noise, and so is the local oscillator. Up to now, a general analytical treatment of phase noise in receivers that use band-pass filtering and then non-linear processing of the signal, as in

Stokes receiver branches, is not available. However, in such receivers, under reasonable hypotheses, phase noise can be considered to be suppressed to a great extent. These hypotheses are:

a) ideal square-law elements, or multipliers;

b) IF filter bandwidth "large enough" with respect to the IF beat linewidth to avoid phase noise to amplitude noise conversion.

The "large enough" criterion means that in general it is necessary to allow for a wider IF bandwidth than the one that would be chosen on the basis of the information signal spectrum only. However, wider IF filter bandwidths mean larger noise power, thus a certain penalty has to be expected. It can be significantly reduced by the use of a postdetection filter [9], [10].

Under a) and b), phase noise cancels out within the non-linear elements, where only products of synchronous signal components are performed. In the following treatment we will assume that a) and b) hold.

However, even under this assumption, in the mathematical expressions of the baseband signals of a generic receiver employing square-law devices or multipliers, phase-noise does not cancel in signal-times-noise terms. This leads to rather involved formulas, although, once the statistical properties of these terms are analyzed, it turns out that they are not altered by the presence of phase noise factors. To get rid of these complex noise expression, and to prove that under a) and b) phase noise no longer affects these systems, we first get back to the original shot noise processes (6). We state that expressing shot noise through

$$\tilde{x}' e^{j(\omega_0 t + \phi_n(t))} = \tilde{x} e^{j\omega_0 t} \tag{7}$$

(where $\phi_n(t)$ represents the phase noise, a random process independent of $\tilde{x}$), the process $\tilde{x}'$ has identical statistical characteristics as $\tilde{x}$. The same holds for $\tilde{y}'$. To prove this, we express x_c', x_s' as a function of x_c, x_s. We get

$$\begin{bmatrix} x_c' \\ x_s' \end{bmatrix} = \boldsymbol{U} \begin{bmatrix} x_c \\ x_s \end{bmatrix} \tag{8}$$

where $\boldsymbol{U}$ is

$$\begin{bmatrix} \cos\phi_n(t) & \sin\phi_n(t) \\ -\sin\phi_n(t) & \cos\phi_n(t) \end{bmatrix}$$

a real unitary stochastic matrix. Now, we resort to the following theorems, whose proof is reported in Appendix A.

Theorem 1a (T1a): given two independent real zero-mean stationary random processes, jointly Gaussian and identically distributed, say a and b, and the transformation:

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \boldsymbol{U} \begin{bmatrix} a \\ b \end{bmatrix} \tag{9}$$

where $\boldsymbol{U}$ is a real unitary matrix, the processes a' and b' are still independent zero-mean Gaussian random processes with the same distributions as a and b.

Theorem 2 (T2): assuming all the hypothesis of (T1a), except for $\boldsymbol{U}$ that is now a stochastic unitary matrix, in the sense that its coefficients are stationary random process independent of a, b, such that

$$\boldsymbol{U}^\dagger(t)\boldsymbol{U}(t) = \mathbf{1}, \tag{10}$$

(T1a) still holds if and only if a, b are white random processes. It is straightforward to verify that (T1a) and (T2) prove (7).

Once this equivalent representation has been adopted, phase noise factors cancel out of noise-times-signal terms in much the same way as they cancel out of signal-times-signal terms. Hence, under a) and b) the systems has a complete immunity with respect to phase noise, and this tolerance is mirrored in baseband formulas where all phase noise related terms disappear.

Let us now assume that the incoming field polarization is linear and aligned with one of the analysis axis, say $\hat{x}$, of the Stokes receiver. Baseband expressions of the Stokes parameters are in this case:

$$\begin{aligned} S_1 &= A^2 + 2Ax_c + x_c^2 + x_s^2 - y_c^2 - y_s^2 \\ S_2 &= 2(x_c y_c + x_s y_s) + 2Ay_c \\ S_3 &= 2(x_s y_c - x_c y_s) - 2Ay_s. \end{aligned} \tag{11}$$

This assumption implies that the signal, intended as a term where noise factors are absent (i.e., A^2), is carried by S_1, while S_2 and S_3 bring noise only. The set (11) is a very neat and readable form of the output of the Stokes receiver. When the signal SOP, however, instead of being linear and aligned with $\hat{x}$, is of a general form, this simple form becomes more complicated and results in expressions where all the S_i's display both noise and signal terms. Nevertheless, we will show that the analysis of noise statistics and, as a consequence, of POLSK systems performance can be carried out on just the simple (11) form, without any loss of generality.

To prove this we first introduce the Cayley–Klein [18] representation of a rotation of the reference system in a three-dimensional space:

$$\boldsymbol{P} = \begin{bmatrix} S_1 & S_2 - jS_3 \\ S_2 + jS_3 & -S_1 \end{bmatrix} \tag{12}$$

$$\boldsymbol{P}' = \begin{bmatrix} S_1' & S_2' - jS_3' \\ S_2' + jS_3' & -S_1' \end{bmatrix} = \boldsymbol{QPQ}^\dagger \tag{13}$$

where $\boldsymbol{P}$ is the matrix representation of a point of coordinates (S_1, S_2, S_3) with respect to the reference set $\hat{s}_1$, $\hat{s}_2$, $\hat{s}_3$, and $\boldsymbol{Q}$ is a rotation matrix such that $\boldsymbol{P}'$ is the matrix representation of $\boldsymbol{P}$ according to a new reference set $\hat{s}_1'$, $\hat{s}_2'$, $\hat{s}_3'$. For a theoretical treatment of this formalism see [18].

In this context we need only to know that $\boldsymbol{Q}$ is capable of performing all possible rotations of the reference axes, and that it is a unitary matrix with the additional constraint of a unit determinant.

A way to represent a generic transformation of the SOP of a fully polarized lighwave, which preserves the degree of polarization, is the following [1], [2]. Given $\vec{E}$ and $\vec{E}'$, the electromagnetic field vectors before and after the transformation, and given their decomposition:

$$\begin{aligned} E_x &= a_x(t)e^{j(\omega_0 t + \phi_x(t))} \qquad & E_x' &= a_x'(t)e^{j(\omega_0 t + \phi_x'(t))} \\ E_y &= a_y(t)e^{j(\omega_0 t + \phi_y(t))} \qquad & E_y' &= a_y'(t)e^{j(\omega_0 t + \phi_y'(t))} \end{aligned}$$

$$\vec{E} = E_x\hat{x} + E_y\hat{y} \qquad \vec{E}' = E'_x\hat{x}' + E'_y\hat{y}' \tag{14}$$

where $\hat{x}$, $\hat{y}$ and $\hat{x}'$, $\hat{y}'$ are transverse reference axis sets (i.e., normal to the direction of propagation), we have

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \boldsymbol{Q}\begin{bmatrix} E_x \\ E_y \end{bmatrix} \tag{15}$$

where $\boldsymbol{Q}$ is a complex matrix with unit determinant called Jones matrix. A subset of Jones matrices, called the set of matrices of birefringence or optical activity, not only preserves the degree of polarization, but also has the additional feature of preserving orthogonality (according to the Hermitian scalar product) between two fields which are orthogonal before the transfomation. Matrices of this kind are complex unitary matrices *with unit determinant.* Throughout this paper, when talking of Jones matrices, we strictly refer to this latter subset.

Calling $\vec{S}$ the SOP vector of the $\vec{E}$ field and $\vec{S}'$ the SOP vector of the $\vec{E}'$ field, with $\vec{S}$ and $\vec{S}'$ expressed according to Stokes references consistent with the $\hat{x}$, $\hat{y}$ and $\hat{x}'$, $\hat{y}'$ geometrical reference axes, we associate to $\vec{S}$ and $\vec{S}'$ their matrix representation $\boldsymbol{P}$ and $\boldsymbol{P}'$, respectively, according to (12).

Under this assumption, we state the following result.

Theorem 3 (T3): given the field vector $\vec{E}$ with associated $\boldsymbol{P}$ SOP, and the field vector $\vec{E}'$ with associated $\boldsymbol{P}'$ SOP, as defined above, we have

$$\boldsymbol{P}' = \boldsymbol{Q}\boldsymbol{P}\boldsymbol{Q}^{\dagger}$$

where $\boldsymbol{Q}$ is the Jones matrix describing the SOP transformation in (15).

The proof of this result, in the framework and with the notations of this paper, is given in Appendix A. In a more general and abstract context it can be found in [34]. (T3) implies that a generic SOP transformation which preserves the degree of polarization is equivalent to a rotation of the reference axes in the Stokes space. If this rotation is expressed through the Cayley–Klein formalism, the corresponding rotation matrix $\boldsymbol{Q}$ coincides with the Jones matrix.

Let us now get back to the original problem of the generality of the (11) form. We assume that $\hat{x}'$ and $\hat{y}'$ coincide with the analysis axes of a Stokes receiver. Consequently, the IF-stage signal vector in the Stokes receiver is, according to the notation (2) and dropping a constant factor:

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \begin{bmatrix} x_{ch} \\ y_{ch} \end{bmatrix} \tag{16}$$

where x, y are the analytic signal representation of the noise processes and x_{ch}, y_{ch} are the noisy signals on the IF-branches of the Stokes receiver.

Now, we assume that:

a) the fiber does not affect the degree of polarization of the field;

b) the fiber does not induce loss of orthogonality between two orthogonal (according to the Hermitian scalar product) input fields.

Assumptions (a) and (b) stem from the results of experimental and theoretical works, such as [12]–[14], [33], which have demonstrated that depolarization effects or polarization-dependent losses are to a great extent negligible in optical fibers.

Under the above-mentioned hypotheses, it is always possible to write (16) as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} + \boldsymbol{Q}\begin{bmatrix} E'' \\ 0 \end{bmatrix} = \begin{bmatrix} x_{ch} \\ y_{ch} \end{bmatrix} \tag{17}$$

where $\boldsymbol{Q}$ is a Jones birefringence matrix that changes the field SOP in such a way to transform a field linearly polarized along $\hat{x}''$ in the arbitrary reference $\hat{x}''$, $\hat{y}''$ into the received field E'_x, E'_y.

From (17) we can write:

$$\boldsymbol{Q}\left(\begin{bmatrix} x'' \\ y'' \end{bmatrix} + \begin{bmatrix} E'' \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x_{ch} \\ y_{ch} \end{bmatrix} \tag{18}$$

with:

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \boldsymbol{Q}^{-1}\begin{bmatrix} x \\ y \end{bmatrix}. \tag{19}$$

Now, we introduce another result, proved in Appendix A, on linear transformations of Gaussian noise processes:

Theorem 1b (T1b): result (T1a) still holds if a, b are complex Gaussian random processes identically distributed in the sense of [29, p. 40], i.e., complying with the additional constraint:

$$E\{a(t+\tau)a(t)\} = E\{b(t+\tau)b(t)\} = 0 \quad \forall\tau \tag{20}$$

and $\boldsymbol{U}$ is a real or complex unitary matrix.

We remark that the complex envelope of all stationary narrow-band jointly Gaussian noise processes, i.e., processes obtained by linear time-invariant filtering of white Gaussian stationary noise, satisfy condition (20).

Due to (T1b) we know that x'', y'' have the same statistical properties of x, y.

Now, it is evident that the very simple form (11), of which we are trying to prove the intrinsic generality, is originated by the IF-signal form:

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} E' \\ 0 \end{bmatrix} = \begin{bmatrix} x_{ch} \\ y_{ch} \end{bmatrix}. \tag{21}$$

The simplicity of (11) is due to the fact that in (21) the x noise term is "aligned" with all the signal, while on the y channel we have noise only. We notice that this structure can always be formally recovered within the left-hand side brackets of (18). However, it is obvious that the Stokes parameters S_i's obtained from (18) have not the simple form of (11) because of the presence in (18) of the multiplying $\boldsymbol{Q}$ unitary matrix. But we know from (T3) that $\boldsymbol{Q}$ simply implies a rotation of the reference axes in the Stokes space. Thus, at the output of the signal processing stage that computes the Stokes parameters, the only difference between the ouput form (11), generated by (21), and the output generated by the IF-signal vector (18) is a change of reference in the Stokes space. This type of transformation doesn't add or subtract noise, nor changes the relationships between signal and noise. Moreover, for a POLSK system to work, a scalar product is needed between the received SOP and a set of vectors representing the possible received signals. The decision algorithm simply chooses the signal maximizing the scalar product. As scalar products give the same results independently of the underlying spatial reference, it turns out that the system performances do not change, whatever the $\boldsymbol{Q}$ matrix in (18) is.

Therefore, performance analyses can be carried out on the basis of Stokes parameters in the form (11), which is equivalent to choosing $\boldsymbol{Q} = 1$ in (18), without any loss of generality.

As far as proposed POLSK schemes are concerned, we should point out that in some of them the scalar product with the two possible received signals is explicit, in others it is implicit.

The system proposed in [5] explicitly performs a scalar product between the received SOP and a reference signal recovered through long-term averages. In the system [3] the scalar product is implicit, in the sense that via electro-optic alignment the $\boldsymbol{Q}$ matrix of (18) is transformed into the matrix:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{22}$$

which ensures that the signal be carried by S_2 only. In theory we should make a scalar product with the vectors $(0, \pm 1, 0)$, which is also equivalent to selecting the S_2 channel. But in this system the other two channels do not even exist, as they are useless, and so there is no need for an explicit scalar product. Finally, the JMPSK system proposed in [6] acts performing the product of (18) with a $\boldsymbol{Q}''$ matrix, calculated through long-term averages and computational algorithms, such that again the output be carried by S_2 only. In principle, this system is equivalent to the previous one. The only difference is that alignment of the received SOP's is electronic instead of electro-optic.

From the present analysis, we can conclude that all so far proposed POLSK schemes have exactly the same performances under shot noise disturbance.

III. Performance of POLSK Modulation Schemes

Let us now compute the performance starting from the expression (11) of the Stokes parameters.

We will try first to characterize the received noisy vector $\vec{S}_N$ with respect to the unnoisy one $\vec{S}_I$. In practice, we are interested in finding the conditional pdf of the receiver output given the ideal unnoisy one

$$f_{\vec{S}_N|\vec{S}_I}\left(\vec{S}_N|\vec{S}_I\right). \tag{23}$$

To do this, we shall take the unnoisy signal unit vector, which according to (11) is $\hat{s}_1$, as the polar axis of a polar reference with respect to which $\vec{S}_N$ will have components (see Fig. 3):

$$\vec{S}_N \equiv (\rho, \theta, \alpha).$$

The angle θ can be expressed as follows

$$\theta = \cos^{-1}\left(\frac{\vec{S}_N \cdot \vec{S}_I}{|\vec{S}_N||\vec{S}_I|}\right). \tag{24}$$

Calling $\vec{S}_{N_\perp}$ the projection of $\vec{S}_N$ over the plane orthogonal to $\vec{S}_I$, and chosen a unit vector over this plane $\hat{s}_\alpha$ we have:

$$\alpha = \cos^{-1}\left(\frac{\vec{S}_N \cdot \vec{S}_I}{\left|\vec{S}_{N_\perp}\right|}\right). \tag{25}$$

We will assume $\hat{s}_\alpha = \hat{s}_2$ in the reference used in (11).

Our aim is to fully describe the statistical properties of the random variables ρ, θ, α.

The magnitude ρ of the demodulated signal vector $\vec{S}_N$ is

$$\rho = (A + x_c)^2 + x_s^2 + y_c^2 + y_s^2. \tag{26}$$

The distribution of ρ is a noncentral chi-square with four degrees of freedom and noncentrality parameter A^2 [17], [15]. Its pdf is

$$f_\rho(y) = \frac{1}{2\sigma^2}\left(\frac{y}{A^2}\right)^{\frac{1}{2}} e^{-\frac{A^2+y}{2\sigma^2}} I_1\left(\sqrt{y}\,\frac{A}{\sigma^2}\right) \quad y > 0 \tag{27}$$

and the cdf

$$F_\rho(y) = 1 - Q_2\left(\frac{A}{\sigma}, \frac{\sqrt{y}}{\sigma}\right) \quad y > 0 \tag{28}$$

where Q_2 is a generalized Marcum function, of index 2.

The angle α is uniformly distributed over the interval $[0, 2\pi]$. To prove this result we first define

$$a = A + x_c \quad c = y_c$$
$$b = x_s \quad d = y_s.$$

Substituting a, b, c, d in S_2, S_3 of (11) we get

$$S_2 = 2(ac + bd)$$
$$S_3 = 2(bc - ad).$$

Then, let us introduce the following result, proved in Appendix A:

Theorem 4 (T4): given four real independent Gaussian random processes a, b, c, d, with equal variance σ^2 and of which at least c, d are zero-mean, a vector in a bi-dimensional space with components

$$(ac + bd, cb - ad) \tag{29}$$

forms an angle with respect to a fixed reference axis on its plane which is uniformly distributed over the interval $[0, 2\pi]$. Alternatively, this statement can be also formulated substituting a, b to c, d in the sentence.

We know that $\vec{S}_{N_\perp} = S_2\hat{s}_2 + S_3\hat{s}_3$. Applying ($T4$) to $\vec{S}_{N_\perp}$ the above-stated result for α is proved.

Calculations concerning θ are much more involved. Through the steps described in detail in Appendix B its pdf turns out to be

$$f_\theta(\theta) = \frac{\sin\theta}{2}\, e^{-\frac{A^2}{4\sigma^2}(1-\cos\theta)}\left[1 + \frac{A^2}{4\sigma^2}(1 + \cos\theta)\right] \quad \theta \in [0, \pi]. \tag{30}$$

Integrating (30) we find the even simpler cdf of θ

$$F_\theta(\theta) = 1 - \frac{1}{2}\, e^{-\frac{A^2}{4\sigma^2}(1-\cos\theta)}(1 + \cos\theta) \quad \theta \in [0, \pi]. \tag{31}$$

Finally, we state the following result. The angle α is statistically independent of both θ and ρ. To show this, we resort to the auxiliary vector representation

$$\vec{v}_1 = a\hat{u}_1 + b\hat{u}_2 \tag{32}$$

with a, b random Gaussian processes. The angle α_1 formed by $\vec{v}_1$ with respect to a reference vector, say $\hat{u}_1$, and the magnitude of $\vec{v}_1, \rho_{v_1}$, are statistically independent random variables (see, for example, [16, p. 200]).

Assuming

$$\vec{v}_1 = (A + x_c)\hat{u}_1 + x_s\hat{u}_2$$
$$\vec{v}_2 = y_c\hat{u}_1 + y_s\hat{u}_2$$

we conclude that $\alpha_1, \alpha_2, \rho_1, \rho_2$ are random processes independent of one another. Now, it is easy to see from (24) and (26) that

$$\theta = \arccos\left(\frac{\rho_{v_1} - \rho_{v_2}}{\rho_{v_1} + \rho_{v_2}}\right)$$

and (see proof of ($T4$) in Appendix A):

$$\alpha = \alpha_1 - \alpha_2.$$

Therefore, since θ depends only on ρ quantities, which in turn are independent of α_1 and α_2, we conclude that θ is independent of α. For the same reason α is independent of $\rho = (\rho_{v_1} + \rho_{v_2})$.

Finally, θ in general is not independent of ρ, unless the signal-to-noise ratio approaches zero ($A \to 0$). Moreover, in that case it can be shown through direct computation of the characteristic function that the pdf of the single Stokes parameter S_i is a Laplace one, with parameter $(1/2\sigma^2)$.

The statistical characterization of ρ, θ, and α obtained in this section makes it possible to study the performance of n-POLSK and d-DPOLSK systems.

In the following sections we will exploit these results to derive the error probability for some specific system configurations.

A. 2-POLSK Systems

The signal set consists of two antipodal points on the Poincaré sphere. We will call the unnoisy received signals $\vec{S}_{I_m}, \vec{S}_{I_s}$ with $\vec{S}_{I_s} = -\vec{S}_{I_m}$. Given a transmitted SOP such that the unnoisy received SOP is S_I chosen within the above-indicated set, and the received vector $\vec{S}_N$, an error occur each time the scalar product $\vec{S}_N \cdot \vec{S}_I$ becomes negative. This is due to the fact that the maximum likelihood criterion implies in this binary case a decision based on the sign of the scalar product. Hence, the error event turns out to be

$$\{E\} = \left\{\theta > \frac{\pi}{2}\right\}$$

with θ defined as in (24). Notice that with this signal set the error event is independent of both ρ and α. Therefore, using (31), we simply obtain

$$P(E) = P\left(\theta > \frac{\pi}{2}\right) = \frac{1}{2}\exp\left(-\frac{A^2}{4\sigma^2}\right). \quad (33)$$

This result is valid for all 2-POLSK systems and in particular it holds for all so far proposed POLSK systems [3]–[6]. The curve representing (33) is drawn in Fig. 4.

Such a result was previously indicated in [8] for the special case of the system originally proposed in [3]. In that scheme, due to electro-optical alignment, the IF signal structure turns out to be (18) with the fixed $\boldsymbol{Q}$ matrix (22). If noise is represented according to (19) we get, on the only implemented channel S_2, a noise and signal structure identical to S_1 in (11). However, if we leave noise in its "natural" representation, as in (17), in this system we get a baseband signal with a DPSK-like noise and signal structure, although with a three dB degraded S/N ratio with respect to DPSK. This correspondence between baseband output of the system [3] and the DPSK one has been exploited in [8] to infer the correct result (33).

Other performance analyses have been carried out assuming a Gaussian distribution of the overall noise terms on the S_i's [5], [6]. These analyses lead to an underestimate of system performance of approximately 3 dB. Completely unjustified even under the above-mentioned Gaussian assumption, seems to be the dependence of the performance of the system originally proposed in [5] on the unnoisy received SOP [6].

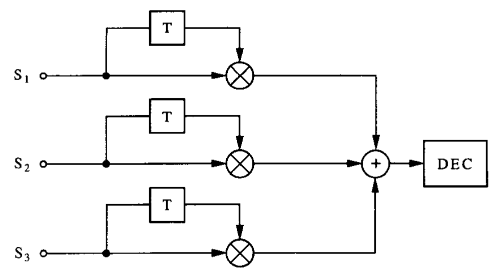

Fig. 2. Differential demodulator structure for binary DPOLSK.

B. 2-DPOLSK System

In this subsection we analyze a POLSK scheme based on differential demodulation that does not require polarization control. It was proposed in [7], together with an approximate performance analysis based on the elimination from the decision variable of all the non-Gaussian noise contributions (i.e., all the terms where products of Gaussian noise variables appear). However, the reported results seem not to be consistent with the declared approach, which actually leads to a three dB worse performance (approx. 8.5 dB below DPSK) with respect to that claimed in the cited paper.

On the contrary, the present analysis is based on the exact statistical distribution of the decision variable.

The system block diagram is shown in Fig. 2. The transmitted signal set is identical to that of 2-POLSK systems. Here, at the receiver stage, instead of performing electro-optic polarization recovery or, after the Stokes parameter extracting part, scalar products with a reference signal set recovered through electronic processing and long-term averages over the S_i, the SOP vector received in the previous symbol-interval is used as a reference vector, in a fashion similar to that of DPSK demodulation. Although this strategy is likely to cause a penalty with respect to 2-POLSK, it results in simpler system implementation and in a fast receiver start-up, the only delay being caused by clock recovery.

The decision signal is thus

$$d = \vec{S}_N \cdot \vec{S}'_N$$

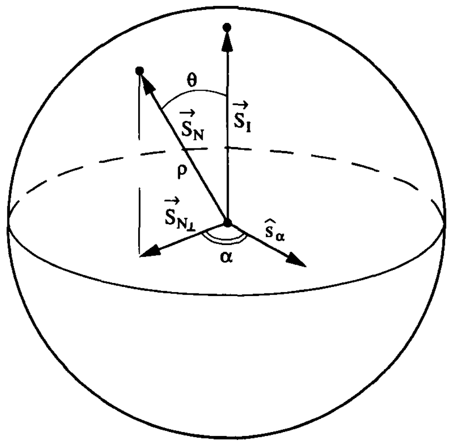

Fig. 3. Representation of the noisy received signal in polar coordinates referred to the unnoisy signal.

where the suffix means "delayed". If d is positive, the receiver assumes transmission of the same signal as in the previous symbol-interval. Otherwise a variation is assumed. Of course, differential data encoding is necessary.

The error conditions are now

$$\left\{d < 0 | \vec{S}_{Im,s}, \vec{S}'_{m,s}\right\} \tag{34}$$

$$\left\{d > 0 | \vec{S}_{Im,s}, \vec{S}'_{s,m}\right\}. \tag{35}$$

As before, we choose as reference the ideal unnoisy signals, and express the actual ones with respect to them, in polar coordinates (see Fig. 3).

As a result, event (34) becomes

$$d = \vec{S}_N \cdot \vec{S}'_N = \rho\rho'(\sin\theta\sin\theta'\cos(\alpha-\alpha') + \cos\theta\cos\theta') < 0 \tag{36}$$

where ρ, α, θ and ρ', α', θ' are the polar coordinates of $\vec{S}_N$ and $\vec{S'_N}$.

Through straightforward calculations, it can be shown that also event (35) can be rewritten in the (36) form. Thus, the two events (34) and (35) have the same probability.

Hence, the error probability becomes

$$P(E) = P(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\beta < 0)$$

with

$$\beta = \alpha - \alpha'.$$

The angle β is uniformly distributed over $[0, 2\pi]$, since it is the difference between two independent random variables uniformly distributed in $[0, 2\pi]$[2]. Manipulating the error condition we get:

$$P(E) = P(\cot\theta\cot\theta' < -\cos\beta) = P(\cot\theta\cot\theta' < \cos\delta) \tag{37}$$

[2]Since β always appears as the argument of a cosine function, we are, in fact, interested in the distribution of the variable $\beta = \mathrm{mod}(\alpha - \alpha', 2\pi)$. This assumption justifies the result stated above. Henceforth, it will be considered implicit in all similar circumstances.

with δ uniformly distributed over $[0, 2\pi]$. Substituting

$$\eta_1 = \cot\theta$$
$$\eta_2 = \cot\theta'$$

we finally have:

$$P(E) = P(\eta_1\eta_2 < \cos\delta). \tag{38}$$

To carry out the calculations involved in (38) we first need to know the pdf of η. By means of the strictly monotonic transformation:

$$\eta = \cot\theta \quad \theta \in [0, \pi] \quad \eta \in [\infty, -\infty]$$
$$\frac{\partial\theta}{\partial\eta} = -\frac{1}{1+\eta^2} < 0$$

we obtain

$$f_\eta(\eta) = \frac{1}{2}\left(\frac{1}{1+\eta^2}\right)^{\frac{3}{2}} e^{\frac{A^2}{4\sigma^2}\left(1-\frac{\eta}{\sqrt{1+\eta^2}}\right)} \cdot \left[1 + \frac{A^2}{4\sigma^2}\left(1 + \frac{\eta}{\sqrt{1+\eta^2}}\right)\right] \quad \eta \in [-\infty, \infty]. \tag{39}$$

The conditional error probability

$$P(\eta_1, \eta_2 < y | y = \cos\delta)$$

must be calculated splitting the integration depending on η_2 being positive or negative [22]

$$P(E|y) = \begin{cases} P(\eta_1 < y/\eta_2) & \text{if } \eta_2 > 0 \\ P(\eta_1 > y/\eta_2) & \text{if } \eta_2 < 0. \end{cases}$$

Therefore

$$P(E|y) = \int_0^\infty f_\eta(\eta_2)\,d\eta_2 \int_{-\infty}^{y/\eta_2} f_\eta(\eta_1)\,d\eta_1 + \int_{-\infty}^0 f_\eta(\eta_2)\,d\eta_2 \int_{y/\eta_2}^{\infty} f_\eta(\eta_1)\,d\eta_1. \tag{40}$$

Finally, we must average on δ

$$P(E) = \int_0^{2\pi} P(E|y = \cos\delta)\,d\delta. \tag{41}$$

Only the first of all the integration necessary to obtain $P(E)$ can be done analytically. Therefore, we have to integrate numerically the following expression:

$$P(E) = \int_0^{2\pi} \frac{1}{2\pi}d\delta \int_0^\infty f_\eta(\eta_2)\left\{1 - F_\theta\left(\mathrm{acot}\left[\frac{\cos\delta}{\eta_2}\right]\right)\right\}d\eta_2 + \int_0^{2\pi}\frac{1}{2\pi}d\delta \int_{-\infty}^0 f_\eta(\eta_2)F_\theta\left(\mathrm{acot}\left[\frac{\cos\delta}{\eta_2}\right]\right)d\eta_2. \tag{42}$$

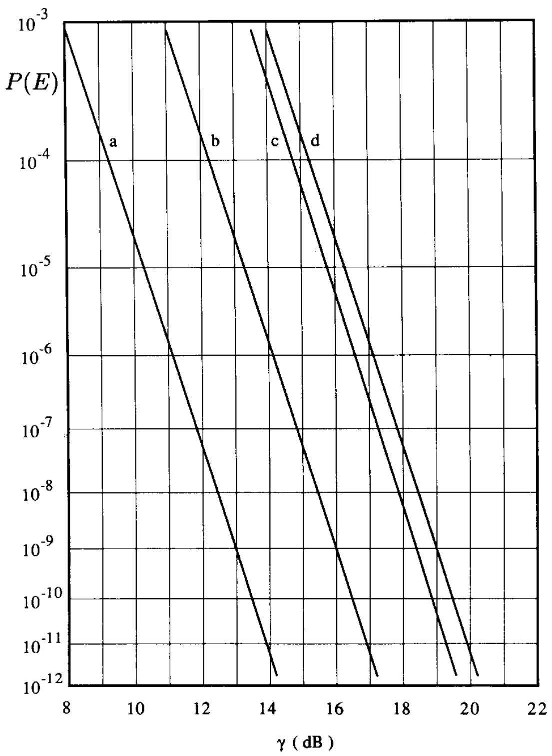

Fig. 4. Error probability as a function of the signal-to-noise ratio $\gamma = \frac{A^2}{2\sigma^2}$ for binary DPSK (curve a), POLSK (curve b), DPOLSK (curve c) and ASK (curve d).

This double integration has been performed using the Romberg algorithm.[3] Iteration was stopped when the relative difference between two successive steps was less than 10^{-4}. The resulting $P(E)$ curve is reported in Fig. 4. The penalty with respect to 2-POLSK systems is approximately 2.4 dB.

C. n-POLSK Systems

In this subsection we will briefly present some preliminary results about multilevel POLSK systems. A more complete analysis of system performances as well as of reference signal set recovery strategies will be given in a specifically dedicated forthcoming paper.

The three dimensional Stokes space is likely to be more efficient for multilevel transmission than the two-dimensional space used for combined amplitude and the phase modulation schemes. This remark has been confirmed by the rough shot-noise performance analyses carried out for three multilevel POLSK schemes. All these systems assume a constant transmitted power. Two of them are 4-POLSK systems. They differ in signal geometry: in one case we have four signals lying on a maximum circle over the Poincaré sphere, while in the other the signals are the vertices of a tetrahedron inscribed into the Poincaré sphere [see Fig. 5(a) and (b)].

[3]Some algebraic manipulations were also necessary to avoid infinite integration intervals.

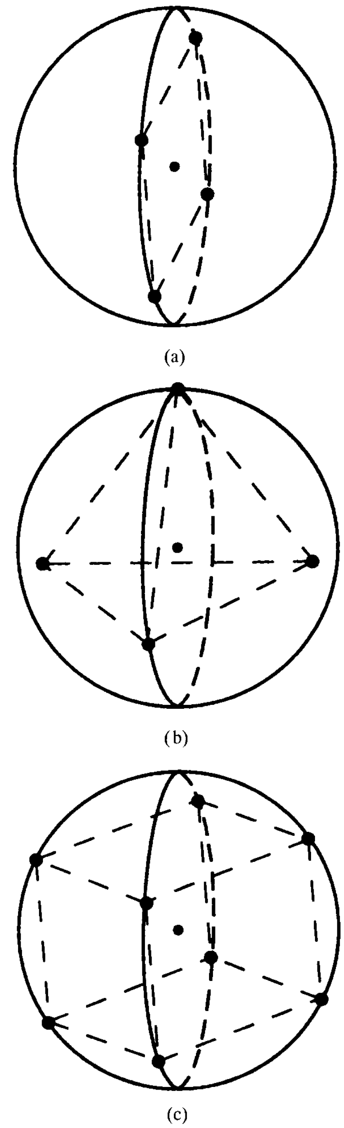

Fig. 5. Signal points constellations for n-POLSK modulation schemes: a) 4-POLSK on a maximum circle of the Poincaré sphere, b) 4-POLSK at the vertices of a tetrahedron inscribed into the Poincaré sphere, and c) 8-POLSK at the vertices of a cube inscribed into the Poincaré sphere.

The third system is an 8-POLSK, where the signal set is made up by the vertices of a cube inscribed into the Poincaré sphere [see Fig. 5(c)].

The formulas we present are rough upper bounds on the actual $P(E)$, in this sense: in all schemes the error probability is a function of both θ and α; we have chosen as representative of the system $P(E)$ the following upper bound:

$$P_{ub}(E) = \max_{\alpha} \{P(E|\alpha)\}. \tag{43}$$

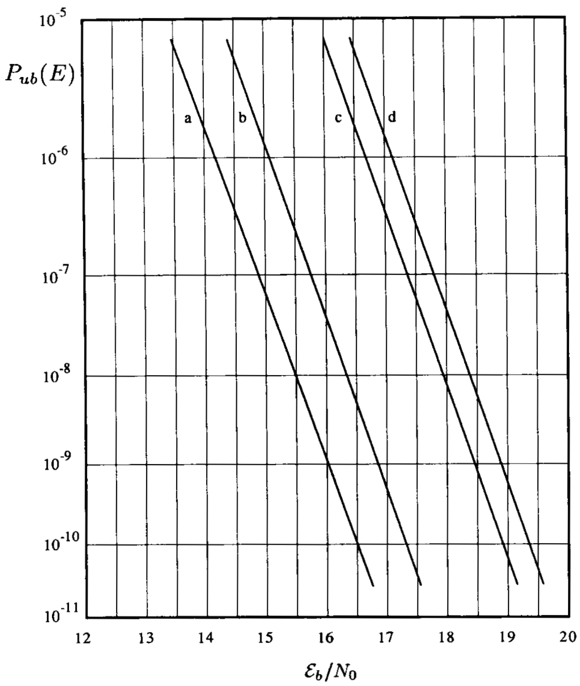

Fig. 6. Error probability as a function of the signal-to-noise ratio $\mathcal{E}_b/N_0$ for binary POLSK (curve a, 4-POLSK *tetrahedron* (curve b), 4-POLSK *circle* (curve c) and 8-POLSK *cube* (curve d).

1) 4-POLSK Circle: We assume as error condition, according to (43):

$$\{E\}_{ub} = \left\{\theta > \frac{\pi}{4}\right\}.$$

Therefore,

$$P_{ub}(E) = 1 - F_\theta(\pi/4) = \frac{1}{2}\left(1 + \frac{\sqrt{2}}{2}\right)e^{-\frac{A^2}{4\sigma^2}\left(1-\frac{\sqrt{2}}{2}\right)}. \tag{44}$$

The error probability curve is reported in Fig. 6 versus $\mathcal{E}_b/N_0$, the ratio between the energy per bit and noise spectral density. The resulting penalty is 2.33 dB with respect to 2-POLSK.

4-POLSK tetrahedron: The error condition is now

$$\{E\}_{ub} = \{\theta > 0.9553\}.$$

This condition can be derived dividing by two the angle subtended between the center of the sphere and two adjacent signal points. We get

$$P_{ub}(E) = 1 - F_\theta(0.9553) = (0.7882)e^{-\frac{A^2}{4\sigma^2}(0.4226)}. \tag{45}$$

The resulting penalty, in $\mathcal{E}_b/N_0$, is 0.7407 dB with respect to 2-POLSK. The $P_{ub}(E)$ curve is shown in Fig. 6.

8-POLSK Cube: The error condition, derived as above, is

$$\{E\}_{ub} = \{\theta > 0.6155\}.$$

and the resulting error probability bound is

$$P_{ub}(E) = 1 - F_\theta(0.6155) = (0.9082)e^{-\frac{A^2}{4\sigma^2}(0.1835)}. \tag{46}$$

Again the curve is reported in Fig. 6. The penalty with respect to 2-POLSK in $\mathcal{E}_b/N_0$ is 2.5924 dB.

These bounds indicate that multilevel transmission in POLSK can be accomplished with relatively small penalties, as suggested above. In particular, we remark that the penalty between 2-DPSK and 2-POLSK is a net 3 dB while the one between 4-DPSK and optimum 4-POLSK is, approximately, only 1.55 dB. Finally, 8-POLSK shows better performances, of approximately 0.6 dB, with respect to 8-DPSK.

IV. Spectral Analysis of POLSK Signals

The expression of the transmitted 2-POLSK signal is the following:

$$\vec{v}_\xi(t) = \mathcal{R}\left\{Ae^{j2\pi f_0 t}\sum_{k=0}^{\infty}\begin{bmatrix}\xi_k\\1-\xi_k\end{bmatrix}u_T(t-kT)\right\}$$
$$\xi_k \in \{0,1\} \tag{47}$$

where a vector form has been exploited to represent the two launched orthogonal polarizations. In (47) $u_T(t)$ is a rectangular waveform of unit amplitude and duration T, f_0 is the optical carrier frequency, T is the symbol duration, and the ξ_k's represent the transmitted binary symbols, forming a sequence of independent, equally likely random variables assuming the values 0,1.

After the transit along the fiber, the signal becomes

$$\vec{v}_\xi(t) = \mathcal{R}\left\{Ae^{j2\pi f_0 t}\sum_{k=0}^{\infty}\boldsymbol{Q}\begin{bmatrix}\xi_k\\1-\xi_k\end{bmatrix}u_T(t-kT)\right\} \tag{48}$$

where

$$\boldsymbol{Q} = \begin{bmatrix}q_{11} & q_{12}\\q_{21} & q_{22}\end{bmatrix} \tag{49}$$

is the Jones matrix accounting for the fiber effects on the SOP's. Expression (48) is such that the two vector components are supposed to represent the fields arriving at the $\hat{x}$ and $\hat{y}$ channel photodetectors, respectively.

Resorting to standard spectral analysis techniques [21, pp. 30–33], the power spectral density of the incident fields turns out to be:

$$\boldsymbol{G}(f) = \frac{A^2}{4T}\left\{\begin{bmatrix}1-2\mathcal{R}(q_{11}q_{12}^*)\\1+2\mathcal{R}(q_{11}q_{12}^*)\end{bmatrix}|X(f-f_0,T)|^2 + T\begin{bmatrix}1+2\mathcal{R}(q_{11}q_{12}^*)\\1-2\mathcal{R}(q_{11}q_{12}^*)\end{bmatrix}\delta(f-f_0)\right\} \tag{50}$$

where

$$X(f,T) = \frac{\sin(\pi fT)}{\pi f}e^{-j\pi f/T} \tag{51}$$

and the relationship $q_{11}q_{12}^* = -(q_{21}q_{22}^*)$ has been applied.

It is evident from (50) that, though dependent on the received SOP's, the magnitude of the power spectral densities on the two channels is ASK-like, with a main lobe of width

$\frac{2}{T}$. The spectra of multilevel POLSK's have more complicated expressions, but also in that case it can be shown that the envelope is the one of an ASK system of bit rate equal to the symbol rate of the systems under consideration.

V. System Considerations

In this section we will try to compare POLSK with other coherent transmission schemes on the basis of some performance indicators.

We concentrate first on binary transmission schemes, taking as representative of the POLSK family the 2-POLSK system using the full Stokes receiver [5].

The following considerations will be based on these performance indicators: necessity of polarization control, shot noise tolerance, phase noise tolerance and bandwidth occupancy.

SOP control is necessary for all the coherent homodyne and heterodyne modulation systems. Various automatic polarization control methods have been proposed [20, ch. 8], having in common the use of rather complicated optics and control electronics. An alternative is polarization diversity [25] where the received lightwave is split into two orthogonal polarizations that are separately heterodyne-detected and then added together after phase adjustment. Polarization diversity usually features relatively simpler optics and no control electronics, but more complicated receiver electronics. The full Stokes receiver 2-POLSK [5] does not require optical polarization control, but instead a feedforward electronic system.

As for shot-noise tolerance, and leaving apart homodyne receivers, POLSK shows 3.5 and 3 dB of penalty with respect to the best performing systems, i.e., heterodyne PSK and heterodyne or phase diversity DPSK respectively [27].

CPFSK with differential demodulation exhibits shot noise figures which depend on the modulation index m. Performance as good as DPSK[4] can be obtained for $m = 0.5$, i.e., the MSK case [24]. Also its phase noise tolerance is a function of m, thus a tradeoff is needed with bandwidth occupancy. Since with narrow-linewidth lasers (10 Mhz linewidth) and at medium-high bit rates (600 M/s–1.2 G/s) high m are needed to ensure acceptable phase noise insensitivity ($m = 4.9$–2.45 respectively, [24]), either bandwidth performances or phase noise tolerance becomes quite poor.

Dual filter envelope FSK shows shot noise performances equal to those of POLSK, whereas single filter envelope FSK and envelope ASK have a three dB degradation.

Laser phase noise tolerance is quite poor for PSK and DPSK [28]. POLSK features a high insensitivity to it, and can be considered part of the class of the so-called phase-noise-canceling heterodyne receivers (PNCHR). The common principle is that of transmitting in some way side information over the optical carrier used in the modulation process, in order to obtain two IF signals with equal phase disturbances and mix them so as to achieve phase noise cancellation [31], [32]. The phase noise cancellation requires IF filters wider than what is strictly needed for the information-bearing waveform, and this in turn degrades the performance due to the increased noise power entering the receiver. However, this undesired effect can be significantly attenuated through the use of a postdetection filter, as recent results have shown [9].

SOP tracking is not required by the DPOLSK scheme described in Section 3. When implementation simplicity and receiver start-up time are a premium, this scheme could be considered a serious candidate, in spite of its slightly poorer shot noise performance, which shows a 2.4 dB degradation with respect to POLSK.

Concerning bandwidth occupation, POLSK systems present a power spectral density in the two orthogonal channels whose magnitude depends on the SOP. The shape of the continuous part of the power spectral density, and thus its bandwidth occupancy, on both channels, is the same as the ASK, PSK, and DPSK ones. Both CPFSK and envelope FSK present a wider spectral occupancy, whose width depends on the modulation index.

Summarizing the above considerations, we can say that binary POLSK modulation schemes represent interesting alternatives to both envelope-based modulations and phase modulations. In most cases a direct comparison between 2-POLSK schemes and other systems highlights both advantages and drawbacks, so that the ultimate choice should take into account in detail technological problems and peculiar requirements of specific applications.

Turning now to multilevel transmission, we can extrapolate from digital radio systems that the best tradeoff between bandwidth and power efficiency is obtained so far using two-dimensional modulation schemes, like PSK, DPSK and QAM. In this paper, we have proposed and preliminarily analyzed multilevel POLSK modulation. Rough upper bounds to the error probability show that the third degree of freedom available in the Stokes space allows the shot-noise tolerance to come closer to that of PSK and DPSK when increasing the number of signal points. As shown in Section (III-C), 8-POLSK (cubic set) is approximately 0.6 dB better than 8-DPSK and only 2.1 dB worse than 8-PSK.

Moreover, PSK and DPSK low phase noise tolerance gets worse in multilevel transmissions. A 4-phase system has a halved phase margin with respect to a 2-phase. In addition, if we want to exploit the multilevel signal set to reduce the symbol rate, keeping the bit rate constant, we must face the problem of a required larger coherence time for the carrier. As an example, if we want to use 4-DPSK to transmit at a certain bit rate, we have eight-times more severe requirements on the IF beat linewidth than for a 2-DPSK transmitting at the same bit rate. On the contrary, multilevel POLSK should remain insensitive to phase noise, at least in the limit of large enough IF filter bandwidths.

For a given bit rate, multilevel transmission reduces the system bandwidth, and thus the required speed for the electronic hardware. This could permit the use of digital signal processing

[4]This point is somewhat controversial. It can be shown that the delay-line receiver permits to use optimum (matched) IF filtering only with DPSK, but not with MSK. This is enough to say that the best performance of MSK cannot reach the best performance (matched filter) of DPSK. However, since part of the penalty can be recovered through post-detection filtering, and the analysis of this latter receiver configuration is not available in closed form, up to now, to our knowledge, no minimum value of this penalty has been given. In the text we refer to the rough exponential $P(E)$ formula which is formally identical for both systems, but does not take into account the above-mentioned circumstance.

applied to forward error correcting codes to achieve coding gains of several dB's.

VI. Conclusion

An exact and rigorous analysis of the performance of coherent optical modulation schemes employing the state of polarization as the modulating parameter has been presented.

Through the statistical characterization of the Stokes parameters extracted from the received signal affected by shot noise, exact expressions of the symbol error probability have been obtained for the POLSK modulation schemes already proposed in the literature, and for a modulation scheme that does not require polarization control.

Encouraging preliminary results have been presented about multilevel modulation schemes based on the state of polarization.

Future research will be devoted to quantify the phase noise sensitivity of POLSK modulation schemes and to deepen the analysis of multilevel systems with respect to their performance and modulator–demodulator structure.

Appendix A

In this Appendix we will give the proof of Theorems 1–5 that have been used in the text.

Theorem 1a

The n-th order joint statistics of the random processes a and b can be fully characterized through the pdf of the random vector

$$z = \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}, \tag{52}$$

i.e.,

$$f(z) = \frac{1}{\pi^{2n} \det \boldsymbol{R}} \exp(-z^\dagger \boldsymbol{R}^{-1} z) \tag{53}$$

being

$$\boldsymbol{R} = Ezz^\dagger = E\begin{bmatrix} a_1a_1^* & \cdots & a_1a_n^* & a_1b_1^* & \cdots & a_1b_n^* \\ a_2a_1^* & \cdots & a_2a_n^* & a_2b_1^* & \cdots & a_2b_n^* \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ b_1a_1^* & \cdots & b_1a_n^* & b_1b_1^* & \cdots & b_1b_n^* \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ b_na_1^* & \cdots & b_na_n^* & b_nb_1^* & \cdots & b_nb_n^* \end{bmatrix} \tag{54}$$

$$a_j = a(t_j) \quad b_j = b(t_j) \qquad 1 \le j \le n \tag{55}$$

where E is the mean operator.

Equations (53), (54), and (55) show that the knowledge of the elements of the matrix

$$E\begin{bmatrix} a_\tau \\ b_\tau \end{bmatrix}[a^* b^*] = \begin{bmatrix} R_{aa}(\tau) & R_{ab}(\tau) \\ R_{ba}(\tau) & R_{bb}(\tau) \end{bmatrix} \tag{56}$$

where a_τ, b_τ represent the processes a, b delayed of τ seconds, respectively, permits a complete characterization of the joint statistics of the two processes a and b.

The same is true for the two processes a' and b', i.e., their statistics can be obtained through the knowledge of the matrix:

$$E\begin{bmatrix} a'_\tau \\ b'_\tau \end{bmatrix}[a'^* b'^*] = \begin{bmatrix} R_{a'a'}(\tau) & R_{a'b'}(\tau) \\ R_{b'a'}(\tau) & R_{b'b'}(\tau) \end{bmatrix}. \tag{57}$$

Using now the relationship (9) between a, b, and a', b', (57) can be written, according to the hypothesis of the theorem, as

$$\begin{aligned} E\boldsymbol{U}\begin{bmatrix} a_\tau \\ b_\tau \end{bmatrix}[a^* b^*]\boldsymbol{U}^\dagger &= \boldsymbol{U}\begin{bmatrix} R(\tau) & 0 \\ 0 & R(\tau) \end{bmatrix}\boldsymbol{U}^\dagger \\ &= R(\tau)\boldsymbol{U}\boldsymbol{1}\boldsymbol{U}^\dagger = R(\tau)\boldsymbol{1} \end{aligned} \tag{58}$$

which proves the theorem.

Theorem T1b

The hypothesis

$$\begin{aligned} E\{a(\tau + t)a(t)\} &= E\{b(t+\tau)b(t)\} \\ &= E\{a(t+\tau)b(t)\} \\ &= 0 \quad \forall\, \tau \end{aligned} \tag{59}$$

allows to extend the statistical description (52), (53), (54) to the case of a, b complex (Grettenberg Theorem, see [29], [30]). Therefore, to prove (T1b) it is sufficient to refer to the proof of (T1a) and to verify that the properties

$$\begin{aligned} E\{a'(t+\tau)a'(t)\} &= E\{b'(t+\tau)b'(t)\} \\ &= E\{a'(t+\tau)b'(t)\} \\ &= 0 \quad \forall\, \tau \end{aligned} \tag{60}$$

hold true, so as to verify that the mentioned statistical description is valid for a', b' too. In fact (60) can be written as

$$\begin{aligned} E\boldsymbol{U}\begin{bmatrix} a_\tau \\ b_\tau \end{bmatrix}[a\ b]\boldsymbol{U}^t &= E\boldsymbol{U}\begin{bmatrix} a_\tau a & a_\tau b \\ b_\tau a & b_\tau b \end{bmatrix}\boldsymbol{U}^t \\ &= \boldsymbol{U}\boldsymbol{0}\boldsymbol{U}^t = \boldsymbol{0} \end{aligned} \tag{61}$$

and the theorem is proved.

Theorem 2

The random processes a, b are independent white Gaussian random processes with the same autocorrelation function $R(\tau) = \frac{N_0}{2}\delta(\tau)$. Using the same steps as in the proof of Theorem 1a, we are led to the following matrix which characterizes the statistics of a', b' conditioned on $\boldsymbol{U}$:

$$\begin{aligned} E_{|U}\boldsymbol{U}(t+\tau)\begin{bmatrix} a_\tau \\ b_\tau \end{bmatrix}[a^* b^*]\boldsymbol{U}^\dagger(t) &= \boldsymbol{U}(t+\tau)\frac{N_0}{2}\delta(\tau)\boldsymbol{1}\boldsymbol{U}^\dagger(t) \\ &= \boldsymbol{U}(t)\frac{N_0}{2}\delta(\tau)\boldsymbol{1}\boldsymbol{U}^\dagger(t) \\ &= \frac{N_0}{2}\delta(\tau)\boldsymbol{1} = R(\tau)\boldsymbol{1} \end{aligned} \tag{62}$$

where $E_{|U}$ means conditional average with respect to U.

Thus the conditional pdf of a', b', is the same as the pdf of a, b, independently of $\boldsymbol{U}$. Averaging with respect to $\boldsymbol{U}$ leaves that pdf unchanged, so that the theorem is proved.

Theorem 3

According to [1, p. 554], we can write the equality

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} [E_x^* E_y^*] = \frac{1}{2}\begin{bmatrix} S_0 + S_1 & S_2 - jS_3 \\ S_2 + jS_3 & S_0 - S_1 \end{bmatrix} = \frac{1}{2}(\boldsymbol{P} + S_0\mathbf{1}) \tag{63}$$

This matrix is also known as the *coherence matrix* of the field $\vec{E}$ [1, p. 545]. Under the hypothesis that propagation losses are independent of the field polarization, they can be factorized and are thus uninfluent. Therefore, we can write the coherence matrix of the received field with $S_0' = S_0$:

$$\begin{bmatrix} E_x' \\ E_y' \end{bmatrix} [E_x'^* E_y'^*] = \frac{1}{2}\begin{bmatrix} S_0 + S_1' & S_2' - jS_3' \\ S_2' + jS_3' & S_0 - S_1' \end{bmatrix} = \frac{1}{2}(\boldsymbol{P}' + S_0\mathbf{1}). \tag{64}$$

Deriving now $\boldsymbol{P}$ and $\boldsymbol{P}'$ from (63) and (64) and noticing that from (15) we can write

$$\begin{bmatrix} E_x' \\ E_y' \end{bmatrix} [E_x'^* E_y'^*] = \boldsymbol{Q}\begin{bmatrix} E_x \\ E_y \end{bmatrix} [E_x^* E_y^*]\boldsymbol{Q}^\dagger \tag{65}$$

it is straightforward to verify that:

$$\boldsymbol{P}' = \boldsymbol{Q}\boldsymbol{P}\boldsymbol{Q}^\dagger \tag{66}$$

which proves the theorem.

Theorem T4

Let us define a plane and a reference system whose unit vectors are $\hat{u}_1$, $\hat{u}_2$. We will call

$$\begin{aligned} \vec{v}_1 &= a\hat{u}_1 + b\hat{u}_2 \\ \vec{v}_2 &= c\hat{u}_1 + d\hat{u}_2 \end{aligned} \tag{67}$$

Let us assume, for the time being, that a, b, c, d, are *all* zero-mean independent Gaussian processes. The angles α_1 and α_2 formed by $\vec{v}_1$, $\vec{v}_2$ with a reference unit vector, say $\hat{u}_1$, in the plane, are uniformly distributed over the interval $[0, 2\pi]$ (see, for example, [16, p. 200]).

The relative angular displacement between $\vec{v}_1$ and $\vec{v}_2$ is given by $\alpha_{12} = \alpha_1 - \alpha_2$. This angle (subtracting 2π if necessary) is still uniformly distributed within $[0, 2\pi]$ as a direct calculation of the simple convolution of the pdf of α_1, α_2 shows.

Defining a vector $\vec{z} = (z_1, z_2)$ as follows:

$$\begin{aligned} z_1 &= \vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1||\vec{v}_2|\cos\alpha_{12} = ac + bd \\ z_2 &= \vec{v}_1 \wedge \vec{v}_2 \cdot (\hat{u}_1 \wedge \hat{u}_2) = |\vec{v}_1||\vec{v}_2|\sin\alpha_{12} = cb - ad \end{aligned}$$

we have that the angle α subtended by this vector with respect to a reference axis, say $\hat{u}_1$, is

$$\alpha = \arctan\left(\frac{z_2}{z_1}\right) = \arctan(\tan(\alpha_{12})) = \alpha_{12}.$$

Therefore, α is uniformly distributed over the interval $[0, 2\pi]$.

This result still holds if one of the two vectors $\vec{v}_1$, $\vec{v}_2$ is generated by nonzero mean Gaussian processes. Assuming it is $\vec{v}_1$, we have that α_1 is no longer uniformly distributed. However α still is. In fact, the conditional pdf of α is:

$$f_\alpha(\alpha|\alpha_1) = \frac{1}{2\pi}.$$

This is to say, it is independent of α_1. This is essentially due to the fact that α_1 and α_2 are independent, and that, dealing with angles, it is possible to reduce their interval of existence to $[0, 2\pi]$. The cdf of α is

$$F_\alpha(\alpha) = \int_0^\alpha f_\alpha(\beta)\,d\beta = \int_0^\alpha \int_0^{2\pi} f_\alpha(\beta|\gamma) f_{\alpha_1}(\gamma)\,d\gamma\,d\beta = \frac{\alpha}{2\pi}$$

Thus, the pdf of α is still $\frac{1}{2\pi}$.

Appendix B

In this Appendix, we present the detailed calculations leading to the pdf of θ as given in (30).

First of all, we define two independent random processes a, b as follows:

$$\begin{aligned} a &= (A + x_c)^2 + x_s^2 \\ b &= y_s^2 + y_c^2 \end{aligned} \tag{68}$$

The distribution of a is noncentral chi-square with two degrees of freedom and noncentrality parameter A^2. The pdf is as follows:

$$f_a(y) = \frac{1}{2\sigma^2} e^{-\frac{A^2+y}{2\sigma^2}} I_0\left(\sqrt{y}\,\frac{A}{\sigma^2}\right) \quad y > 0. \tag{69}$$

The distribution of b is a central chi-square with two degrees of freedom. The pdf is the following:

$$f_b(y) = \frac{1}{2\sigma^2} e^{-\frac{y}{2\sigma^2}} \quad y > 0. \tag{70}$$

Let us now define the random process γ as follows:

$$\gamma = \frac{a}{b} \quad \gamma \in [0, \infty]. \tag{71}$$

We want to calculate the pdf of γ. Its cumulative function is

$$F_\gamma(z) = P\left(\frac{a}{b} \le z\right)$$

and its formal expression is [22]:

$$F_\gamma(\gamma) = \int_0^\infty d\beta \int_0^{\gamma\beta} f_a(\delta) f_b(\beta)\,d\delta. \tag{72}$$

Taking the derivative of (72) with respect to γ we get

$$f_\gamma(\gamma) = \frac{\partial}{\partial\gamma} F_\gamma(\gamma) = \int_0^\infty \beta f_a(\gamma\beta) f_b(\beta)\,d\beta. \tag{73}$$

Let us now define the following transformation:

$$\gamma = \frac{1+k}{1-k} \quad \gamma \in [0, \infty], \qquad k \in [-1, 1]. \tag{74}$$

It is a strictly monotonous transformation, as

$$\frac{\partial \gamma}{\partial k} = \frac{2}{1-k^2} > 0 \quad \forall\, k \in [-1, 1].$$

Due to this feature of (74) we can resort to the simple formula [22]

$$f_k(k) = f_\gamma(\gamma(k)) \left| \frac{\partial \gamma(k)}{\partial k} \right|$$

resulting in

$$f_k(k) = \frac{2}{1-k^2} \int_0^\infty \beta f_a\left(\frac{1+k}{1-k}\beta\right) f_b(\beta)\, d\beta. \tag{75}$$

The expression of k as a function of a, b turns out to be the following:

$$k = \frac{a-b}{a+b}.$$

From the definition of a and b it is straightforward to verify that k is exactly the argument of the arcosine appearing in (24), so that

$$\theta = \arccos(k).$$

A last step is needed to obtain the formal expression of the pdf of θ:

$$k = \cos\theta \quad k \in [-1, 1], \qquad \theta \in [0, \pi].$$

This transformation too is a monotonous one, as

$$\frac{\partial k}{\partial \theta} = -\sin\theta < 0 \quad \forall\, \theta \in [0, \pi]$$

so we may write

$$f_\theta(\theta) = f_k(k(\theta)) \left| \frac{\partial k(\theta)}{\partial \theta} \right|$$

resulting in the following formal expression for the pdf of θ:

$$f_\theta(\theta) = \frac{2\sin(\theta)}{1-\cos(\theta)^2} \int_0^\infty \beta f_a\left(\frac{1+\cos(\theta)}{1-\cos(\theta)}\beta\right) f_b(\beta)\, d\beta. \tag{76}$$

Substituting in (76) the actual expressions of the pdf's within the integral, we obtain

$$f_\theta(\theta) = \frac{2\sin(\theta)}{1-\cos(\theta)^2} \frac{1}{4\sigma^2} \int_0^\infty e^{-\frac{\left[A^2 + \left(1+\frac{1+\cos\theta}{1-\cos\theta}\right)x\right]}{2\sigma^2}} \cdot I_0\left(\frac{A}{\sigma^2}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}x}\right) x\, dx. \tag{77}$$

To solve the integral in (77) we shall resort to a series of results, all of which can be found in [19]. We will report the page and formula numbers used in that book. Adapting to our needs result [19, 6.643–2 p. 720] we obtain

$$g(\alpha, \beta) = \int_0^\infty x e^{-\alpha x} I_0(2\beta\sqrt{x})\, dx = \beta^{-1}\alpha^{-\frac{3}{2}} M_{-\frac{3}{2},0}\left(\frac{\beta^2}{\alpha}\right) \tag{78}$$

where M is a Whittaker function. Then using result [19, 9.220-2 p. 1059] we easily find

$$g(\alpha, \beta) = \left(\frac{\beta^2}{\alpha}\right)^{\frac{1}{2}} e^{-\frac{\beta^2}{2\alpha}} \Phi\left(2, 1; \frac{\beta^2}{\alpha}\right) \tag{79}$$

where Φ is a degenerate hypergeometric confluent function of the first kind. Using the recurrency relation [19, 9.212-4 p. 1058], we get

$$\Phi\left(2, 1; \frac{\beta^2}{\alpha}\right) = \left(\frac{\beta^2}{\alpha} + 1\right) \Phi\left(1, 1; \frac{\beta^2}{\alpha}\right). \tag{80}$$

But

$$\Phi(\gamma, \gamma; z) = e^z \tag{81}$$

according to [19, 9.215-1 p. 1059]. After minor algebra we finally get

$$g(\alpha, \beta) = \int_0^\infty x e^{-\alpha x} I_0(2\beta\sqrt{x})\, dx = \left(1 + \frac{\beta^2}{\alpha}\right) \frac{1}{\alpha^2} e^{\frac{\beta^2}{\alpha}}. \tag{82}$$

Substituting in (82)

$$\alpha = \frac{1 + \frac{1+\cos\theta}{1-\cos\theta}}{2\sigma^2}$$
$$\beta = \frac{A}{2\sigma^2}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

and comparing the result to (77), it is straightforward to obtain (30).

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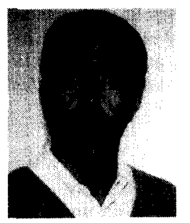

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