Problem Set #4: Solutions

1. $\bar{B} = B_z \hat{z} \qquad \bar{j} = j_x \hat{x}$

ELECTRONS: $n_e = \frac{1}{6} \cdot 10^{17} \text{ cm}^{-3}$ $\qquad \bar{\bar{m}} = \begin{pmatrix} 0.98 m_0 & 0 & 0 \\ 0 & 0.19 m_0 & 0 \\ 0 & 0 & 0.19 m_0 \end{pmatrix}$

$$\bar{F} = e\bar{E} + \frac{e}{c}\bar{v} \times \bar{B} - \frac{\bar{\bar{m}}\bar{v}}{\tau} = 0 \quad \text{at steady state}$$

$$\therefore e\tau\bar{E} = \bar{\bar{m}}\bar{v} - \frac{e\tau}{c}\bar{v}\times\bar{B} = \left(\bar{\bar{m}} - \frac{e\tau}{c}\bar{\bar{B}}\right)\bar{v} \quad \text{defined by:}$$

$$\bar{v}\times\bar{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B_z \end{vmatrix} = v_y B_z \hat{x} - v_x B_z \hat{y} = \begin{pmatrix} 0 & B_z & 0 \\ -B_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\bar{v} = \bar{\bar{B}}\bar{v}$$

$$\bar{j} = ne\bar{v} = ne^2\tau\left[\bar{\bar{m}} - \frac{e\tau}{c}\bar{\bar{B}}\right]^{-1}\cdot\bar{E}$$

$$\Downarrow$$

$$\bar{\bar{\sigma}} = ne^2\tau\left[\bar{\bar{m}} - \frac{e\tau}{c}\bar{\bar{B}}\right]^{-1} = \frac{ne^2\tau}{m_x m_y + \frac{e^2\tau^2}{c^2}B^2}\begin{pmatrix} m_y & +\frac{e\tau}{c}B & 0 \\ -\frac{e\tau}{c}B & m_x & 0 \\ 0 & 0 & \frac{m_x m_y + (\frac{e\tau}{c}B)^2}{m_z} \end{pmatrix}$$

$$\bar{\bar{\sigma}}_{total} = \Sigma\, \bar{\bar{\sigma}}_{pocket} \qquad \text{define } \alpha = \frac{e\tau}{c}B$$

$$= 2n_e e^2\tau\left[\begin{pmatrix} \frac{m_t}{m_t^2+\alpha^2} & \frac{+\alpha}{m_t^2+\alpha^2} & 0 \\ \frac{-\alpha}{m_t^2+\alpha^2} & \frac{m_t}{m_t^2+\alpha^2} & 0 \\ 0 & 0 & \frac{1}{m_l} \end{pmatrix} + \begin{pmatrix} \frac{m_t+m_l}{m_t m_l+\alpha^2} & +2\frac{\alpha}{m_t m_l+\alpha^2} & 0 \\ -2\frac{\alpha}{m_t m_l+\alpha^2} & \frac{m_t+m_l}{m_t m_l+\alpha^2} & 0 \\ 0 & 0 & \frac{2}{m_t} \end{pmatrix}\right]$$

$$R_{Hall} = \frac{E_y}{j_x B} \qquad \begin{aligned} j_x &= \sigma_{xx}E_x + \sigma_{xy}E_y \\ j_y &= \sigma_{yx}E_x + \sigma_{yy}E_y = 0 \end{aligned}$$

$$E_y = -\sigma_{yx}/\sigma_{yy}\, E_x \;; \quad j_x = \left(\sigma_{xx} - \frac{1}{\sigma_{yy}}\sigma_{xy}\sigma_{yx}\right)E_x$$

$$R_{Hall} = \frac{\sigma_{yx}}{\sigma_{xy}\sigma_{yx} - \sigma_{xx}\sigma_{yy}}\cdot\frac{1}{B}$$

For simplification in the semiclassical limit $\omega_c \tau \ll 1$ leads to $\alpha^2 \ll m_0^2$

$$\bar{\bar{\sigma}}_{total} = \frac{2 n_e e^2 \tau}{m_t^2 m_l} \begin{pmatrix} m_t^2 + 2 m_t m_l & +\alpha(m_l + 2m_t) & 0 \\ -\alpha(m_l + 2m_t) & m_t^2 + 2 m_t m_l & 0 \\ 0 & 0 & m_t^2 + 2 m_t m_l \end{pmatrix}$$

$$R_{Hall} = \frac{m_t^2 m_l \cdot +\alpha(m_l + 2m_t)}{2 n_e e^2 \tau B \left[(m_t^2 + 2 m_t m_l)^2 + \alpha^2 (m_l + 2m_t)^2\right]}$$

$$= \frac{+3 m_t^2 m_l (m_l + 2m_t)}{6 n e c \left[(m_t^2 + 2 m_t m_l)^2 + \alpha^2 (m_l + 2m_t)^2\right]}$$

$$\approx R_{0\,Hall} \cdot \frac{3 m_l (m_l + 2m_t)}{(m_t + 2m_l)^2}$$

$$\Delta \rho_{xx} = \frac{1}{\rho_{xx}(0)} \left(\rho_{xx}(B) - \rho_{xx}(0)\right) = \cancel{\sigma_{yy}(0)\left(\frac{1}{\sigma_{yy}(B)} - \frac{1}{\sigma_{yy}}\right)}$$

$$= \frac{1}{\sigma_{yy}(0)} \left(\sigma_{yy}(B) - \sigma_{yy}(0)\right)$$

$$\sigma_{yy}(B) \approx 2 n_e e^2 \tau \; \frac{2 m_t^2 m_l + m_t^3 + \alpha^2 (m_l + 2m_t)}{m_t^3 m_l}$$

$$\Delta \rho_{xx} \approx \frac{(2m_t + m_l)\alpha^2}{m_t^2 (2m_l + m_t)} \approx 0$$

HOLES : $m_{hh}^* = 0.5\, m_0 \qquad m_{lh}^* = 0.16\, m_0$

$$n = 2 n_{hh} + n_{lh} = 10^{17}\ \text{cm}^{-3} \qquad \frac{n_{hh}}{n_{lh}} = \left(\frac{m_{hh}}{m_{lh}} \right)^{3/2}$$

$$\bar{\bar{\sigma}}_{pocket} = \frac{n_i e^2 \tau}{m_i (1+\omega_i^2 \tau^2)} \cdot \begin{pmatrix} 1 & \omega_i \tau & 0 \\ -\omega_i \tau & 1 & 0 \\ 0 & 0 & 1+\omega_i^2 \tau^2 \end{pmatrix} \qquad \alpha_i = \frac{n_i}{m_i}$$

$$\bar{\bar{\sigma}}_{total} = e^2 \tau \begin{pmatrix} 2\, \alpha_h/(1+\omega_h^2\tau^2) + \alpha_l/(1+\omega_l^2\tau^2) & \text{same positive} & 0 \\ -2\, \alpha_h \omega_h \tau/(1+\omega_h^2\tau^2) - \alpha_l \omega_l \tau/(1+\omega_l^2\tau^2) & \text{same} & 0 \\ 0 & 0 & 2\alpha_h + \alpha_l \end{pmatrix}$$

$$R_{Hall} = \frac{\sigma_{yx}}{\sigma_{xy}\sigma_{yx} - \sigma_{xx}\sigma_{yy}} \cdot \frac{1}{B}$$

as before, but it is difficult to obtain a nice expression in terms of nec.

$$\Delta \rho_{xx} \approx 0$$

2. a)

Region I	Region II	Region III
Incident + Reflected	Incident + Reflected + decay	Incident
$\lvert A_+\rvert$ $\quad$ $\lvert A_-\rvert$	$\lvert B_+\rvert$ $\quad$ $\lvert B_-\rvert$	$\lvert C_+\rvert$

$$\psi_I = A_+ e^{i\frac{\omega}{c}z} e^{-i\omega t} + A_- e^{-i\frac{\omega}{c}z} e^{-i\omega t}$$

from now on neglect time dependent part:

$$\psi_{II} = B_+ \exp\left(i\frac{n\omega}{c}z - k\frac{\omega}{c}z\right) + B_- \exp\left(-i\frac{n\omega}{c}z + k\frac{\omega}{c}z\right)$$

$$\psi_{III} = C\exp\left(i\frac{\omega}{c}z\right)$$

$$\begin{cases} A_+ + A_- = B_+ + B_- \\ B_+ \exp\left(i\frac{n\omega}{c}T - k\frac{\omega}{c}T\right) + B_- \exp\left(-i\frac{n\omega}{c}T + k\frac{\omega}{c}T\right) = C\exp\left(i\frac{\omega}{c}T\right) \\ A_+ i\frac{\omega}{c} + A_-\left(-i\frac{\omega}{c}\right) = B_+\left(i\frac{n\omega}{c}\right) - B_-\left(i\frac{n\omega}{c}\right) \\ B_+\left(i\frac{n\omega}{c} - k\frac{\omega}{c}\right)\exp\left(i\frac{n\omega}{c}T - k\frac{\omega}{c}T\right) + B_-\exp\left(-i\frac{n\omega}{c}T + k\frac{\omega}{c}T\right)\left(-i\frac{n\omega}{c} + k\frac{\omega}{c}\right) = Ci\frac{\omega}{c}\exp\left(i\frac{\omega}{c}T\right) \end{cases}$$

$$\begin{cases} A_+ + A_- = B_+ + B_- \\ A_+ - A_- = nB_+ - nB_- \\ B_+\theta + B_-\frac{1}{\theta} = C\exp\left(i\frac{\omega}{c}T\right) \\ B_+\alpha\theta - B_-\alpha\frac{1}{\theta} = \left(i\frac{\omega}{c}\right)C\exp\left(i\frac{\omega}{c}T\right) \end{cases}$$

$$\theta = \exp\left(i\frac{n\omega}{c}T - k\frac{\omega}{c}T\right)$$

$$\alpha = \left(i\frac{n\omega}{c} - k\frac{\omega}{c}\right)$$

$$B_+\left(\alpha\theta - i\frac{\omega}{c}\theta\right) = B_-\left(i\frac{\omega}{c}\cdot\frac{1}{\theta} + \alpha\frac{1}{\theta}\right)$$

$$B_- = B_+\theta^2\left(\frac{\alpha - i\omega/c}{\alpha + i\omega/c}\right) = B_+\theta^2\delta$$

$$\begin{cases} A_+ + A_- = B_+(1+\theta^2\delta) \\ A_+ - A_- = nB_+(1-\theta^2\delta) \end{cases}$$

$$A_+ - A_- = n(A_+ + A_-)\left(\frac{1-\theta^2\delta}{1+\theta^2\delta}\right)$$

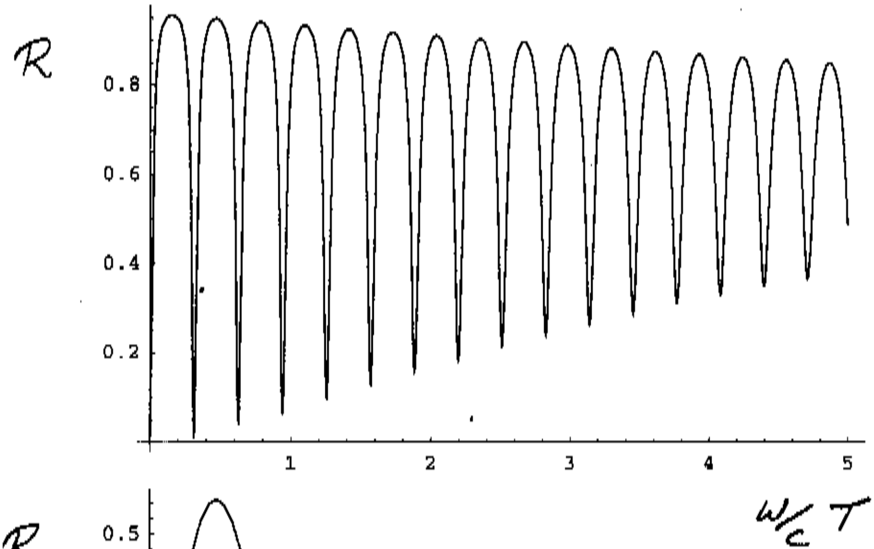
$$A_-\left[n\left(\frac{1-\theta^2\delta}{1+\theta^2\delta}\right) + 1\right] = A_+\left[1 - n\left(\frac{1-\theta^2\delta}{1+\theta^2\delta}\right)\right]$$

$$\mathcal{R} = \left| \frac{A_-}{A_+} \right|^2 = \left| \frac{1 - n\left(\dfrac{1-\theta^2\delta}{1+\theta^2\delta}\right)}{1 + n\left(\dfrac{1-\theta^2\delta}{1+\theta^2\delta}\right)} \right|^2$$

by MATHEMATICA

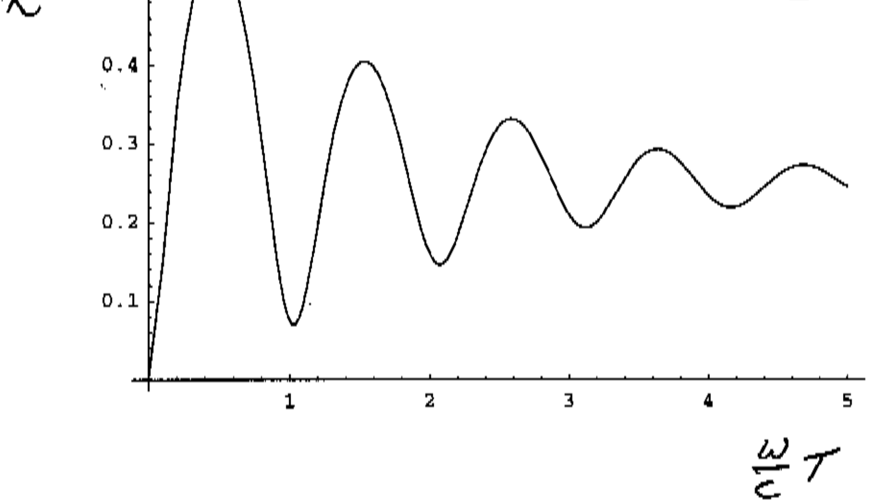
$$\mathcal{R} = \frac{(k-i(n+1))(n-1)\, e^{2(k-in)\frac{\omega}{c}T} - (k-i(n-1))(n+1)}{(k-i(n+1))(n+1)\, e^{2(k-in)\frac{\omega}{c}T} - (k-i(n-1))(n-1)} \times$$

$$\times \frac{(k+i(n-1))(n+1) - (k+i(n+1))(n-1)\, e^{2(k+in)\frac{\omega}{c}T}}{(k+i(n-1))(n-1) - (k+i(n+1))(n+1)\, e^{2(k+in)\frac{\omega}{c}T}}$$



$n = 10$

$k = 0.07$



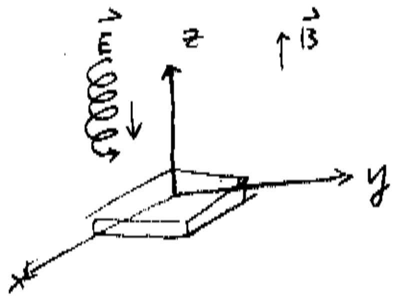
$n = 3$

$k = 0.3$

c). Suppose now we have the magnetic field applied along z direction, that is, $\vec{B} = B\hat{z}$. First, let us find out the motion of carriers under the magnetic field and electromagnetic wave.

For simplicity, we assume the mass is isotropic. The equation of motion is

$$m^* \frac{d\vec{V}}{dt} = \frac{m^*\vec{V}}{\tau} + \frac{e}{c}\vec{V}\times\vec{B} + e\vec{E}(t)$$



For a right incident circularly polarized light moving in $-z$ direction, the electric field can be written as

$$\vec{E}(t) = (E\hat{x} - iE\hat{y})\, e^{-i\omega t}$$
$$= (\hat{x} - i\hat{y})\, E e^{-i\omega t} \equiv \vec{E}_r\, e^{-i\omega t}$$

For a left circularly polarized light,

$$\vec{E}(t) = (E\hat{x} + iE\hat{y})\, e^{-i\omega t}$$
$$= (\hat{x} + i\hat{y})\, E e^{-i\omega t} \equiv \vec{E}_l\, e^{-i\omega t}$$

The velocity $\vec{V}$ of the carrier driven by the electric field is

$$\vec{V}(t) = \vec{V}_0\, e^{-i\omega t} = (V_x\hat{x} + v_y\hat{y})\, e^{-i\omega t}$$

Substitute these into the equation of motion, we got:

$$\begin{cases} -i\omega m^* V_x = m^* V_x/\tau + \frac{1}{c} e V_y B + eE_x \\ -i\omega m^* V_y = m^* V_y/\tau - \frac{1}{c} e V_x B + eE_y \end{cases}$$

Solve for V_x, V_y, we have

$$(1 - i\omega\tau)V_x = \omega_c \tau V_y + \frac{e\tau}{m} E_x \qquad \left(\omega_c = \frac{eB}{m^* c}\right)$$
$$(1 - i\omega\tau)V_y = -\omega_c \tau V_x + \frac{e\tau}{m} E_y$$

$$\begin{pmatrix} 1-i\omega\tau & -\omega_c\tau \\ \omega_c\tau & 1-i\omega\tau \end{pmatrix}\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \frac{e\tau}{m^*}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\therefore \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \frac{e\tau/m^*}{(1-i\omega\tau)^2+(\omega_c\tau)^2}\begin{pmatrix} 1-i\omega\tau & \omega_c\tau \\ -\omega_c\tau & 1-i\omega\tau \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

For right circularly polarized light, $E_x = E$, $E_y = -iE$

$$\therefore \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \frac{e\tau/m^*}{(1-i\omega\tau)^2+(\omega_c\tau)^2}\begin{pmatrix} 1-i(\omega+\omega_c)\tau \\ -i(1-i(\omega+\omega_c)\tau) \end{pmatrix} E$$

$$= \frac{e\tau}{m^*}\cdot\frac{1-i(\omega+\omega_c)\tau}{(1-i\omega\tau)^2+(\omega_c\tau)^2}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\therefore \sigma_r(\omega) = \frac{ne^2\tau}{m^*}\frac{1-i(\omega+\omega_c)\tau}{(1-i\omega\tau)^2+(\omega_c\tau)^2} = \frac{ne^2\tau}{m^*}\frac{1}{1-i(\omega-\omega_c)\tau}$$

Similarly, for left circularly polarized light, $E_x = E$, $E_y = iE$

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \frac{e\tau/m^*}{(1-i\omega\tau)^2+(\omega_c\tau)^2}\begin{pmatrix} 1-i(\omega-\omega_c)\tau \\ i(1-i(\omega-\omega_c)\tau) \end{pmatrix} E$$

$$= \frac{e\tau}{m^*}\cdot\frac{1}{1-i(\omega+\omega_c)\tau}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\therefore \sigma_l(\omega) = \frac{ne^2\tau}{m^*}\cdot\frac{1}{1-i(\omega+\omega_c)\tau}$$

For $B=0$, $\sigma(\omega) = \frac{ne^2\tau}{m^*}\frac{1}{1-i\omega\tau}$.

Thus, we see that the effect of magnetic field is to change $\omega \to \omega \pm \omega_c$ in $\sigma(\omega)$.

Therefore, if the plasma frequency at $B=0$ is $\tilde{\omega}_p$, than the plasma fregeuency for right circular polarized light can be obtained by

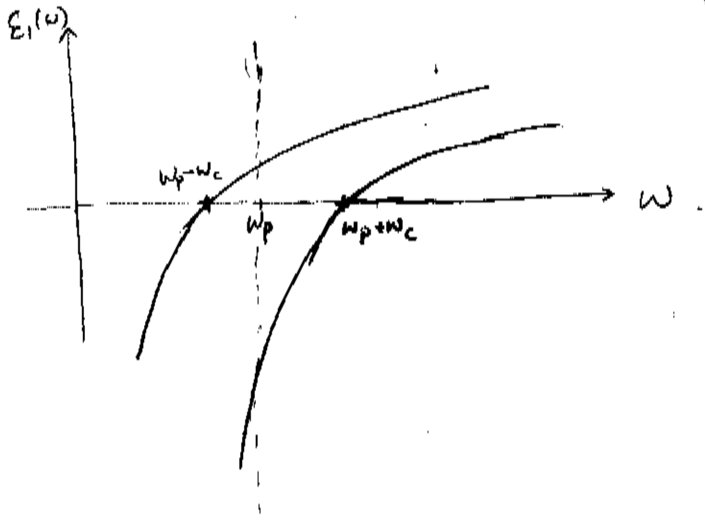
$$\tilde{\omega}_p = \omega_{p(r)} - \omega_c \qquad \therefore \omega_{p(r)} = \omega_p + \omega_c$$

Similarly, for left circular polarized light:

$$\tilde{\omega}_p = \omega_{p(l)} + \omega_c \qquad \therefore \omega_{p(l)} = \omega_p - \omega_c$$

$\therefore$ The effect of the magnetic is to change the plasma frequency by $\pm \omega_c$. There are now two different plasma frequencies $\tilde{\omega}_p \pm \omega_c$, corresponding to two different circularly polarized lights.

If we plot $\varepsilon_1(\omega) = Re\{\varepsilon_{complex}\}$, we may have something like this.



4

12 hole pockets
6 electron pockets

$m_t = 0.06\, m_0, \quad m_e = m_0, \quad E_f^e = 50\,\text{meV}$

$$@T=0 \begin{cases} n = 6\,\frac{3}{2}\,\frac{1}{2\pi^2}\left(\frac{2E_f^e}{\hbar^2}\right)^{3/2} m_t\sqrt{m_e} \\ p = 12\,\frac{3}{2}\,\frac{1}{2\pi^2}\left(\frac{2E_f^h}{\hbar^2}\right)^{3/2} m_t\sqrt{m_e} \end{cases}$$

$$n = p \Rightarrow E_f^h = 2^{-2/3} E_f^e = 31.5\,\text{meV}$$

$$n = p = 1.8 \times 10^{19}\,\text{cm}^{-3}$$

$E \perp z$:

$$\sigma = \frac{(n+p)e^2\tau}{m_t(1-i\omega\tau)}$$

$$\varepsilon = \varepsilon_0 + \frac{4\pi i \sigma}{\omega}$$

$$\text{Re}\,\varepsilon = \varepsilon_0 - \frac{4\pi(n+p)e^2\tau^2}{m_t(1+\omega^2\tau^2)} = 0$$

$$\omega_p^2 = \frac{4\pi(n+p)e^2}{m_t\varepsilon_0} - \frac{1}{\tau^2}$$

$E \parallel z$: similar, m_e instead of m_t

$$\omega_p^2 = \frac{4\pi(n+p)e^2}{m_e\varepsilon_0} - \frac{1}{\tau^2}$$

consider $\varepsilon_0 = 1, \tau = \infty$

then $\omega_p(E \perp z) = 1.38 \times 10^{15}\,\text{s}^{-1}$

$\omega_p(E \parallel z) = 3.37 \times 10^{14}\,\text{s}^{-1}$