Problem Set #9
Solutions

1a. $\langle l|x^2|l'\rangle = \sum_n \langle l|x|n\rangle\langle n|x|l'\rangle$

$$= \frac{\hbar}{2m_c^* \omega_c^*}\cdot\left(\sqrt{l+1}\,\delta_{n,l+1} + \sqrt{l}\,\delta_{n,l-1}\right)\cdot$$

$$\cdot\left(\sqrt{n+1}\,\delta_{l',n+1} + \sqrt{n}\,\delta_{l',n-1}\right)$$

$$= \frac{\hbar}{2m_c^*\omega_c^*}\cdot\left(\sqrt{(l+1)(n+1)}\,\delta_{l',n+1}\delta_{n,l+1} + \sqrt{l(n+1)}\,\delta_{l',n+1}\delta_{n,l-1} + \sqrt{(l+1)n}\,\delta_{l',n-1}\delta_{n,l+1} + \sqrt{nl}\,\delta_{l',n-1}\delta_{n,l-1}\right)$$

$$= \frac{\hbar}{2m_c^*\omega_c^*}\left(\sqrt{(l+1)(l+2)}\,\delta_{l',l+2} + \sqrt{l^2}\,\delta_{l',l} + \sqrt{(l+1)^2}\,\delta_{l',l} + \sqrt{l(l-1)}\,\delta_{l',l-2}\right)$$

$$= \frac{\hbar}{2m_c^*\omega_c^*}\left(\sqrt{(l+1)(l+2)}\,\delta_{l',l+2} + (2l+1)\,\delta_{l',l} + \sqrt{l(l-1)}\,\delta_{l',l-2}\right)$$

Similarly,

$$\langle l|p^2|l'\rangle = \frac{\hbar m_c^*\omega_c^*}{2}\left(\sqrt{(l+1,l+2)}\,\delta_{l',l+2} + (2l+1)\,\delta_{l',l} + \sqrt{l(l+1)}\,\delta_{l',l-2}\right)$$

b. $\langle E_p\rangle = \frac{1}{2}m_c^*\omega_c^{*2}\langle x^2\rangle = \frac{\hbar\omega_c^*}{4}(2l+1)$

$\langle E_k\rangle = \frac{\langle p^2\rangle}{2m_c^*} = \frac{\hbar\omega_c^*}{4}(2l+1)$

$\Big\}\ \langle E\rangle = \hbar\omega_c^*\left(l+\frac{1}{2}\right)$

2. a. At $B = 0T$, all pockets have the same occupation

$$n = N/6 = 1.67 \cdot 10^{17} \text{ cm}^{-3}$$

$$\omega_c = \frac{eE(10T)}{mc\mu_B}$$

At $B = 10T$, the lowest Landau level energy is

$$E^{\parallel}_{k_z=0} = \frac{1}{2}\hbar\omega_c^* = \frac{1}{2}\hbar\frac{eB}{cm_c^*} = \frac{1}{2}\hbar\frac{eB}{cm_t} = \frac{\hbar}{2 \cdot 0.19} \cdot 2 \cdot 10^{12} = 3.5 \text{ meV} \qquad \omega_c = \frac{2E(10T)}{\hbar c}$$

$$E^{\perp}_{k_z=0} = \frac{1}{2}\hbar\frac{eB}{cm_c^*} = \frac{1}{2}\hbar\frac{eB}{c\sqrt{m_t m_l}} = 1.5 \text{ meV}$$

$$n = \text{4 pockets} \times \text{2 spin} \times \frac{m_c\omega_c}{2\pi\hbar} \times \int_{1.5}^{E_F} 1D_{DOS} \cdot dE$$

$$= \frac{4 m_c\omega_c}{\pi\hbar} \cdot \frac{1}{2\pi} \cdot 2\sqrt{\frac{2m_z(E_F - 1.5)}{\hbar^2}}$$

$$= \frac{4eB}{\pi^2\hbar^2 c}\sqrt{2m_z(E_F - 1.5)} = 4 \cdot 10^{34}\sqrt{E_F - 1.5e} = 10^{24}$$

$$E_F = 1.5 \cdot 1.6 \cdot 10^{-19} + 6.2 \cdot 10^{-22} \text{ J} = 2.4 \cdot 10^{-19} \text{ J} = 1.504 \text{ eV}$$

Indeed only <u>one</u> Landau level of the perpendicular pockets is occupied.

$\Downarrow$

2 pockets are empty, 4 pocket have $2.5 \cdot 10^{17} \text{ cm}^{-3}$ carriers

b. $E^{\parallel}_{k_z=0} = E_F \qquad n = \frac{4eB}{\pi^2\hbar^2 c}\sqrt{2(E_F - E^{\perp}_{k_z=0})}$

$$\frac{1}{2}\hbar\omega_c^{\parallel} = \frac{1}{2}\hbar\frac{eB}{cm_t} = \frac{1}{2}\left(\frac{n\pi^2\hbar^2 c}{4eB}\right)^2 + \frac{1}{2}\hbar\frac{eB}{cm_c^{\perp}}$$

c. $(1,1,1)$

$$\hbar\omega_0\left(\frac{1}{0.19} - \frac{1}{\sqrt{0.19 \cdot 0.98}}\right) = \frac{1}{\omega_0^2} \cdot 9 \cdot 10^{-28}$$

$$\omega_0^3 = 2.88 \cdot 10^6 \,;\quad \omega_0 = 142 \implies B = 7.1 \cdot 10^{-10} \text{ T}$$

3a. Depth of well $= \frac{3}{4}\Delta E_g = 0.21\ eV = 3.36\cdot10^{-20}\ J$

$$E_{n=2} = \frac{4\pi^2\hbar^2}{2mL^2} = \frac{3.41\cdot10^{-36}}{L^2} = 3.36\cdot10^{-20} \qquad L = 100\ \mathring{A}$$

$$E_{n=3} = \frac{7.67\cdot10^{-36}}{L^2} = 3.36\cdot10^{-20} \qquad L = 151\ \mathring{A}$$

$$100\mathring{A} < L_{2\,bound} < 151\mathring{A} \quad \checkmark$$

$$\hbar\omega_c = \frac{\hbar eB}{m_c c} = 3\cdot10^{-21}\ J$$

$$E_{1,80\mathring{A}} = \frac{\pi^2\hbar^2}{2mL^2} = 1.33\cdot10^{-20}\ J \qquad E_{2,80\mathring{A}} = 5.33\cdot10^{-20}\ J$$

$$E_1 + \frac{1}{2}\hbar\omega_c = 1.48\cdot10^{-20}\ J$$

$$E_1 + \frac{3}{2}\hbar\omega_c = 1.78\cdot10^{-20}\ J$$

$$\vdots$$

$$\frac{\frac{3}{4}\Delta E_g - (E_1 + \frac{1}{2}\hbar\omega_c)}{\hbar\omega_c} = \frac{1.88\cdot10^{-20}}{0.3\cdot10^{-20}} = 6.26$$

6 Landau levels at $B = 10T$

$$DOS = \frac{m\omega}{2\pi\hbar} = 2.76\cdot10^{15}\ m^{-2}$$

$$n_{2D} = n\cdot8\cdot10^{-9} = 8\cdot10^{13}\ m^{-2}$$

1 level occupied, 3% filled.

b. $DOS = 3n_{2D} = 2.4\cdot10^{14}\ m^{-2} \Longrightarrow B = 10\cdot\frac{2.4\cdot10^{14}}{2.76\cdot10^{15}} = 0.87T$

$$\hbar\omega_c = 2.6\cdot10^{-22}\ J$$

$$E_1 + \hbar\omega_c = 3.36\cdot10^{-20}\ J \rightarrow E_1 = 3.33\cdot10^{-20}\ J$$

$$\rightarrow a = 50\ \mathring{A}$$