Physics 8.07, Fall 1999
Homework #3.1 (revised version)

Reading for Tuesday, September 28: Griffiths pp. 160–179

Reading for Thursday, September 30: Griffiths pp. 179–201

Problem Set #3.1 (revised version)

Due **Friday October 1** by 5:00 PM in the 8.07 homework box in 4-339B.

1. The electrostatic potential $V(x, y, 0)$ in the x-y plane is given by

$$V(x, y, 0) = \frac{1C}{4\pi\epsilon_0\sqrt{x^2 + y^2 + (1\text{m})^2}} .$$

There are no charges in the upper half-space $z > 0$. Find the potential $V(x, y, z)$ everywhere in the upper half-space $z > 0$.

2. A "pure" dipole with dipole moment $\mathbf{p} = p\hat{z}$ is placed at the center of a conducting sphere of radius R.

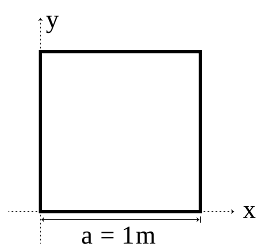
(a) Determine the potential everywhere in the interior of the sphere.

(b) Find the charge density induced on the sphere.

(c) What is the dipole moment of the induced charge on the sphere?

3. Griffiths, problem 3.18

4. The next two problems (4 and 5) are concerned with the following boundary value problem: A square pipe extends infinitely in the z direction. The sides of the square are of length 1 m and are positioned in the x-y plane as shown:



The potential V on the 4 sides of the pipe is independent of z and is given by (all distances x, y in this problem are measured in meters)

$$\begin{aligned} V(x,0) &= 0, \quad 0 \le x \le 1 \\ V(0,y) &= 0, \quad 0 \le y \le 1 \\ V(x,1) &= u(x), \quad 0 \le x \le 1 \\ V(1,y) &= w(y), \quad 0 \le y \le 1 \end{aligned}$$

where everywhere except in part 4a, $u(x) = \lambda x$ and $w(y) = \lambda y$ with $\lambda = 1$ V/m.

(a) Find an exact series expansion for the potential everywhere inside the pipe in terms of general boundary conditions $u(x)$ and $w(y)$.

(b) What is the form of this expansion for the particular functions $u(x) = \lambda x$, $w(y) = \lambda y$?

(c) Find a polynomial expression for the potential $V(x, y)$, either by simple reasoning (easier) or by summing the series (harder).

(d) Determine the exact value of the potential $V(0.5, 0.5)$ at the center of the pipe. (Hint: Use symmetry and linearity if you didn't get part (c).)

(e) Compute the partial sum of the first $2k$ modes (the first k pairs of modes) of the series at the center point and compare to your answer from (d) for $k = 1, 3, 5, 10, 20$.

(f) Graph the partial sums of the expansion for $k = 5$ and $k = 20$ along the following lines from $x = 0$ to $x = 1$:

  (i) $V(x, 1)$

  (ii) $V(x, x)$

  (iii) $V(x, 1 - x)$

  If you found the solution to (c) include the exact curves in your graph for comparison.

5. This problem involves a numerical solution of the previous boundary value problem. Using your favorite math package or programming language, write a program or set of function definitions that implement the relaxation algorithm on a square lattice of size $(N+1)\times(N+1)$ for $N = 4$ and $N = 10$. Start with initial conditions given by

$$\begin{aligned} u(n, m; 0) &= 0, \quad \text{if } n + m \leq N \\ u(n, m; 0) &= (n + m)/N - 1, \quad \text{if } n + m > N \ . \end{aligned}$$

(Note that these initial conditions agree with the boundary conditions from problem 4(b–f) using $x = n/N, y = m/N$.) At each time-step t calculate

$$u(n, m; t) = \frac{u(n + 1, m; t - 1) + u(n - 1, m; t - 1) + u(n, m + 1; t - 1) + u(n, m - 1; t - 1)}{4}$$

for the interior values $0 < n, m < N$ while keeping the boundary conditions fixed.

(a) Calculate the values of $u(N/2, N/2; t)$ at time-steps $t = 5, 10, 25, 50$ for $N = 4$ and $t = 10, 30, 100, 200$ for $N = 10$. Compare these to the predicted exact value from 4d. How many time-steps does it take for $u(N/2, N/2; t)$ to be within 1% of the exact value for each value of N?

(b) Plot your approximations to the function $V(x, 1 - x)$ on a graph, including the exact curve for comparison, for the same values of t and N as in part (a). (Put the data for $N = 4$ on one graph and make a second graph of the data for $N = 10$.)

(c) What is the largest value of N for which your computer can calculate N^2 time-steps on a grid of size $(N + 1)^2$ in 1 minute of real time? (You don't need to find an exact value for N — within a factor of 2 will be sufficient.)

(d) How does your answer to (a) change if all the initial values in the interior are set to 0?