Physics 8.07, Fall 1999
Homework #5

Reading for Tuesday, October 12: Griffiths pp. 234–254.

Reading for Thursday, October 14: Griffiths pp. 255–274.

Problem Set #5

Due **Thursday, October 14** by 9:30 AM in the 8.07 homework box in 4-339B.

1. Griffiths, problem 5.9.

2. Griffiths, problem 5.10.

3. Griffiths, problem 5.13.

4. Griffiths, problem 5.19.

5. A particle moves in the magnetic dipole field

$$\mathbf{B} = \alpha \frac{3(\hat{\mathbf{r}} \cdot \hat{z})\hat{\mathbf{r}} - \hat{z}}{r^3}$$

where $\alpha = 1000\,\mathrm{T} \cdot \mathrm{m}^3$.

(a) What is the strength of the magnetic field on the circle C of points defined by

$$x^2 + y^2 = (10\,\mathrm{m})^2\,, \quad z = 0\ ?$$

(b) What is the approximate frequency ω and radius R describing the cyclotron motion of a particle of charge $Q = 1\,\mathrm{C}$ and mass $m = 1\,\mathrm{kg}$ in this magnetic field if the particle has initial position $\mathbf{r} = (10\,\mathrm{m}, 0, 0)$ and initial velocity $\mathbf{v} = (1\,\mathrm{m/s}, 0, 0)$?

(c) What is the derivative $\partial_x B_z(10\,\mathrm{m}, 0, 0)$ of the magnetic field strength at the point $(10\,\mathrm{m}, 0, 0)$?

(d) Using the formula derived in class, calculate the approximate drift velocity of the particle from part (b) around the circle C. What is the frequency with which the orbit drifts around the circle?

6. In this problem we will do a numerical integration of the trajectory of a charged particle in the dipole field of problem 5. As mentioned in class, this is essentially a simulation (with different numerical constants) of a charged particle trapped in the Van Allen belts surrounding the earth. To develop the correct numerical technique, parts (a) and (b) refer to a particle in a constant magnetic field.

(a) The simplest algorithm for numerically integrating the Lorentz force law

$$\mathbf{F} = Q\Big(\mathbf{v} \times \mathbf{B}(\mathbf{x})\Big)$$

is the Euler method. In general, for a force law whose equation of motion is of the form

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$$

it is convenient to rewrite the equation of motion as a pair of first-order ODE's

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{F}(\mathbf{x}, \mathbf{v}) \ . \end{aligned}$$

In the Euler method we numerically approximate these ODE's with the discrete time analog

$$\begin{aligned} \mathbf{X}(n+1) &= \mathbf{X}(n) + \mathbf{V}(n)\Delta t \\ \mathbf{V}(n+1) &= \mathbf{V}(n) + \mathbf{F}\Big(\mathbf{X}(n), \mathbf{V}(n)\Big)\Delta t \end{aligned}$$

where $\mathbf{X}(n)$ approximates $\mathbf{x}(n\Delta t)$. Use the Euler method to approximate the motion of a charge 1 , mass 1 kg particle moving in a constant magnetic field $\mathbf{B} = -1T\hat{z}$, if the initial position and velocity of the particle are taken to be

$$\begin{aligned} \mathbf{X}(0) &= \mathbf{x}(0) = (1\,\text{m}, 0, 0) \\ \mathbf{V}(0) &= \mathbf{v}(0) = (0, 1\,\text{m/s}, 0) \end{aligned}$$

Use a time step $\Delta t = 0.01\,\text{s}$, and graph the trajectory of the particle in the x-y plane over the time of 10 cyclotron cycles. Is the particle executing perfect cyclotron motion according to the prediction of theory?

(b) As you saw in part (a), the Euler algorithm is numerically unstable and is not very accurate. An improved technique, known as Runge-Kutta, incorporates extra information about the derivatives of the force function by evaluating the force function at points along the trajectory. The simplest Runge-Kutta method, known as the "midpoint method" essentially involves using the force associated with the position and velocity of the particle halfway along the next time step in phase space. Mathematically, this algorithm is described by the equations

$$\begin{aligned}
\mathbf{W}(n+1) &= \mathbf{V}(n)\Delta t \\
\mathbf{U}(n+1) &= \mathbf{F}\Big(\mathbf{X}(n), \mathbf{V}(n)\Big)\Delta t \\
\mathbf{X}(n+1) &= \mathbf{X}(n) + \Big(\mathbf{V}(n) + \tfrac{1}{2}\mathbf{U}(n+1)\Big)\Delta t \\
\mathbf{V}(n+1) &= \mathbf{V}(n) + \mathbf{F}\left(\mathbf{X}(n) + \tfrac{1}{2}\mathbf{W}(n+1), \mathbf{V}(n) + \tfrac{1}{2}\mathbf{U}(n+1)\right)\Delta t
\end{aligned}$$

Note that in this algorithm the velocity and acceleration used to update the position and velocity, respectively, are taken at what would be the midpoint of the next Euler step.

Use this second-order Runge-Kutta algorithm to repeat the previous part of the problem and graph your results. You should see a much better cyclotron motion.

(c) Use the midpoint method to numerically integrate the trajectory of the charged particle of problem 5(b). Use a time step $\Delta t = 0.05\,\mathrm{s}$, and run to time $500\,\mathrm{s}$ (10,000 time steps). Graph the motion of the particle on the x-y plane. What is the approximate radius of the cyclotron motion of the particle? How does this relate to your answer for 5(b)? How long does it take for the particle to complete one rotation around the origin? How does this compare to your answer for 5(d)?

(d) Now give the particle some initial velocity along the z direction. For the same dipole configuration as in the previous part, start with initial position and velocity

$$\begin{aligned}
\mathbf{X}(0) &= \mathbf{x}(0) = (10\,\mathrm{m}, 0, 0) \\
\mathbf{V}(0) &= \mathbf{v}(0) = (0, 0.5\,\mathrm{m/s}, 0.7\,\mathrm{m/s})\ .
\end{aligned}$$

Use a time step $\Delta t = 0.005$ and run for $300\,\mathrm{s}$ (60,000 time steps). Graph your trajectory on the x-y plane, the θ-ϕ plane, and the s-z plane (s is the cylindrical coordinate $s = \sqrt{x^2 + y^2}$). Does the trajectory correspond with what you would expect for a charged particle moving in the Van Allen belts around Earth?