8.942 Pset #1 Solution

1. (Dodelson , 2−3)

(a)
$$g_{\mu\nu} = \begin{pmatrix} -(1+2\phi) & & & \\ & 1-2\phi & & \\ & & 1-2\phi & \\ & & & 1-2\phi \end{pmatrix}$$

$$\Gamma^0{}_{00} = \frac{1}{2} g^{00} \partial_0 g_{00} = \frac{1}{2}(1+2\phi)^{-1} 2 \frac{\partial\phi}{\partial t} \approx \frac{\partial\phi}{\partial t}$$

$$\Gamma^i{}_{00} = -\frac{1}{2} g^{ij} \partial_j g_{00} = \frac{1}{2}(1-2\phi)^{-1} \delta^{ij}\, 2 \frac{\partial\phi}{\partial x^j} \approx \delta^{ij} \frac{\partial\phi}{\partial x^j}$$

(b) We need one more connection

$$\Gamma^0{}_{i0} = \frac{1}{2} g^{00} \partial_i g_{00} = +\frac{1}{2}(1+2\phi)^{-1} 2 \frac{\partial\phi}{\partial x^i} \approx \frac{\partial\phi}{\partial x^i}$$

In nonrelativistic limit,

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu = (1+2\phi)\, dt^2 - (1-2\phi)(d\vec{x})^2$$
$$\approx dt^2 (1 - v^2 + 2\phi) \quad , \text{where } v^2 = \left(\frac{d\vec{x}}{dt}\right)^2$$

So

$$\frac{dt}{d\tau} \approx 1 + \frac{1}{2}v^2 - \phi$$

$$\frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} \approx \frac{dx^i}{dt}$$

$$\frac{d^2t}{d\tau^2} = \frac{d}{dt}\left(\frac{dt}{d\tau}\right)\frac{dt}{d\tau} \approx \frac{d}{dt}\left(1+\frac{1}{2}v^2-\phi\right) = \frac{1}{m}\frac{d}{dt}(p^0 - m\phi), \text{ where } p^0 \text{ is the particle's energy.}$$

the time component of geodesic equation reads

$$\frac{d^2t}{d\tau^2} + \Gamma^0{}_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\frac{1}{m}\frac{d}{dt}(p^0 - m\phi) + \Gamma^0{}_{00}\left(\frac{dt}{d\tau}\right)^2 + 2\Gamma^0{}_{i0}\frac{dx^i}{d\tau}\frac{dt}{d\tau} = 0$$

(note: $\Gamma^0{}_{ij} \sim \frac{\partial\phi}{\partial t}\delta_{ij}$, so $\Gamma^0{}_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \sim \frac{\partial\phi}{\partial t}v^2$ is a higher order term)

$$\text{LHS} \approx \frac{1}{m}\frac{d}{dt}(p^0 - m\phi) + \frac{\partial\phi}{\partial t} + 2\frac{\partial\phi}{\partial x^i}\frac{dx^i}{dt}$$
$$= \frac{1}{m}\frac{d}{dt}(p^0 - m\phi) + 2\left(\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x^i}\frac{dx^i}{dt}\right) - \frac{\partial\phi}{\partial t}$$
$$= \frac{1}{m}\frac{d}{dt}(p^0 - m\phi + 2m\phi) - \frac{\partial\phi}{\partial t}$$
$$= \frac{1}{m}\frac{d}{dt}(p^0 + m\phi) - \frac{\partial\phi}{\partial t}$$

$$\Rightarrow \quad \frac{1}{m}\frac{d}{dt}(p^0 + m\phi) = \frac{\partial\phi}{\partial t} .$$

If the potential ϕ does not explicitly depend on time, $\partial\phi/\partial t = 0$,

then $\frac{d}{dt}(p^0 + m\phi) = 0$,

that is, $p^0 + m\phi$ is conserved.

(c) $$\frac{d^2x^i}{d\tau^2} = \frac{d}{dt}\left(\frac{dx^i}{d\tau}\right)\frac{dt}{d\tau} \approx \frac{d^2x^i}{dt^2}$$

The space components of the geodesic equation reads

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\text{LHS} \approx \frac{d^2x^i}{dt^2} + \delta^{ij}\frac{\partial\phi}{\partial x^j}\left(\frac{dt}{d\tau}\right)^2$$

$$\approx \frac{d^2x^i}{dt^2} + \delta^{ij}\frac{\partial\phi}{\partial x^j}$$

(Note: $\Gamma^i{}_{0j} \sim \frac{\partial\phi}{\partial t}\delta^i_j$, $\Gamma^i{}_{jk} \sim \frac{\partial\phi}{\partial\vec{x}}$,

so $\Gamma^i{}_{0j}\frac{dt}{d\tau}\frac{dx^j}{d\tau}$ and $\Gamma^i{}_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau}$ are high order terms.)

So,

$$\frac{d^2x^i}{dt^2} = -\delta^{ij}\frac{\partial\phi}{\partial x^j}.$$

2. RW metric

- Define conformal time s.t. $d\tau = \frac{dt}{a(t)}$. Call $a(\tau) = a(t)$.

  Introduce a new coordinate χ so that $d\chi^2 = \frac{dr^2}{1-kr^2}$.

  So

$$ds^2 = a^2(\tau)\left[-d\tau^2 + d\chi^2 + r^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

For $K>0$,

(for upper hemisphere in 4D space) (for the whole sphere in 4D)

$$\chi \doteq \frac{1}{\sqrt{K}} \arcsin(\sqrt{K}r), \quad \text{or} \quad r(\chi) \doteq \frac{1}{\sqrt{K}} \sin(\sqrt{K}\chi)$$

Note that $r \to \frac{1}{\sqrt{K}}$ is a coordinate singularity, not physical.

thus $0 \leq \chi \leq \frac{\pi}{\sqrt{K}}$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$

For $K<0$,

$$\chi = \frac{1}{\sqrt{|K|}} \operatorname{arcsinh}(\sqrt{|K|}r), \quad \text{or} \quad r(\chi) = \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}\chi)$$

$0 \leq \chi < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$

For $K=0$,

$$\chi = r, \quad \text{or} \quad r(\chi) = \chi$$

$0 \leq \chi < \infty$

- Introduce a new coordinate $\bar{r}$ so that $r = \frac{\bar{r}}{1+\frac{1}{4}k\bar{r}^2}$;

$$dr = \frac{1-\frac{1}{4}k\bar{r}^2}{(1+\frac{1}{4}k\bar{r}^2)^2}\, d\bar{r}, \quad 1-kr^2 = \frac{(1-\frac{1}{4}k\bar{r}^2)^2}{(1+\frac{1}{4}k\bar{r}^2)^2}$$

So

$$\frac{dr^2}{1-kr^2} = \frac{d\bar{r}^2}{(1+\frac{1}{4}k\bar{r}^2)^2}$$

So

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \frac{d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)}{(1+\frac{1}{4}k\bar{r}^2)^2}\right]$$

For $K>0$,

$$\bar{r} = \frac{2}{kr}\left(1+\sqrt{1-kr^2}\right), \quad \frac{2}{\sqrt{K}} < \bar{r} < \infty, \text{ for the lower hemisphere}$$

and

$$\bar{r} = \frac{2}{kr}\left(1-\sqrt{1-kr^2}\right); \quad 0 \leq \bar{r} < \frac{2}{\sqrt{K}}, \text{ for the upper hemisphere in 4D space}$$

For $k<0$, $\qquad \bar{r} = \frac{2}{|k| r}\left(\sqrt{1+|k| r^2} - 1\right)$; $\quad 0 \leq \bar{r} < \frac{2}{\sqrt{|k|}}$

For $k=0$, $\qquad \bar{r} = r$, $\qquad 0 \leq \bar{r} < \infty$

- let
$$x = \bar{r} \sin\theta \cos\phi$$
$$y = \bar{r} \sin\theta \sin\phi$$
$$z = \bar{r} \cos\theta$$

So that $\quad dx^2+dy^2+dz^2 = d\bar{r}^2 + \bar{r}^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$, and $\quad x^2+y^2+z^2 = \bar{r}^2$

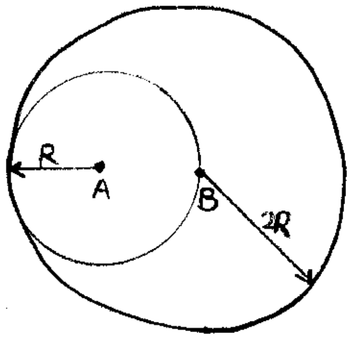
So
$$ds^2 = -dt^2 + a^2(t) \frac{(dx^2+dy^2+dz^2)}{\left[1+\frac{1}{4}k(x^2+y^2+z^2)\right]^2}$$

For $k>0$ and $k=0$, $\qquad -\infty < x, y, z < \infty$

For $k<0$, $\qquad -\frac{2}{\sqrt{|k|}} < x, y, z < \frac{2}{\sqrt{|k|}}$

3. Isotropy implies Homogeneity

a)



Consider a spherical shell of radius R around the center A. Point B is on this shell. By isotropy with respect to A, this shell is uniform at the same time. The points on this shell are away from point B with distance ranging from 0 to 2R. Now, isotropy with respect to B implies the homogeneity within a sphere of radius 2R around the center B. By infinitely increasing the size R, one concludes that the whole universe must be homogeneous if it is isotropic at all locations at the same time.

b) Isotropy implies not only that there is only radius-dependence in the spatial coordinates, but that there is no preferred axis for other attributes. For the Einstein tensor, this means that $G_i^{\ j} = f(t,r)\,\delta_i^{\ j}$.

Apply Bianchi identity for 3D hypersurface.

$$0 = \nabla_j G_i^{\ j} = (\partial_j f)\,\delta_i^{\ j} = \partial_i f \quad , \text{ for all } i.$$

On the hypersurface, $G_i^{\ j} = R_i^{\ j} - \frac{1}{2}\delta_i^{\ j} R^{(3D)}$, where $R^{(3D)} \equiv R_i^{\ i}$.

$$G^{(3D)} \equiv G_i^{\ i} = R^{(3D)} - \frac{3}{2}R^{(3D)} = -\frac{1}{2}R^{(3D)}$$

so

$$R_i^{\ j} = G_i^{\ j} - \delta_i^{\ j} G^{(3D)}$$

so

$$\partial_k R_i^{\ j} = (\partial_k f)\delta_i^{\ j} - \delta_i^{\ j}\,\partial_k(3f) = 0$$

that is, the curvature must be uniform on the hypersurface.

A more rigorous way is to consider all spacetime indices. Then $G_i^{\ j} = f(t,r)\,\delta_i^{\ j}$, $G_0^{\ i} = 0$, $G_0^{\ 0}(t,r)$, by isotropy. Since $G_0^{\ 0}$ is a scalar on the hypersurface, the argument in part (a) applies to $G_0^{\ 0}$, so $G_0^{\ 0} = G_0^{\ 0}(t)$. Apply Bianchi identity $0 = \nabla_\mu G_i^{\ \mu} = \nabla_j G_i^{\ j} + \Gamma^0_{\ 0j} G_i^{\ j} - \Gamma^0_{\ 0i} G_0^{\ 0}$. The connection $\Gamma^0_{\ 0i} = 0$ because it is odd under parity. So, again we get $\nabla_j G_i^{\ j} = 0$, and this implies $\partial_i f = 0$. Since $G_\mu^{\ \nu} = R_\mu^{\ \nu} - \frac{1}{2}\delta_\mu^{\ \nu} R$, we have $G \equiv G_\nu^{\ \nu} = -R$, so $R_i^{\ j} = G_i^{\ j} - \frac{1}{2}\delta_i^{\ j}(G_k^{\ k} + G_0^{\ 0})$. Again we have $\partial_k R_i^{\ j} = 0$.

4 Rotating Flat Space time

(a) $$ds^2 = -dt^2 + dr^2 + dz^2 + r^2(d\phi' - \omega dt)^2$$

$$= -(1-\omega^2 r^2)\,dt^2 + dr^2 + dz^2 + r^2 d\phi'^2 - 2\omega r^2\, dt\, d\phi'$$

In new coordinates $x^\mu = (t, r, z, \phi')$

$$g_{\mu\nu} = \begin{pmatrix} -1+\omega^2 r^2 & & & -\omega r^2 \\ & 1 & & \\ & & 1 & \\ -\omega r^2 & & & r^2 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & -\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\omega & 0 & 0 & \frac{1}{r^2} - \omega^2 \end{pmatrix}$$

(b) The non-zero Christoffel symbols are as follows.

$$\Gamma^1{}_{00} = -\omega^2 r$$

$$\Gamma^1{}_{03} = \Gamma^1{}_{30} = \omega r$$

$$\Gamma^1{}_{33} = -r$$

$$\Gamma^3{}_{01} = \Gamma^3{}_{10} = -\omega/r$$

$$\Gamma^3{}_{13} = \Gamma^3{}_{31} = 1/r$$

(c) $$\left.\begin{aligned} R_{\mu\nu} &= 0, \\ R &= 0 \end{aligned}\right\}$$ This is expected since the spacetime is manifestly flat in old coordinate system.

(d) Thus $$G_{\mu\nu} = 0 = 8\pi G\, T_{\mu\nu}$$

So $$T_{\mu\nu} = 0$$