8.942 Pset #2 Solution

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1. Quasar Absorption Line Statistics

$$n(z) = n_0 (1+z)^3 \;,\quad \text{where } n_0 = 10^{-3}\ \mathrm{Mpc}^{-3}$$

$$\frac{dl}{dz} = \frac{c\,dt}{dz} = \frac{c\,da}{aH\,dz} = -\frac{c}{(1+z)H(z)} = -\frac{D_H}{(1+z)E(z)} ,$$

where $D_H = \frac{c}{H_0}$, $E(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}$

For

$$\frac{d\tau}{dz} = 3 \quad \text{at } z = 2 ,$$

$$\sigma = \frac{|d\tau/dz|}{n(z)\,|dl/dz|} = \frac{|d\tau/dz|\,E(z)}{n_0 D_H\,(1+z)^2} = \frac{3\times\sqrt{0.3\times 3^3 + 0.7}}{10^{-3}\times 3000\times(0.7)^{-1}\times 3^2} \times 10^6\,\mathrm{kpc}^2$$

$$= 2.3\times 10^5\ \mathrm{kpc}^2 .$$

2. Direct observations of relative expansion

For the same particular object (assuming no peculiar velocity / acceleration), it has fixed comoving distance from the observer. The radiation emitted at two times t_1 and $t_1+\Delta t_1$ will be observed at later times t_0 and $t_0+\Delta t_0$, related by

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{dt}{a(t)}\,, \qquad (1)$$

where $a(t) = (1+z)^{-1}$ is normalized to unity at zero redshift when $t = t_0$.

For $\Delta t/t \ll 1$ (and thus $\Delta z/z \ll 1$), (1) yields $\Delta t_1 = \frac{a(t_1)}{a(t_0)} \Delta t_0$.

The observed redshift drift

$$\Delta z^{obs.} = \frac{a(t_0+\Delta t_0)}{a(t_1+\Delta t_1)} - \frac{a(t_0)}{a(t_1)}$$

$$\approx \left[\frac{a(t_0)}{a(t_1)} H(t_0) - H(t_1)\right] \Delta t_0$$

For ΛCDM model, $H(z) = H_0 \sqrt{\Omega_M (1+z)^3 + \Omega_\Lambda}$ (for flat universe)

$$\Delta z^{obs.} \approx H_0 \left[1+z_1 - \sqrt{\Omega_M (1+z_1)^3 + \Omega_\Lambda} \right] \Delta t_0$$

So, to detect $\Delta z^{obs.} \sim 10^{-6}$, one has to wait for

$$\Delta t_0 = \frac{\Delta z^{obs.}}{H_0 \left[1+z_1 - \sqrt{\Omega_M (1+z_1)^3 + \Omega_\Lambda} \right]} \approx 5.8 \times 10^4 \text{ yrs}.$$

Alternatively, if $\Delta z^{obs.} \sim \frac{10^{-6}}{\sqrt{N}} = \left(\frac{10^{-6}}{\Delta t_0}\right) \Delta t_0'$

$$N = \left(\frac{\Delta t_0}{\Delta t_0'}\right)^2 = \left(\frac{58490}{1}\right)^2 \sim 3.4 \times 10^9 .$$

3. Supernova rates

(a)

The comoving horizon is $D = \int_0^{t_0} \frac{c\,dt}{a(t)} = \int_0^1 \frac{c\,da}{a^2 H} = D_H \int_0^{\infty} \frac{dz'}{E(z')}$

where $D_H = \frac{c}{H_0} = 3000\, h^{-1}\, \mathrm{Mpc}$,

$E(z) = \sqrt{\Omega_M (1+z)^3 + \Omega_\Lambda}$. (for flat universe).

For flat universe, the comoving horizon volume is

$$V_c = \frac{4\pi}{3} D^3$$

So, the Supernova rate in the present 3D hypersurface of constant time is

$$SNR_{II} \times V_c$$

$$= SNR_{II} \times \frac{4\pi}{3} \left[D_H \int_0^{\infty} \frac{dz'}{\sqrt{\Omega_M (1+z')^3 + \Omega_\Lambda}} \right]^3$$

$$\approx 5.5 \times 10^8 \ \mathrm{SNe/yr} .$$

\# explosion in an hour $\approx \frac{5.5 \times 10^8}{365 \times 24} = 6.3 \times 10^4$

(b) The physical volume of a shell at scale factor a is (for a flat universe)

$$dV(a) = 4\pi \left(a D(a) \right)^2 c\, dt$$

where $D(a)$ = comoving distance at a

$= \int_a^1 \frac{c\, da'}{a'^2 H(a')} = D_H \int_a^1 \frac{da'}{a'^2 E(a')} \equiv D_H\, g(a)$, where $g(a) \equiv \int_a^1 \frac{da'}{a'^2 E(a')}$

$a D(a)$ = physical distance at a

$c\, dt = \frac{c\, da}{a H(a)}$

so,

$$dV(a) = \frac{4\pi a D^2(a)\, c\, da}{H(a)} = \frac{4\pi D_H\, a\, D^2(a)\, da}{E(a)} = \frac{4\pi D_H^3\, a\, g^2(a)\, da}{E(a)}$$

So, Supernova rate in past light cone

$$= \frac{\text{events}}{\text{conformal time}} = \sum_a \frac{\text{events}}{(\text{phys. vol}) \times (\text{phys. time})} \times (\text{phys. vol.}) \times \underset{\substack{\uparrow \\ \text{time dilation}}}{a}$$

$$= \int SNR_{II}(a)\, dV(a) \times a$$

$$= D_H^3 \int_0^1 4.6\times10^{-5}\left(\frac{SNe}{yr\, Mpc^3}\right)\left[x^5 e^{-x}/2 + (1-e^{-x})\right] \times \frac{4\pi a^2 g^2(a)\, da}{E(a)}$$

$$\equiv \int_0^1 f(a)\, da,$$

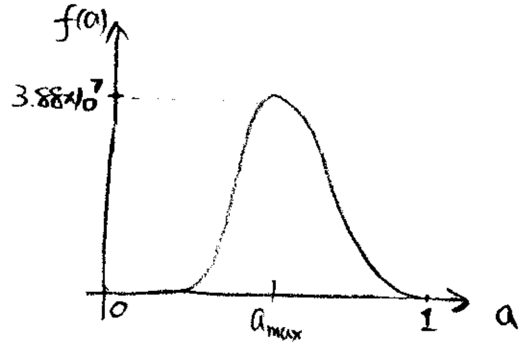
where $$f(a) = \left(4.6\times10^{-5} \times (3000 h^{-1})^3 \frac{SNe}{yr}\right) \times \frac{4\pi a^2 g^2(a)}{E(a)} \left[x^5 e^{-x}/2 + (1-e^{-x})\right],$$

$$x = 20.3\, a^{3/2}$$

Numerical integration gives SNe rate $= 1.1\times10^7$ SNe/yr.

($\simeq 1.2\times10^3$ SNe/hr.)

(c)



Numerically solve $f'(a) = 0$ gives $a_{max} = 0.428$, which contributes to the rate in (b) the greatest.

4 Distance Ratios

For Row 1 and 2, $D_L = (1+z) D_C$ (flat universe)

where $$D_C = D_H \int_0^z \frac{dz'}{E(z')},$$

and $$E(z) = \sqrt{\Omega_M (1+z)^3 + \Omega_\Lambda}$$

For Row 3, $D_L = (1+z) D_C$ (flat universe)

but $$E(z) = \sqrt{\Omega_M (1+z)^3 + \Omega_\Lambda (1+z)^{3(1+w_{DE})}}$$

For Row 4, $D_L = (1+z) D_H \frac{1}{\sqrt{\Omega_K}} \sinh\left[\sqrt{\Omega_K}\, D_C/D_H\right]$, (for $\Omega_K > 0$)

and $E(z) = \sqrt{\Omega_M (1+z)^3 + \Omega_K (1+z)^2 + \Omega_\Lambda}$

Ω_M	Ω_{DE}	W_{DE}	Ω_K	$d_L\,(z=0.1)$	$d_L\,(z=0.5)$	$\frac{d_L(z=0.5)}{d_L(z=0.1)}$
0.30	0.70	-1	0	460.6	2834.9	6.15
0.25	0.75	-1	0	462.4	2885.8	6.24
0.30	0.70	-0.9	0	458.4	2789.8	6.09
0.30	0.65	-1	0.05	459.6	2811.8	6.12

(Distances in Mpc)

Suppose we use a particular type of source as our standard candle, two of which are located at $z=0.5$ and $z=0.1$. Its apparent magnitude m, absolute magnitude M and distance are related by

$$5 \log_{10} \frac{D_L}{\text{kpc}} = m - M - 10$$

So

$$5 \log_{10} \frac{D_L(z=0.5)}{D_L(z=0.1)} = m(z=0.5) - m(z=0.1)$$

The four models above can be distinguished if $\Delta\left(\frac{d_L(0.5)}{d_L(0.1)}\right) \lesssim 0.03$, which maps to the precision in m as

$$\sqrt{2}\,\Delta m = 5 \times \frac{1}{\left|\frac{D_L(0.5)}{D_L(0.1)}\right| \ln 10} \Delta\left(\frac{D_L(0.5)}{D_L(0.1)}\right).$$

(The factor of $\sqrt{2}$ is due to the assumption that two measurements of apparent magnitude are uncorrelated.) So $\Delta m \lesssim \frac{5}{\sqrt{2}} \times \frac{0.03}{\ln 10 \times 6.15} = 0.0075$.

(or $\sqrt{2}\,\Delta m \sim 0.01$)