Problem Set #3

(37/40)

Due: Thursday, March 9, 2006, in class

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1. Neutrino Mass Densities

Consider three light (but not massless) neutrinos with degenerate masses ($m_{\nu_e} \simeq m_{\nu_\mu} \simeq m_{\nu_\tau}$).

a) Calculate the mass of the neutrinos so that the current total cosmological mass density in neutrinos is equal to $\Omega_\nu h^2 = 0.12$, where h is the dimensionless Hubble Constant, $H_0 = 100\,h\,\mathrm{km/s/Mpc}$.

b) Calculate the scale factor, a, when neutrinos of this mass become non-relativistic, by equating the energy density contributed by the mass to that contributed by the neutrino momentum.

c) How far in comoving distance do the neutrinos travel between a scale factor of $a = 10^{-9}$ and the scale factor calculated above. You may assume a flat universe with the energy densities contributed just by photons and neutrinos.

2. Comoving Entropy Densities

a) Find an expression for the comoving entropy density, S, for a non-relativistic weakly interacting gas in local thermal equilibrium with temperature, T. Show that this is negligible compared to a relativistic gas at the same temperature.

b) Find an expression for the entropy of vacuum energy with $\rho = -P$. Can you explain this behavior?

3. Quark/anti-quark Asymmetry

The baryon-to-photon ratio, η, is defined as the ratio at the conclusion of Big Bang Nucleosynthesis. Current measurements of light elements as well as data from WMAP constrain this ratio to approximately $\eta = 6 \times 10^{-10}$. Relate this ratio to the quark/anti-quark asymmetry (consider just up and down quarks) present before the quark/hadron transition: $\delta_q = \frac{q}{\bar{q}} - 1$. Assume 3 quarks per baryon, and account for relativistic particles that will contribute to the photon number density between quark/anti-quark annihilation and electron/positron annihilation.

4. Expansion History with Decaying Dark Matter

a) Calculate the relic abundance of a Weakly Interacting Massive Particle (WIMP) with a rest mass of 250 GeV. Assume a thermally averaged cross-section of $< \sigma\, v >= 10^{-25}\,\mathrm{cm^3/s}$ and the effective relativistic degrees of freedom, $g_* = 100$. Express your answer as a comoving mass density, $\rho * a^3$.

b) If the WIMP is unstable and decays (with end products of photons) with a rate of 0.2 Gyr^{-1}, what is the present cosmological energy density of WIMPs? For this part, you can assume the age of the universe is 13.7 Gyr.

c) Calculate and plot the age of the universe starting from the epoch of matter/radiation equality as a function of scale factor. You may assume a flat universe with energy contribution from WIMPS, photons and massless neutrinos, and that $H_0 = 70\,\mathrm{km/s/Mpc}$. Include a plot of ρ_γ and ρ_{WIMP} as a function of scale factor.

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1. Neutrinos

(a) The energy density of three generations of massive ν and $\bar{\nu}$ is

$$\rho_\nu = m_\nu n_\nu \quad , \text{ assuming non-relativistic.}$$

Since both ν and photons have the same behavior $n \propto a^{-3}$, we can calculate n_ν by considering $\frac{n_\nu}{n_\gamma}$ when ν is relativistic ($T_\gamma \gg m_\nu$).

$$\frac{n_\nu}{n_\gamma} = \frac{(3/4) \times 2 \times 3}{2} \times \left(\frac{T_\nu}{T_\gamma}\right)^3$$

$$= \frac{9}{4} \times \left(\frac{4}{11}\right) = \underset{\substack{\uparrow \\ 3 \text{ generations}}}{3} \times 3/11$$

Since $$(n_\gamma)_{today} = \frac{\zeta(3)}{\pi^2} \times 2 \times T_\gamma^3$$

$$= \frac{1.202}{\pi^2} \times 2 \times \left(\frac{2.725\ k}{11605\,k/eV}\right)^3 ,$$

and $$\rho_{crit} = 8.098\, h^2 \times 10^{-11}\ eV^4 ,$$

$$\Omega_\nu = \frac{\left(\frac{m_\nu}{eV}\right) \times \frac{1.202 \times 2}{\pi^2} \left(\frac{2.725}{11605}\right)^3 \times \frac{9}{11}\, eV^4}{8.098\, h^2 \times 10^{-11}\ eV^4} = (m_\nu/eV)\, h^{-2} \times 0.0319 ,$$

or $$m_\nu = 31.4\, eV \times \Omega_\nu h^2 .$$

(Note : if there is only one generation of massive ν ,

$$m_\nu = 3 \times 31.4\, eV\ \Omega_\nu h^2 = 94\, \Omega_\nu h^2\ eV \quad)$$

For $\Omega_\nu h^2 = 0.12$, $m_\nu = 31.4 \times 0.12 = 3.77\ eV$

(b) Energy density contributed by the momentum is

$$\rho_{\nu(p)} = \frac{g_\nu}{(2\pi)^3} \int \frac{d^3p}{e^{E(p)/T}+1}\, p \approx \frac{7}{8} \times \frac{\pi^2}{30}\, g_\nu T_\nu^4 ,$$

while that contributed by the mass is

$$\rho_{\nu(m)} = m_\nu n_\nu \approx m_\nu \times \frac{3}{4} \frac{\zeta(3)}{\pi^2}\, g_\nu T_\nu^3 .$$

$$\rho_{\nu(p)} = \rho_{\nu(m)} \Rightarrow T_\nu = \frac{180\,\zeta(3)}{7\pi^4} m_\nu$$

Note that $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma = \left(\frac{4}{11}\right)^{1/3} \times \frac{2.725}{11605}\,\mathrm{eV} \times a_*^{-1}$.

So $$a_* = \left(4/11\right)^{1/3} \frac{2.725}{11605} \times \frac{7\pi^4}{180\,\zeta(3)} \times \frac{1}{m_\nu/\mathrm{eV}}$$

$$= 1.4 \times 10^{-4}$$

(c) For $a < a_*$, $H(a) \approx H_0\sqrt{\Omega_\gamma a^{-4} + \Omega_{\nu 0} a_* a^{-4}} = H_0\sqrt{\Omega_\gamma + \Omega_{\nu 0} a_*}\, a^{-2}$.

The second term in $\sqrt{\ }$ is because, as $a < a_*$, density $\propto a^{-4}$, while density $\propto a^{-3}$ as $a > a_*$. So, $\Omega_{\nu 0} = \Omega_\nu^* a_*^3$, and $\Omega_\nu a^4 = \Omega_\nu^* a_*^4$ $\Rightarrow$ $\Omega_\nu = \Omega_{\nu 0} a_* a^{-4}$. The comoving distance

$$\chi = \int_{10^{-9}}^{a_*} \frac{c\,da}{a^2 H(a)} = \frac{c}{H_0}\left(\Omega_\gamma + \Omega_{\nu 0} a_*\right)^{-1/2} \left(a_* - 10^{-9}\right)$$
$$\approx \frac{c}{H_0}\left(\Omega_\gamma + \Omega_{\nu 0} a_*\right)^{-1/2} a_*$$

Note that $h^2 = 1 \times h^2 = (\Omega_\nu + \Omega_\gamma) h^2 = \Omega_\nu h^2 + \Omega_\gamma h^2$,

and $\Omega_{\nu 0} h^2 = 0.12$

$\Omega_\gamma h^2 = 2.47 \times 10^{-5}$ (Dodelson eq (2.70))

So $h^2 \approx 0.12 \Rightarrow h = 0.346$

$\Rightarrow \Omega_{\nu 0} \approx 1.0$, $\Omega_\gamma \approx 2.058 \times 10^{-4}$

$\Rightarrow \chi = 3000\,h^{-1}\,\mathrm{Mpc} \times \left(2.058 \times 10^{-4} + 1.0 \times 1.4 \times 10^{-4}\right)^{-1/2} \times 1.4 \times 10^{-4}$

$= 65\ \mathrm{Mpc}$

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2. Comoving Entropy Densities

(a) $$S_{\text{non-rel}} = \frac{(\rho + P)}{T} a^3$$

$$= \frac{(nm + nT)}{T} a^3 \quad \text{, for non-relativistic weakly interacting gas}$$

$$= n\left(\frac{m}{T} + 1\right) a^3$$

$$\approx \frac{nma^3}{T} \quad \text{, for non-relativistic weakly interacting gas}$$

$$= g\left(\frac{mT}{2\pi}\right)^{3/2} \frac{m}{T} a^3 e^{-(m-\mu)/T} \propto T^{-5/2} e^{-(m-\mu)/T}$$

while $S_{\text{rel}} = \left(\frac{\pi^2}{30}\right)\frac{4}{3} g_* T^3 a^3 = \text{constant}$, for relativistic gas.

So, for the same T, $S_{\text{non-rel}}$ is exponentially suppressed and thus can be ignored compared to S_{rel}.

(b) $$S_{\text{vac}} = \frac{(\rho + P)}{T} a^3 = 0 \quad \text{, for vacuum energy } P = -\rho$$

but what about $T=0$? — excluded by therm. 3rd law.

This is because vacuum energy has only one state (the ground state).

$\Rightarrow$ $S = \ln \Omega = 0$

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3. $q/\bar{q}$ Asymmetry

OK for 10 MeV, Quark/Hadron transition is at ≈ 1 GeV

The mass of u and d quarks are 4–8 MeV. So consider the $q\bar{q}$ annihilation at the energy scale of ~ 10 MeV. The relativistic gas includes $\mu^\pm$, $\tau^\pm$, [illegible] $+ \gamma$, $e^\pm$, ν_e, ν_μ, ν_τ, u, d (and $\bar{\nu}$s, $\bar{u}$, $\bar{d}$), gluons, before them and γ, $e^\pm$, ν_e, ν_μ, ν_τ (and $\bar{\nu}$s) after them.

The conservation of comoving entropy reads $g_{*s}^{(\text{before})} T_{\text{before}}^3 a_{\text{before}}^3 = g_{*s}^{(\text{after})} T_{\text{after}}^3 a_{\text{after}}^3$

where $$g_*^{(\text{before})} = 2 + \frac{7}{8} \times 4 \times 3 + \frac{7}{8} \times 6 + \frac{7}{8} \times 24 + 2 \times 8 = \cancel{191/4}\ [illegible]\ 219/4$$

$$g_*^{(\text{after})} = 2 + \frac{7}{8} \times 4 + \frac{7}{8} \times 6 = 43/4$$

$$\Rightarrow (T_{\text{before}}/T_{\text{after}})^3 = g_*^{(\text{after})}/g_*^{(\text{before})} = \cancel{43/191}\ 43/219$$

The baryon-to-photon ratio $\eta = \dfrac{N_{\Delta q}/3}{n_\gamma^{(after)} a_{after}^3}$, (where N is comoving density.)

So, the $q/\bar{q}$ asymmetry before the annihilation

$$\delta_q = \frac{N_{\Delta q}}{n_q^{(before)} a_{before}^3} = 3\eta \frac{n_\gamma^{(after)} a_{after}^3}{n_q^{(before)} a_{before}^3}$$

$$= 3\eta \frac{2 \times (T_{after} a_{after})^3}{3/4 \times 24 \times (T_{before} a_{before})^3}$$

$$= \frac{\cancel{8}\,\cancel{181}}{43\,\cancel{129}} \eta \approx \cancel{1.48\eta} = \cancel{8.9 \times 10^{-10}}.$$

$$1.70\eta = 10.2 \times 10^{-10}.$$

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4. Decaying Dark Matter

(a) The comoving mass density $\rho_X = \rho_1 a_1^3 = m n_1 a_1^3$, where a_1 is a time sufficiently late that $Y \approx Y_\infty$. $n_1 = Y_\infty T_1^3 = \dfrac{x_f}{\lambda} T_1^3 = \dfrac{x_f H(m)}{m^3 \langle \sigma v \rangle} T_1^3$

$H(m) = \left(\dfrac{8\pi G}{3}\right)^{1/2} \left(\dfrac{\pi^2}{30} g_*(m)\, m^4\right)^{1/2}$, where $g_*(m)$ includes all relativistic particles at $T = m$; $g_*(m) \approx 100$.

So, $$\rho_X = \left(\frac{4\pi^3 G g_*}{45}\right)^{1/2} \frac{x_f}{\langle \sigma v \rangle} \left(\frac{T_1 a_1}{T_0}\right)^3 T_0^3$$

Note that photons are heated by the annihilation of particles with masses between 1 MeV and 250 GeV; so the conservation of entropy tells that

$$\left(\frac{T_1 a_1}{T_0}\right)^3 = \frac{g_*(today)}{g_*(T_1 \sim 10\,GeV)} = \frac{\{\gamma, \nu_e, \nu_\mu, \nu_\tau\}}{\{\gamma, e^\pm, \mu^\pm, \tau^\pm, \nu_e, \nu_\mu, \nu_\tau \,\&\, \bar{\nu}s, u, d, s, c, b, \text{gluons}\}}$$

$$\approx \frac{2 + \frac{7}{8} \times 6 \times \left(\frac{4}{11}\right)}{100} = 0.039 \;\left(\simeq \frac{1}{25.6}\right)$$

So $$\rho_X = \left(\frac{4\pi^3 G g_*}{45}\right)^{1/2} \frac{x_f T_0^3}{25.6 \langle \sigma v \rangle}$$

estimate x_f with $H(x_f) = \Gamma(x_f)$.

for $g_* \approx 100$, $\langle \sigma v \rangle = 10^{-25}\ cm^3 s^{-1}$, $x_f \sim 20$. $T_0 = 2.725\,K$

$\rho_X = \cancel{1.86}\ 3.8 \times 10^{-30}\ g\ cm^{-3}$ $\qquad$ $H(x_f) = \Gamma(x_f)$ gives $x_f \approx 24$!

or $\rho_X / \rho_{cr} \approx$ ~~[illegible]~~ $0.02\, h^{-2}$

(b) $$\Omega_{WIMP} = (\rho_X/\rho_{cr.})\, e^{-\alpha t_{age}}$$

$$= 0.01 h^{-2} \times e^{-13.7\times 0.2}$$

$$= 6\times10^{-4} h^{-2}$$

(c) At the epoch of matter/radiation equality,

$$\rho_{WIMP} = \rho_\gamma + \rho_\nu \equiv \rho_{eq}$$

The comoving density is $\Omega_0 = \frac{1}{\rho_{cr.}}\left(\rho_{WIMP} a_{eq}^3 + \rho_{rad}\, a_{eq}^4\right)$

$$\approx \rho_{eq}\, a_{eq}^3 / \rho_{cr.}$$

Assume $\Omega_0 = 1$, $h = 0.7$, and $a_{eq} \sim 10^{-3}$; then $\rho_{eq}/\rho_{cr.} = a_{eq}^{-3}$

At $t = 0$, $a \approx a_{eq}$

$$\rho_{WIMP} = \rho_{eq}$$

$$\rho_\gamma = \frac{8}{29}\rho_{eq}, \quad \rho_\nu = \frac{21}{29}\rho_{eq}, \quad \text{since } \frac{\rho_\gamma}{\rho_\nu} = \frac{2}{\frac{7}{8}\times 6} = \frac{8}{21}$$

When $t > 0$,

$$\rho_{WIMP}(t) = \rho_{eq}\, e^{-\alpha t}\left(\frac{a_{eq}}{a}\right)^3 = \rho_{cr.}\, e^{-\alpha t} a^{-3}$$

$$\rho_\gamma(t) = \left[\frac{8}{29}\rho_{eq}\, a_{eq}^4 + \rho_{eq}(1 - e^{-\alpha t})\, a_{eq}^3\right] a^{-4}$$

$$= \left(\frac{8}{29} a_{eq} + 1 - e^{-\alpha t}\right)\rho_{cr.}\, a^{-4}$$

$$\rho_\nu(t) = \frac{21}{29}\rho_{eq}(a_{eq}/a)^4 = \frac{21}{29}\rho_{cr.}\, a_{eq}\, a^{-4}$$

Friedman eq. $\Rightarrow$

$$\frac{H}{H_0} = \sqrt{\frac{\rho}{\rho_{cr.}}}$$

$$= \left[e^{-\alpha t} a^{-3} + \left(\frac{8}{29} a_{eq} + 1 - e^{-\alpha t}\right) a^{-4} + \frac{21}{29} a_{eq}\, a^{-4}\right]^{1/2}$$

$$H_0 \frac{dt}{da} = \frac{1}{a}\left[e^{-\alpha t}(a^{-3} - a^{-4}) + (1 + a_{eq})a^{-4}\right]^{-1/2}$$

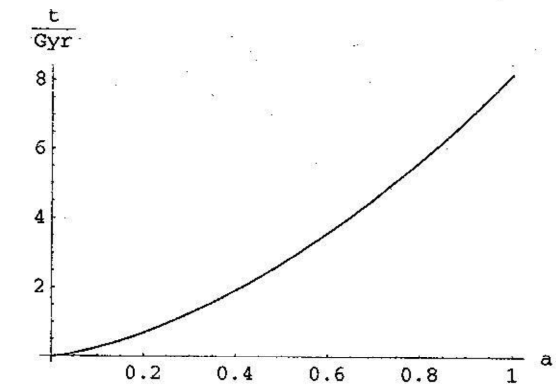
$$\boxed{\frac{d(t/H_0^{-1})}{da} = \left[e^{-\alpha t}(a^{-1} - a^{-2}) + (1 + a_{eq})\, a^{-2}\right]^{-1/2} \equiv f(t, a)}$$

with
$1/H_0 = 1.4\times10^{10}$ [illegible] yr
≈ 0.0716 [illegible]
$\alpha t =$ [illegible]

Numerically solve this differential eqn., and plot as attached.

$t(a=1) = 8.22\ \mathrm{Gyr}.$

In[2]:=
```
solution =
  NDSolve[{0.0716 t'[a] == 1 / Sqrt[Exp[-0.2 * t[a]] * (1 / a - 1 / a^2) + (1 + 0.001) / a^2],
   t[0.001] == 0}, t, {a, 0.001, 1}];
Plot[t[a] /. solution, {a, 0.001, 1}, AxesLabel → {a, t / Gyr}]
```



Out[2]= - Graphics -

In[3]:=
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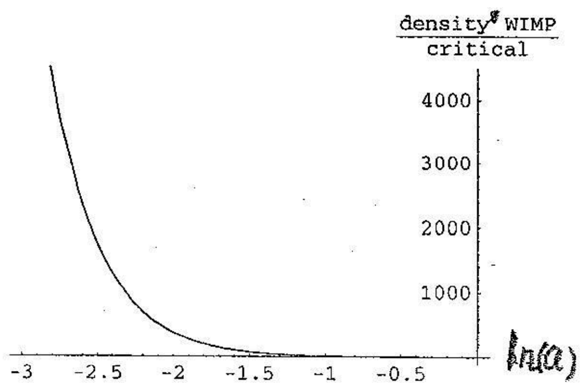
t[1] /. solution
```

Out[3]= {8.21839}

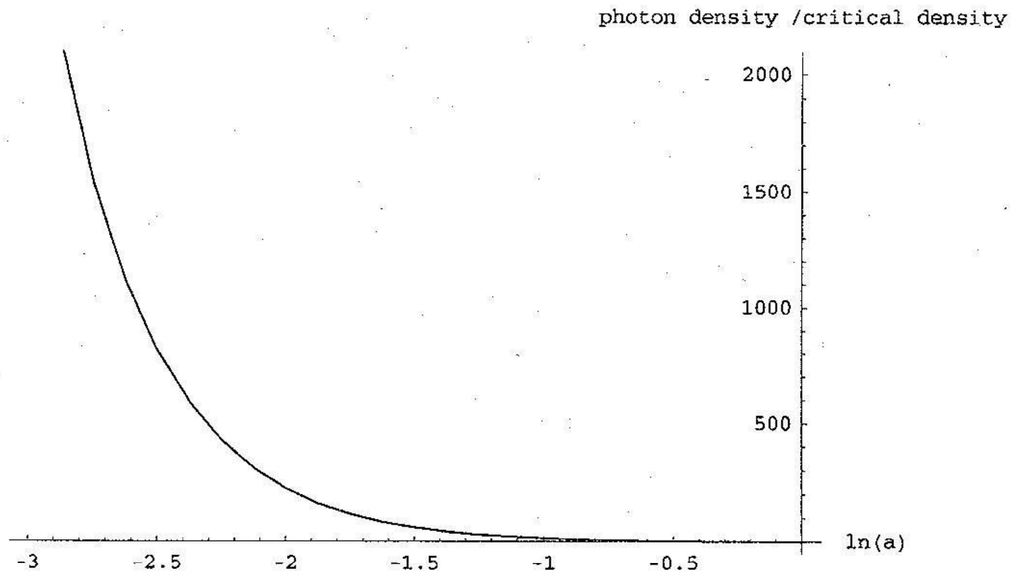
In[6]:=
```
Plot[Exp[-0.2 * t[Exp[lna]]] * Exp[lna] ^ (-3) /. solution,
 {lna, -3, 0}, AxesLabel → {ln a, WIMP density / critical density}]
```



Out[6]= - Graphics -

In[13]:=

```
Plot[(8 / 29 * 0.001 + 1 - Exp[-0.2 * t[Exp[lna]]]) * Exp[lna] ^ (-4) /. solution,
  {lna, -3, 0}, AxesLabel → {"ln(a)", "photon density /critical density"}]
```



Out[13]= - Graphics -