Problem Set #4

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Due: Thursday, April 13, 2006, in class

Five (5) problems from Dodelson, Modern Cosmology:

1. Chapter 5, Exercise 3
   There is a typo in Eq. 2.101 mentioned in the errata for the textbook:
   "Page 56, Eq 2.101: In the denominator P_0 should be P^0
   (superscripted instead of subscripted). It matters if you want to
   show that there are no factors of Φ, Ψ in the stress energy tensor.
   Thanks to Eduardo Rozo."
2. Chapter 5, Exercise 5
3. Chapter 5, Exercise 14
4. Chapter 4, Exercise 7
   (A solution is available in Appendix A of Dodelson.
   Proceed through the exercise referring to the printed solution if necessary).
5. Chapter 4, Exercise 10

8.942 Pset 4 Solution

1. Dodelson Ch. 5, ex. 3.

In scalar metric perturbation,

$$g_{00} = -(1+2\Psi)\,, \quad g_{0i} = 0\,, \quad g_{ij} = a^2(1+2\Phi)\delta_{ij}\,,$$

$$(-\det g_{\mu\nu})^{-1/2} = \left[a^6(1+2\Psi)(1+2\Phi)^3\right]^{-1/2} = a^{-3}(1+2\Psi)^{-1/2}(1+2\Phi)^{-3/2}$$

The proper momentum is $p^2 \equiv g^{ij}P_iP_j = a^{-2}(1+2\Phi)^{-1}P_iP_i$

so, $$p = a^{-1}(1+2\Phi)^{-1/2}P\,, \quad \text{where } P = (P_iP_i)^{1/2}\,.$$

$$d^3p = a^{-3}(1+2\Phi)^{-3/2}\,dP_1\,dP_2\,dP_3$$

Note that $-m_i^2 = P^2 \equiv g_{\mu\nu}P^\mu P^\nu = g^{00}(P_0)^2 + g^{ij}P_iP_j = -(1+2\Psi)^{-1}(P_0)^2 + p^2$

$$\Rightarrow (1+2\Psi)^{-1}(P_0)^2 = p^2 + m_i^2 = E_i^2(p) \quad \Rightarrow \quad P_0 = (1+2\Psi)^{1/2}E_i(p)$$

$$T^0{}_0\Big|_{\text{species } i} = g_i\int \frac{dP_1\,dP_2\,dP_3}{(2\pi)^3}(-\det g_{\mu\nu})^{-1/2}\frac{P^0P_0}{P^0}f_i(\vec{x},\vec{p},t)$$

$$= -g_i\int\frac{d^3p}{(2\pi)^3}\,a^3(1+2\Phi)^{3/2}a^{-3}(1+2\Psi)^{-1/2}(1+2\Phi)^{-3/2}(1+2\Psi)^{1/2}E_i(p)f_i(\vec{x},\vec{p},t)$$

$$= -g_i\int\frac{d^3p}{(2\pi)^3}E_i(p)f_i(\vec{x},\vec{p},t)$$

or, $$T^0{}_0(\vec{x},t) = -\sum_{\substack{\text{all species}\\ i}} g_i\int\frac{d^3p}{(2\pi)^3}E_i(p)f_i(\vec{x},\vec{p},t) \qquad \text{(Dodelson 5.22)}$$

Note that $$P_i = a(1+2\Phi)^{1/2}p_i$$

$$T^0{}_i\Big|_{\text{species }\alpha} = g_\alpha\int\frac{dP_1\,dP_2\,dP_3}{(2\pi)^3}(-\det g_{\mu\nu})^{-1/2}\frac{P^0P_i}{P^0}f_\alpha(\vec{x},\vec{p},t)$$

$$= g_\alpha\int\frac{d^3p}{(2\pi)^3}\,a^3(1+2\Phi)^{3/2}a^{-3}(1+2\Psi)^{-1/2}(1+2\Phi)^{-3/2}\,a(1+2\Phi)^{1/2}p_i$$

$$\times f_\alpha(\vec{x},\vec{p},t)$$

$$= a\,g_\alpha\int\frac{d^3p}{(2\pi)^3}(1-\Psi+\Phi)\,p_i\,f_\alpha(\vec{x},\vec{p},t)$$

$$= a\,g_\alpha(1-\Psi(t,\vec{x})+\Phi(t,\vec{x}))\int\frac{d^3p}{(2\pi)^3}p_i\,f_\alpha(\vec{x},\vec{p},t)$$

To linear order, $T^0{}_i\Big|_{\text{species } \alpha} = a g_\alpha \int \frac{d^3p}{(2\pi)^3} p_i f_\alpha(\vec{x}, \vec{p}, t)$

$$+ a g_\alpha(\Phi - \Psi) \int \frac{d^3p}{(2\pi)^3} p_i f_\alpha^{(0)}(p)$$

Since $f_\alpha^{(0)} = f_\alpha^{(0)}(p)$ is an even function of $\vec{p}$, the zero-order integral $\int \frac{d^3p}{(2\pi)^3} p_i f_\alpha^{(0)}(p) = 0$. So

$$T^0{}_i\Big|_{\text{species } \alpha} = a g_\alpha \int \frac{d^3p}{(2\pi)^3} p_i f_\alpha(\vec{x}, \vec{p}, t)$$

2. Dodelson Ch.5, ex.5.

$$\begin{aligned}
R_{0i} &= \Gamma^\alpha{}_{0i,\alpha} - \Gamma^\alpha{}_{0\alpha,i} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{0i} - \Gamma^\alpha{}_{\beta i}\Gamma^\beta{}_{0\alpha} \\
&= \Gamma^0{}_{0i,0} + \Gamma^j{}_{0i,j} - \Gamma^0{}_{00,i} - \Gamma^j{}_{0j,i} + \Gamma^j{}_{0j}\Gamma^0{}_{0i} + \Gamma^0_{j0}\Gamma^j{}_{0i} + \Gamma^k_{jk}\Gamma^j{}_{0i} \\
&\quad - \Gamma^0_{ji}\Gamma^j{}_{00} - \Gamma^j{}_{0i}\Gamma^0{}_{0j} - \Gamma^j{}_{ki}\Gamma^k{}_{0j} \\
&= ik_i\Psi_{,0} + ik_j\delta_{ij}\Phi_{,0} - ik_i\Psi_{,0} - 3ik_i\Phi_{,0} + 3Hik_i\Psi + \delta_{ij}Hik_j\Psi \\
&\quad + i\delta_{ij}H\Phi[k_j + 3k_j - k_j] - \delta_{ij}a^2H\frac{ik_j}{a^2}\Psi - \delta_{ij}Hik_j\Psi \\
&\quad - i\delta_{kj}H\Phi[\delta_{ij}k_k + \delta_{jk}k_i - \delta_{ik}k_j] \\
&= 2ik_i(H\Psi - \Phi_{,0})
\end{aligned}$$

$$\begin{aligned}
G^0{}_i &= g^{00}G_{0i} = g^{00}R_{0i} \\
&= -(1+2\Psi)^{-1} \times 2ik_i(H\Psi - \Phi_{,0}), \\
&= 2ik_i\left(\frac{\dot{\Phi}}{a} - H\Psi\right)
\end{aligned}$$

In exercise 3, $T^0{}_i = a\sum_{\text{all species } \alpha} g_\alpha \int \frac{d^3p}{(2\pi)^3} p_i f_\alpha(\vec{p}, \vec{x}, t)$

For CDM $\quad g\int \frac{d^3p}{(2\pi)^3} p^i f_{dm} = n_{dm} m_{dm} v^i = \rho_{dm} v^i$

For baryon $\quad g\int \frac{d^3p}{(2\pi)^3} p^i f_b = n_b m_b v_b^i = \rho_b v_b^i$

For photon $\quad f_\gamma = \dfrac{1}{\exp\left[\dfrac{p}{T(t)(1+\Theta(\vec{x},\vec{p},t))}\right] - 1}$

$$\simeq f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta,$$

where $f^{(0)} = \dfrac{1}{e^{p/T} - 1}$

$$g_\gamma \int \frac{d^3p}{(2\pi)^3}\, p_i f_\gamma = 0 - g_\gamma \int \frac{d^3p}{(2\pi)^3}\, \hat{p}_i\, p^2 \frac{\partial f^{(0)}}{\partial p}\, \Theta \quad ,$$

In Fourier space, $\hat{k}^i \left(g_\gamma \int \frac{d^3p}{(2\pi)^3} p_i f_\gamma \right) = -g_\gamma \int \frac{d^3p}{(2\pi)^3} p^2 \frac{\partial f^{(0)}}{\partial p} \mu \Theta = -\underbrace{g_\gamma \int_0^\infty \frac{dp}{2\pi^2} p^4 \frac{\partial f^{(0)}}{\partial p}}_{\substack{-4\rho_\gamma \\ \text{integrate by parts}}} \underbrace{\int_{-1}^{1} \frac{d\mu}{2} \mu\, \Theta(\mu)}_{-i\Theta_1}$

$$= -4i \rho_\gamma \Theta_1$$

similarly, for massless neutrinos, $\hat{k}^i \left(g_\nu \int \frac{d^3p}{(2\pi)^3} p_i f_\nu \right) = -4i \rho_\nu N_1$

So, $\hat{k}^i \cdot$ (Einstein Eqn) $\Rightarrow 2ik\left(\frac{\dot{\Phi}}{a} - H\Psi\right) = 8\pi G a \left[\rho_{dm} v + \rho_b v_b - 4i \rho_\gamma \Theta_1 - 4i \rho_\nu N_1 \right]$

or, $\boxed{\dot{\Phi} - aH\Psi = \frac{4\pi G a^2}{ik}\left[\rho_{dm} v + \rho_b v_b - 4i \rho_\gamma \Theta_1 - 4i \rho_\nu N_1 \right]}$

3. Dodelson Ch.5, ex. 14.

Most generally, scalar perturbations to the metric are

$$g_{00} = -(1+2A)$$

$$g_{0i} = -a B_{,i}$$

$$g_{ij} = a^2 \left(\delta_{ij} [1 + 2\psi] - 2E_{,ij} \right) .$$

The most general coordinate transformation is generated by

$$t \rightarrow \tilde{t} = t + \xi^0 (t, \vec{x})$$

$$x^i \rightarrow \tilde{x}^i = x^i + \delta^{ij} \xi_{,j} (t, \vec{x}) .$$

Start from $\quad \tilde{g}_{\alpha\beta}(\tilde{x}) \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} = g_{\mu\nu}(x) . \qquad (*)$

For $\mu = \nu = 0$: $\quad -(1 + 2\tilde{A})(1 + \xi^0{}_{,0})^2 = -(1+2A)$

$$\Rightarrow A \rightarrow \tilde{A} = A - \frac{\dot{\xi}^0}{a}$$

For $\mu = 0, \nu = i$: $\quad -\xi^0{}_{,i} - a\tilde{B}_{,j} \delta^j{}_i + a^2 \delta_{jk} (\xi_{,j})_{,t} \delta^k{}_i = -a B_{,i}$

$$\Rightarrow \tilde{B}_{,i} = B_{,i} - \frac{1}{a} \xi^0{}_{,i} + \dot{\xi}_{,i}$$

$$\Rightarrow B \rightarrow \tilde{B} = B - \frac{1}{a}\xi^0 + \dot{\xi}$$

For $\mu = i, \nu = j$: LHS$(*)$ $= a^2(\tilde{t}) \left(\delta_{kl} [1 + 2\tilde{\psi}] - 2\tilde{E}_{,kl} \right) \left(\delta^k{}_i + \xi_{,ki} \right) \left(\delta^l{}_j + \xi_{,lj} \right)$

Note that $\quad a(\tilde{t}) = a(t) + \frac{da}{dt} \xi^0 = a(t) + aH\xi^0 = a(t)(1 + H\xi^0)$

So, $\quad$ LHS of $(*)$ $= a^2(t)(1 + 2H\xi^0)\left([1 + 2\tilde{\psi}]\delta_{ij} - 2\tilde{E}_{,ij} + 2\xi_{,ij}\right)$

$$= a^2(t) \left(\delta_{ij} [1 + 2\tilde{\psi} + 2H\xi^0] - 2(\tilde{E} - \xi)_{,ij} \right)$$

RHS of $(*)$ $= a^2(t) \left(\delta_{ij} [1 + 2\psi] - 2E_{,ij} \right)$

$$\Rightarrow \psi \rightarrow \tilde{\psi} = \psi - H\xi^0$$

$$E \rightarrow \tilde{E} = E + \xi$$

Under the coordinate transformation

$$\tilde{\Phi}_A = \tilde{A} + \frac{1}{a}\frac{\partial}{\partial\eta}\left[a(\dot{\tilde{E}} - \tilde{B})\right]$$
$$= A - \frac{1}{a}\dot{\xi}^0 + \frac{1}{a}\frac{\partial}{\partial\eta}\left[a(\dot{E} + \cancel{\dot{\xi}} - B + \frac{1}{a}\xi^0 - \cancel{\dot{\xi}})\right]$$
$$= \cancel{A - \frac{1}{a}\dot{\xi}^0} + \frac{1}{a}\frac{\partial}{\partial\eta}\left[a(\dot{E} - B)\right] + \cancel{\frac{1}{a}\dot{\xi}^0}$$
$$= A + \frac{1}{a}\frac{\partial}{\partial\eta}\left[a(\dot{E} - B)\right]$$
$$= \Phi_A$$

$$\tilde{\Phi}_H = -\tilde{\Psi} + aH(\tilde{B} - \dot{\tilde{E}})$$
$$= -\Psi + H\xi^0 + aH\left(B - \frac{1}{a}\xi^0 + \cancel{\dot{\xi}} - \dot{E} - \cancel{\dot{\xi}}\right)$$
$$= -\Psi + \cancel{H\xi^0} + aH(B - \dot{E}) - \cancel{H\xi^0}$$
$$= -\Psi + aH(B - \dot{E})$$
$$= \Phi_H$$

So, Φ_A and Φ_H are gauge invariant quantities.

4. Dodelson Ch 4 Ex. 7

Start from Dodelson Eq. (4.49)

$$C[f(\vec{p})] = \frac{\pi}{4m_e^2 p}\int\frac{d^3q}{(2\pi)^3} f_e(\vec{q})\int\frac{d^3p'}{(2\pi)^3 p'} |\mathcal{M}|^2 \left\{\delta(p-p') + \frac{(\vec{p}-\vec{p}')\cdot\vec{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'}\right\}$$
$$\times\left[f(\vec{p}') - f(\vec{p})\right].$$

Now $\quad |\mathcal{M}|^2 = 6\pi\sigma_T m_e^2\left[1 + \cos^2\langle\hat{p},\hat{p}'\rangle\right]$

$$= |\mathcal{M}_0|^2 + 2\pi\sigma_T m_e^2\left[3\cos^2\langle\hat{p},\hat{p}'\rangle - 1\right], \text{ where } |\mathcal{M}_0|^2 = 8\pi\sigma_T m_e^2$$

$$3\cos^2\langle\hat{p},\hat{p}'\rangle - 1 = 2P_2(\cos\langle\hat{p},\hat{p}'\rangle)$$
$$= 2\times\frac{4\pi}{5}\sum_{m=-2}^{2} Y_{2m}(\hat{p})Y_{2m}^*(\hat{p}') \quad \text{(Dodelson Eq. (C12))}$$

So,

$$\delta|\mathcal{M}|^2 = |\mathcal{M}|^2 - |\mathcal{M}_0|^2 = \frac{16\pi^2\sigma_T m_e^2}{5}\sum_m Y_{2m}(\hat{p})Y_{2m}^*(\hat{p}')$$

Integrate over $d^3\vec{q}$: we have

$$\delta C[f(\vec{p})] = \frac{4\pi^3 n_e \sigma_T}{5p} \sum_m \int \frac{d^3p'}{(2\pi)^3 p'} Y_{2m}(\hat{p}) Y^*_{2m}(\hat{p}') \times$$

$$\left\{ \delta(p-p') + (\vec{p}-\vec{p}')\cdot\vec{V}_b \frac{\partial}{\partial p'} \delta(p-p') \right\} \times \left(f(\vec{p}') - f(\vec{p}) \right)$$

Since $f(\vec{p}') = f^{(0)}(p') - p' \frac{\partial f^{(0)}}{\partial p'} \Theta(\hat{p}')$ has no azimuthal dependence, only $m=0$ term contributes ; other $m \neq 0$ terms integrate to zero.

$$\delta C[f(\vec{p})] = \frac{4\pi^3 n_e \sigma_T}{5p} Y_{20}(\hat{p}) \int \frac{dp' p'}{(2\pi)^3} \int d\Omega' \; Y_{20}(\hat{p}') \times$$

$$\times \left\{ \delta(p-p') + (\vec{p}-\vec{p}')\cdot\vec{V}_b \frac{\partial}{\partial p'} \delta(p-p') \right\} \times \left(f(\vec{p}') - f(\vec{p}) \right)$$

The only term nonzero in the angular integration is proportional to $f(\vec{p}')\,\delta(p-p')$. In Fourier space

$$\delta C[f(\vec{p})] = \frac{2\times 4\pi^3 n_e \sigma_T}{(2\pi)^2\, 5p} \times \frac{5}{4\pi} P_2(\mu) \int dp'\, p' \int \frac{d\mu'}{2} P_2(\mu')$$

$$\delta(p-p') \times \left[f^{(0)}(p') - p' \frac{\partial f^{(0)}}{\partial p'} \Theta(\mu') \right]$$

$$= -\frac{\sigma_T n_e}{2p} P_2(\mu) \int dp\, p'^2 \frac{\partial f^{(0)}}{\partial p'} \delta(p-p') \int \frac{d\mu'}{2} P_2(\mu') \Theta(\mu')$$

$$= -\frac{\sigma_T n_e}{2p} P_2(\mu) \left(p^2 \frac{\partial f^{(0)}}{\partial p} \right) \times (-i)^2 \Theta_2$$

$$= \frac{n_e \sigma_T}{2} p \frac{\partial f^{(0)}}{\partial p} P_2(\mu) \Theta_2$$

So, $$C[f(\vec{p})] = -p \frac{\partial f^{(0)}}{\partial p} n_e \sigma_T \left[\Theta_0 - \Theta(\mu) + \mu V_b \underbrace{- \tfrac{1}{2} P_2(\mu) \Theta_2}_{\text{new!}} \right]$$

So, this adds "new!" part to Dodelson Eq. (4.63) and it becomes

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau} \left[\Theta_0 - \Theta + \mu V_b - \tfrac{1}{2} P_2(\mu) \Theta_2 \right]$$

This is Dodelson Eq. (4.100) without polarization.

5. Dodelson Ch.4, ex. 10

The zero-order part of Dodelson Eq(4.08) is

$$\frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial E}\frac{da/dt}{a}\frac{p^2}{E} = 0$$

Multiply both sides by $\frac{d^3p}{(2\pi)^3}E(p)$ and integrate.

$$\frac{\partial}{\partial t}\int\frac{d^3p}{(2\pi)^3}E(p)f(p,t) + \frac{1}{a}\frac{\partial}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}p^i f(p,t) - \frac{da/dt}{a}\int\frac{d^3p}{(2\pi)^3}p^2\frac{\partial f}{\partial E} = 0$$

the second term vanishes because the integrand is an odd function of p^i.

Integrate the third term by parts

$$\begin{aligned}\int\frac{d^3p}{(2\pi)^3}p^2\frac{\partial f(p,t)}{\partial E} &= \int\frac{4\pi\,dp}{(2\pi)^3}p^4\frac{\partial f(p,t)}{\partial E}\\ &= \int\frac{4\pi\,dp}{(2\pi)^3}p^3E\frac{\partial f(p,t)}{\partial p}\\ &= -\int\frac{4\pi\,dp}{(2\pi)^3}\frac{d(p^3E)}{dp}f(p,t) + (\text{surface term})\\ &= -\int\frac{d^3p}{(2\pi)^3}\left(\frac{p^2}{E}+3E\right)f(p,t)\end{aligned}$$

where we used the fact that $\frac{dE}{dp} = \frac{p}{E}$ because $E=\sqrt{p^2+m^2}$

Note that

$$\rho = g\int\frac{d^3p}{(2\pi)^3}E(p)f(p)$$

$$P = g\int\frac{d^3p}{(2\pi)^3}\frac{p^2}{3E}f(p).$$

We have

$$\frac{\partial\rho}{\partial t} + \frac{da/dt}{a}(3\rho+3P) = 0.$$

This is exactly Eq. (3.55) of Dodelson.