Board Game Topics

Analysis Using Graph Theory

February 24, 2009

Introduction

Today we look at how we can use graph theory, induction and other mathematical concepts to describe strategies and possible solutions to popular one and two player board games and puzzles. We will look at the two-player game of Hex and the one-player puzzle of sudoku. As we go through the examples, if there are any unfamiliar graph theory definitions, see the cheat sheet at the end for help.

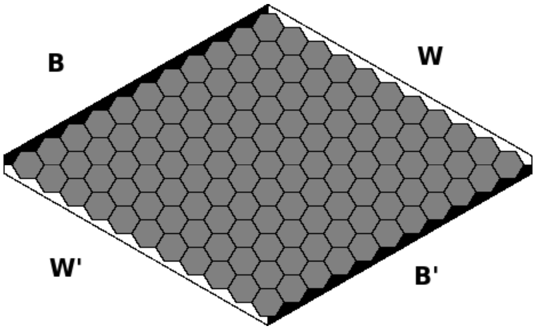
Hex

History

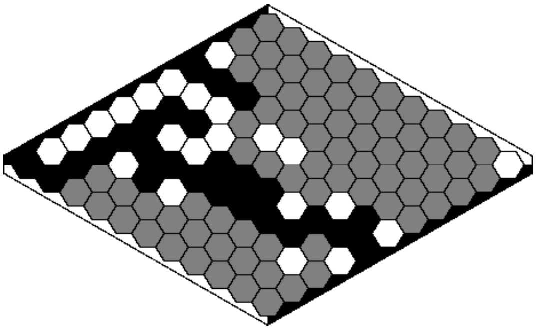
Hex is a simple to learn but difficult to solve board game that was originally invented by Piet Hien, a Danish mathematician, in 1942. It was later rediscovered by John Nash, the game theoriest and mathematician we have already met, in 1948. It was originally called "John," possibly after Nash, or because of the fact that it was played on bathroom floor tiles. Either way, Parker Brothers marketed the game under the name "Hex" and it stuck.

Rules

Hex is played on a rhombus-shaped board consisting of hexagonal shaped spaces. The goal of the two players, black (B) and white (W), are to form a *chain* from one edge of the board to the opposite edge. B's goal is to make a chain of black pieces connecting B and B', and W wants to make a chain of white pieces connecting W and W'. Below is a picture of the empy game board:



To play the game, the black and white players alternate placing pieces in one of the hexagonal cells. There are no restrictions as to where pieces can be played. The first player to create a chain as described above wins. The game ends when a player wins or when the board is filled.

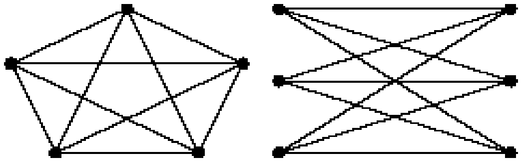


Black has won this game.

Solution

Hex is an example of a weakly solved game. It is known that the first player can always have a winning strategy on any $n \times n$ (the standard is 11 by 11) board, but the actual strategy is far from obvious. We will go through an argument called ***strategy stealing*** to show why the first player can always win, but first we will note a famous theorem and a couple of semi-obvious lemmas:

Kuratowski's Theorem: A graph is planar iff it does not contain a subgraph that is a subdivision of K_5 (the complete graph on five vertices) or $K_{3,3}$ (the bipartite graph on six vertices)



K_5 and $K_{3,3}$

We will not prove this theorem, but note that a subdivision of a graph means a graph that can be formed by inserting or removing vertices from edges.

Lemma 1: A game of Hex cannot end in a draw. (we define a draw as a game ending with no unique winner)

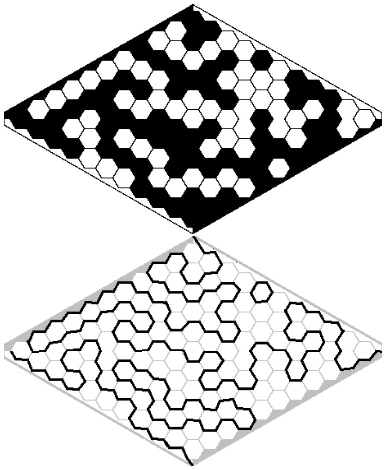
Proof:
If a game can't end in a draw, that means we must show:

1. Both players can't win.
2. Both players can't lose.

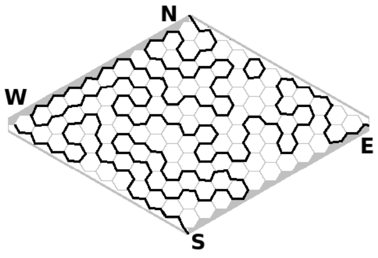
To do so, we will transform the Hex game board into a graph.

Consider the Hex board as a planar graph G of nodes all degree either two or three (we assume here that all of the cells are filled by either the black or the white player). Think of the nodes as the corners of the cells on a Hex board. A "cell" is then surrounded by 6 nodes and 6 edges. Each internal node touches 3 cells.

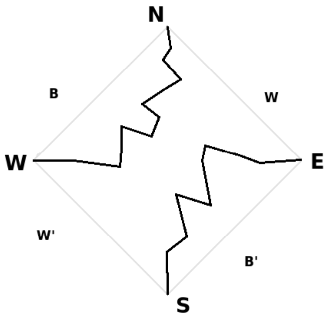
Now make the graph G' by coloring all the edges between differently occupied cells. We can see that all of the inner vertices of G have degree 0 or 2 in G', 0 in the case that the node was surrounded by cells of all the same color, or 2 in the case that a node was surrounded by 2 cells of one color and 1 of the other. The four corners have degree 1.

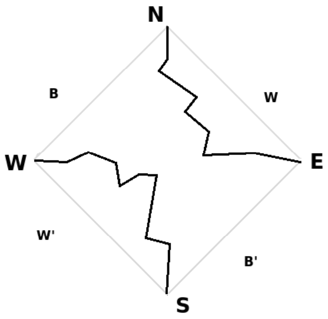


We can see that G' consists purely of disjoint simple cycles, simple paths, and isolated points, and therefore must be planar. Since there are four vertices of degree 1, there must be exactly two simple paths, with two of the degree 1 vertices forming the ends of each path. The graph is planar, so the two paths can't cross. Let's label the four corners N, S, E, W:



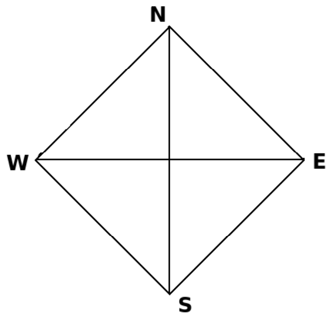
The two paths must either connect N-W and S-E, N-E and S-W, or N-S and E-W. The first two combinations give graphs similar to one of the following:





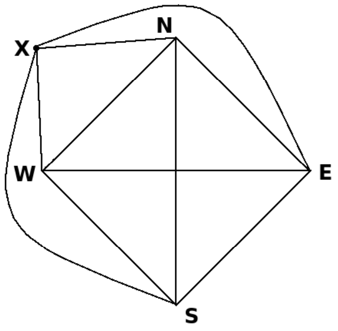
In the first case, a winning chain for white exists between the two paths, and in the second case a winning chain exists for black. So if either of the first two pairs of vertices are matched, there must be a winner, since some player has a winning chain.

Now we look at the third possible pair of paths: between N-S and E-W. These two edges together with the edges of the board form K_4, a complete graph on the four vertices:



It is legitimate to add the edges of the board because the edges of our Hex graph actually mark the edges of groups of pieces of one color, and an edge of the game board forms one of the boundaries of a group of pieces.

We then add an outside neutral vertex and connect it to each of the four corners, which we can do as shown below:



Then we should have a complete graph on five vertices (each vertex N,S,E,W, extra, is connected to each of the others). But by Kuratowski's Theorem, any K_5 graph cannot be planar. So therefore there cannot exist a path from

N to S and E to W. The two paths must be between either N-E, W-S or N-W S-E, and therefore there must be exactly one winner. The game cannot end in a draw □.

Lemma 2: In a game of Hex, a player is never harmed by having extra pieces on the board.

Proof: We note that pieces either contribute to a chain or are useless, but can never *hurt* a player.

Theorem: In a game of Hex, if each player plays a perfect game, then player 1 wins.

Proof (by contradiction):
By Lemma 1, a game of Hex cannot end in a draw, so one player must have a winning strategy. Now assume the second player has a winning strategy. Now let the first player adopt that same winning strategy. His first move will be an arbitrary move, and after that he will essentially become the second player, and can use the second player's original winning strategy. Since by Lemma 2 no extra pieces on the board could be detrimental to player 1, he can win the game. He has "stolen" player 2's strategy, and therefore can always win.

Analysis

To analyze Hex, we will first introduce a couple of definitions (we will talk about everything from Black's point of view, but the argument is symmetric from White's point of view):

- ***Group*** A group is a group of adjacent cells occupied by the same color. In the picture below, there is a group X of white cells:

- ***Subgame*** A subgame is denoted by a triple (x, A, y) where:

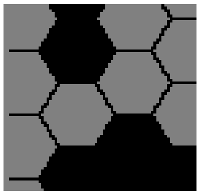
  - x and y are ***ends***, which are groups of cells.
  - A is a ***carrier***, which consists of empty cells beween the two ends.

  - Thus a subgame consists of trying to connect the two ends, x and y, across the carrier A.

- ***Virtual Connection*** A VC is a subgame where Black has a winning strategy even if White moves next.
- ***Virtual Semi-Connection*** A VSC is a subgame where Black has a winning strategy only if it moves before White.

Notations for a virtual connection (left) and virtual semi-connection (right).

- ***Depth*** The depth of a virtual connection is the number of moves ahead the result of the subgame can be determined.. This is also the depth of the game tree search required to establish the virtual connection. So VC's with depth d have info about the nodes of the game tree d moves ahead. To illustrate two obvious cases, we note the depth of two adjacent cells is 0, where a 2-bridge, as depicted below, has depth 2:



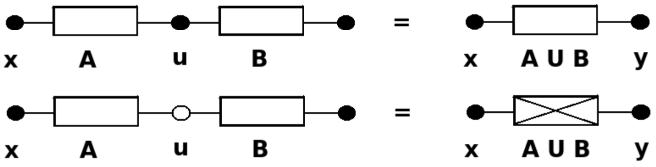
2-bridge.

The result of this subgame is known two moves before it is over. If white moves into one of the spots between the two groups of black cells, black can move in the other, and successfully create the bridge.

Now we will introduce two deduction rules:

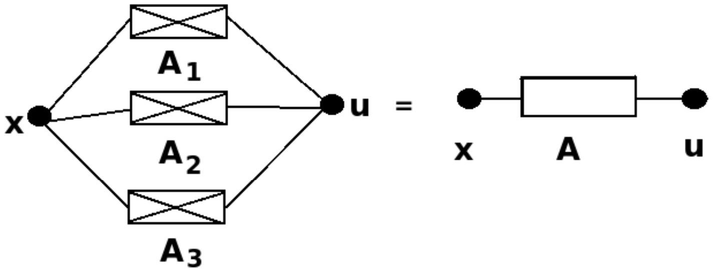
AND Deduction Rule: If a game has virtual connections (x, A, u) and (u, B, y):

- If u is black, the subgame $(x, A \cup B, y)$ is a virtual connection.
- If u is unoccupied, the subgame $(x, A \cup u \cup B, y)$ is a virtual semi-connection.



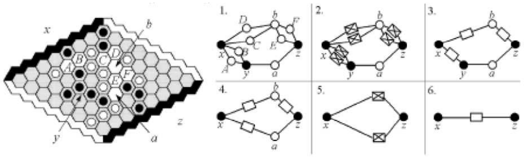
OR Deduction Rule: If a game has the virtual semi-connections $(x, A_k, y), k = 1, 2 \ldots n, n > 1$, then if $A_1 \cap A_2 \cap \ldots \cap A_n = \emptyset$, then the subgame (x, A, y) is a virtual connection, where $A = A_1 \cup A_2 \cup \ldots \cup A_n$.

We can see this by noting that if White occupies a cell $a \in A_i$, then black can move to A_j and make that a virtual connection.



We can combine the AND and OR deduction rules to show winning connections on nearly every Hex board (not all connections can be found using these rules, but they will be fine for our purposes).

The example below is from “A Hierarchical Approach to Computer Hex” by Vadim V. Anshelevich:



The first graph is constructed directly from the Hex board, with each black node representing a black group of pieces and the empty nodes representing carriers between the groups. The second graph adds in the VCS notation to show that the black nodes are connected to each other by VCS’s. The third graph uses the OR rule to reduce each double edge to a VC. The fourth graph uses the AND rule to eliminate y. The fifth again uses the AND rule, and the sixth uses the OR rule, with the end result of showing that black has a VC connecting both sides of the board, and therefore has a winning strategy.

Complexity

So how can we find a winning strategy? One (naive) way would be to enumerate all possible sequences of moves. We could start by numbering the cells of an $n \times n$ board $1, 2, \ldots n^2$. Then each possible game would consist of a permutation of the these numbers, where the order of the numbers is the order in which the cells are played, so a game might look like: $[1, 2, 3, \ldots n^2, n^2 - 5]$, where Black plays cell 1, white cell 2, etc. until the end. So listing them in this manner gives $n^2!$ possible games, or the number of permutations of n^2 things.

There is so far a lot of repetition in our counting scheme. We know that a lot of games will be finished before all the cells are covered, so we’ll say we’ll end up with about half as many of the possible moves as a rough estimate. Another half of the games are covered by rotation, since the board is symmetric. So we actually end up counting the ways one player can occupy a quarter of the cells times the number of ways another player can occupy a quarter of the cells, and then divide this result by 2, since the colors of the players can be switched and give us the same result. This gives us:

$$\frac{\binom{n^2}{\frac{1}{4}n^2}\binom{\frac{3}{4}n^2}{\frac{1}{4}n^2}}{2}$$

possible games. If we (very) roughly estimate each game to take an average of $\frac{n^2}{2}$ moves, finding the full solution to each possible game takes:

$$\frac{n^2!(\frac{3}{4}n^2)!n^2}{(\frac{1}{4}n^2)!(\frac{3}{4}n^2)!(\frac{1}{4}n^2)!(\frac{1}{2}n^2)!4}$$

moves.

If humans were playing the games and we estimate (for very fast humans) one move per second, then a 2×2 game solution would take 12 seconds, a 3×3 2400 seconds, and a standard 11×11 board would take a whopping 10^{54} seconds, or $3 \cdot 10^{46}$ years! Even if we let a computer work on the problem, and let it do a million moves per second, we are still looking at 10^{40} years (compared to our solar system's age, about 4-5 billion or so). Since the brute force method is intractable, we have to be smarter when we play Hex: good players develop strategies to help them win.

Strategy

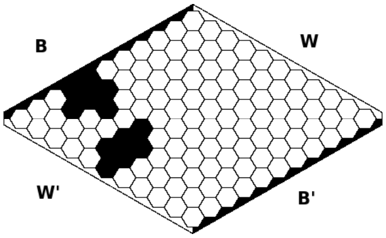
There are several strategies to keep in mind while playing Hex, but only practice and concentration will make you a better player.

A good player:

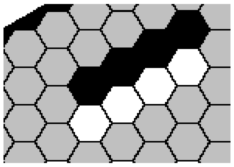
- Blocks advancing chains.
- Works on establishing his own connections
- Looks ahead to prepare traps and escapes

A couple of configurations to be aware of:

- ***Bridges*** Bridges form virtual connections, as the two black (white) squares on either side of the bridge can be connected no matter which player moves first:



- ***Ladders*** These can turn what looks like a winning strategy into a quick loss as one player chases the other down the board.



- ***Templates*** Seasoned Hex players can recognize certain end-game templates that appear over and over again.

In general, you should try to focus on establishing a presence in different regions until you have a virtual connection, and then switch your attention to other regions. Claiming global territory is the key, instead of trying to build an isolated chain starting at one end. You should get some practice today during class!

Sudoku

Introduction

Sudoku was originally invented by American architect Howard Garns in 1979 as the puzzle “Number Place,” and later became popular in Japan in 1986 as *sudoku,* meaning “single number.” Since about 2005, the puzzle has become an international phenomenon.

		1						
		2		3				4
			5			6		7
5			1	4				
	7						2	
				7	8			9
8		7			9			
4				6		3		
						5		

An example of a sudoku puzzle

Just as a refresher, sudoku is a puzzle played on a 9 by 9 grid. To solve the puzzle, you must fill in the grid with the numbers 1 through 9 such that each digit appears once in each row, once in each column, and once in each 3 by 3 subgrid, given a partially filled in grid.

Analysis

A rank n sudoku is a puzzle with n rows and n columns of $n \times n$ grid. Common sudokus are rank 3. We will define a ***zone*** as a subset of the n^4 cells of an n^2 by n^2 sudoku board such that all members of the same zone are either in the same row, column, or box. Thus the familiar 9 by 9 sudoku board has 27 zones.

Any sudoku puzzle can be mapped to a unique graph $X_n = (V, E)$. Each vertex of X_n is connected to all vertices it shares a zone with. So each vertex is connected to the $n^2 - 1$ other cells in its row, column, and box, for a total of $3(n^2 - 1)$. But $n - 1$ are counted as both row and box adjacent cells, and $n - 1$ are counted as both column and box adjacent cells, so the degree of each vertex in the graph will be $3(n^2 - 1) - 2(n - 1) = 3n^2 - 2n - 1$. We call X_n a ***regular*** graph, since all vertices have the same degree.

If we label a grid cell in the ith row and jth column as (i, j), then two cells (i, j) and (i', j') are connected in the graph X_n if $i = i', j = j'$, or $\lceil \frac{i}{n} \rceil = \lceil \frac{i'}{n} \rceil$ and $\lceil \frac{j}{n} \rceil = \lceil \frac{j'}{n} \rceil$.

j/i	1	2	3	4	5	6	7	8	9
1			1						
2			2		3				4
3				5			6		7
4	5			1	4				
5		7						2	
6					7	8			9
7	8		7			9			
8	4				6		3		
9							5		

Now the problem of solving a sudoku is equivalent to finding a 9-coloring of the graph X_n given a partial coloring of a subset of the graph, since two cells have an edge between their corresponding nodes only if they are not the same "color," or in our case, number.

In general, graph coloring is a very hard problem (in a class of problems called NP-complete), but since the sudoku board only contains 81 cells, we will be able to find some helpful facts and strategies.

We will now prove some theorems that we can apply to our sudoku graphs. First we'll try to find out whether a sudoku puzzle has a unique solution or not.

We will take the following theorem about graph coloring as a given without proof: the number of ways to color a graph $G = (V, E)$ with λ colors is a polynomial in λ of degree $|V|$.

Theorem: Given a partial coloring $C = (V', E')$, $V' \subseteq V, E' \subseteq E$, of $G = (V, E)$, the number of ways to complete the proper coloring using λ colors is also a polynomial in λ if $\lambda \geq$ the number of colors in C, d_0.

Proof:

We will denote $|V| = v$ and $|V'| = t$. Let $p_{G,C}(\lambda)$ be the number of ways to complete a proper coloring of G using only λ colors. Then we will show that this is a polynomial with integer coefficients of degree $v - t$ for $\lambda \geq d_0$. To do so, we will split the proof up into three cases:

1. **Case 1** There is more than one vertex of G not in C.

   We prove this case by strong induction on the number of edges.

   Let e be an edge connecting two nodes, at most one of which is in C. Then $G - e$ is the graph G without the edge e, that still contains the endpoint vertices of e. Let G/e be the subgraph of G containing e and its endpoints.

   Let $P(x)$: for a graph with x edges, given a partial coloring C, the number of ways to complete the proper coloring of the graph using λ colors is a polynomial in λ. We can use as a base case a graph with 0 edges, which can clearly be colored using any n colors, $n \geq 1$, since no nodes are connected.

   Each proper coloring of G is also a proper coloring of $G - e$. A proper coloring of $G - e$ is only a proper coloring of G if it gives distinct colorings to the endpoints of e. Since by strong induction $G - e$ and G/e have fewer edges than G, we can assume each, given C, can be colored in a number of ways that is a polynomial in λ. So:

   $$p_{G,C}(\lambda) = p_{G-e,C}(\lambda) - p_{G/e,C}(\lambda)$$

2. **Case 2** There is exactly one vertex in G not in C.

   Suppose G has one vertex v_0 not in C. If v_o is not adjacent to any vertices in C, then $G = C \cup v_0$ is a disjoint union of C and v_0, and so v_0 can be colored with any color from 1 to λ, and so $p_{G,C}(\lambda) = \lambda$.

   If v_0 is adjacent to d vertices of C, then these vertices use d_0 colors, and $p_{G,C}(\lambda) = \max(\lambda - d_0, 0)$. So if d_0 is greater than λ, then there is no way to properly color the entire graph G using λ colors. Otherwise, the coloring can be done in $\lambda - d_0$ ways, the same as the number of remaining colors.

3. **Case 3** Every vertex of G is also in C.

   In this case, C already gives a proper coloring of G in λ or less colors, so we are done, and $p_{G,C} = 1$.

Thus we have proven the theorem, since any graph G given a partial coloring C can be properly colored with λ colors in a number of ways equal to a polynomial in λ $\square$.

So now let's apply the theorem to our Sudoku graph, X_3. The number of ways to complete X_3 is given by $p_{X_3,C}$, so there is a unique solution to a Sudoku graph if p is equal to 1. When is p not equal to 1? This is a suprisingly complicated problem. We'll investigate further.

Theorem: Given a graph G with a partial coloring C using only $\chi(G) - 2$ colors, where $\chi(G)$ is the chromatic number of G, or the smallest number of colors needed to properly color it, there are at least 2 ways to extend the coloring to all of G.

Proof: There are 2 colors not used in C. For any solution to a proper coloring of G, another solution can be formed by interchanging the two unused colors $\square$.

Therefore, we know that to have a unique solution to a sudoku, at least 8 of the 9 numbers must be given somewhere. In general, a board X_n needs to have at least $n^2 - 1$ distinct numbers specified.

So how many cells can be filled in and still not give a unique solution? The board below has $n^2 - 4$ cells specified, but can have two solutions:

9	2	6	5	7	1	4	8	3
3	5	1	4	8	6	2	7	9
8	7	4	9	2	3	5	1	6
5	8	2	3	6	7	1	9	4
1	4	9	2	5	8	3	6	7
7	6	3	1			8	2	5
2	3	8	7			6	5	1
6	1	7	8	3	5	9	4	2
4	9	5	6	1	2	7	3	8

This puzzle can be filled in using either one of the rectangles below:

9	4
4	9

or

4	9
9	4

The question of the minimum number of givens to specify a unique solution is still unsolved, but has been shown to be at most 17 (these puzzles are usually in the "fiendish" category of sudokus").

So where does this leave us when we are unsure of whether a puzzle has unique solution, or a solution at all? We know sudokus can be colored in a polynomial in λ ways, with $\lambda = 9$, given a partial coloring C. There is only a unique solution when this polynomial is equal to 1, but that doesn't do us a whole lot of good, since we don't know how to find these polynomials specifically, just that they exist. We do know that if at least 8 of the 9 numbers aren't specified somewhere, there is no hope of finding a unique solution, and that with as little as 17 entries (or possibly even fewer) there might be a unique solution. But there could be 77 entries specified, and still no unique solution! For now we will have to trust the puzzle-publishers, who test their puzzles using brute force on a computer, only give us solvable sudokus.

Complexity

Just some food for thought: the number of possible Sudokus is about $6.721 \cdot 10^{21}$. When we take into account the fact that we can create 9! Sudokus from any one by permuting the order of the numbers, and each transpose of a Sudoku is another Sudoku we get around $5.47 \cdot 10^9$ possible games. Still a ton! Don't think we will run out of games any time soon...

Strategy

We will look at one possible argument for solving sudokus, i.e., 9-coloring our graph X_n given a partial coloring C.

We defined a zone above to be a subset of n^2 cells of the Sudoku grid such that all the cells of a given zone are either in the same row, column, or box. We will first define a ***Permutation Bipartite Graph (PBG)*** for each one as a vector of possible number assignments to the n vertices such that there exists at least one match between vertices and the set of numbers $D = \{1, 2, \ldots n\}$. Each number can only be assigned to a vertex that contains the possibility of being assigned to that number. So a possible assignment to the PBG below:

$$\{1,2,3,7\},\{3,6\},\{3,4\},\{1,4,\},\{5,6,7\},\{4,6\},\{2,7\},\{8,9\},\{8,9\}$$

is $(2, 3, 4, 1, 5, 6, 7, 8, 9)$ respectively.

Building each PBG requires a few simple rules. (and as a note, most Sudokuers do these steps mentally or with only a small amount of scratch work, but we will go through the process systematically).

There are two initial steps to building the PBGs:

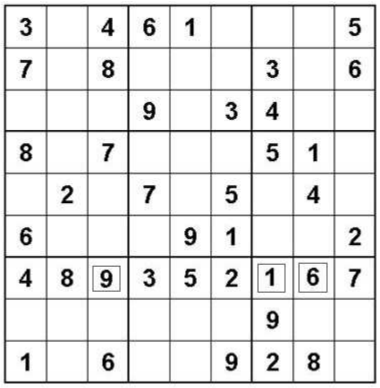
1. Find loners: Find cells that, because all the other digits are either in the same row, column, or box, have only one possibility. "Color" these cells with the appropriate number.

3		4	6	1				5
7		8				3		6
			9		3	4		
8		7				5	1	
	2		7		5		4	
6				9	1			2
4	8	×	3	5	2			7
						9		
1		6			9	2	8	

In the puzzle above, look at the cell a position 7,3. The numbers 1, 6, 8, and 4 are used in its box already, so it can't be those. The numbers 3, 5, 2, and 7 are in the same row, and 7, 8, and 4 are in the same column. This leaves only 9 as a possibility:

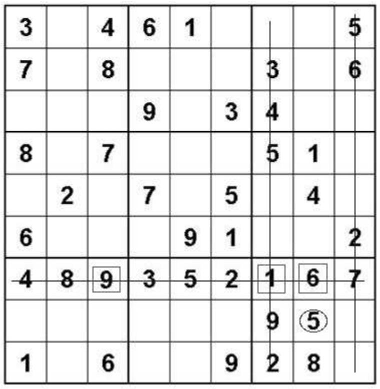
3		4	6	1				5
7		8				3		6
			9		3	4		
8		7				5	1	
	2		7		5		4	
6				9	1			2
4	8	9	3	5	2			7
						9		
1		6			9	2	8	

We can fill in a couple more loners that we just created. Filled in loners will be denoted by boxes.



3		4	6	1				5
7		8				3		6
			9		3	4		
8		7				5	1	
	2		7		5		4	
6				9	1			2
4	8	9	3	5	2	1	6	7
						9		
1		6			9	2	8	

2. "Slice and Dice" to find where a particular number must go. Also color these cells. For example, let's decide where the 5 should go in the bottom right box of the puzzle we've been looking at. We can cross out two of the columns and one of the rows that already contain a 5:



3		4	6	1				5
7		8				3		6
			9		3	4		
8		7				5	1	
	2		7		5		4	
6				9	1			2
4	8	9	3	5	2	1	6	7
						9	⑤	
1		6			9	2	8	

This leaves only one possible cell to put the 5 in.

Now we can write an example of the PBG for the first block:

`{3},{9},{4},{7},{1,5,9},{8},{2,5},{1,5,6},{1,2,5}`

Each PBG is constantly updated during the following processes.

We will use two rules to deal with these PBGs:

1. ***Pile Exclusion Rule (Hidden Pairs)*** If each member of a subset of k numbers only occurs in k vertices of the PBG, then the extra

numbers contained in those vertices can be deleted. For example, if 7, 8, and 9 only appear in $\{1,7,8,9\}$, $\{5,7,8,9\}$, and $\{6,7,8,9\}$, we can rewrite those vertices as $\{7,8,9\}$, $\{7,8,9\}$, and $\{7,8,9\}$.

2. ***Chain Exclusion Rule (Locked Candidates)*** If k vertices in the PBG contain only members of a set of k numbers, these k numbers can be erased from all the other vertices. So for example, if $\{3,6\}$, $\{3,4\}$, $\{4,6\}$ are three vertices, then only the numbers 3, 4, and 6 can be assigned across those vertices, so erase 3, 4, and 6 from all the other vertices.

Let's go through an example of this process on the puzzle above. First, we know for any vertex in the PBG that only contains one possible number, the vertex must be assigned to that number. So 3, 4, 7, 8, and 9 can be removed from any other vertices, since we know exactly where they go.

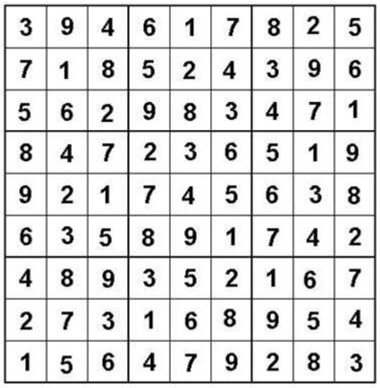
`{3},{9},{4},{7},{1,5},{8},{2,5},{1,5,6},{1,2,5}`

By slice and dice, 6 only has one possibility, so we can erase 1 and 5 from that vertex.

`{3},{9},{4},{7},{1,5},{8},{2,5},{6},{1,2,5}`

Now we can use the chain exclusion rule. We know that 1, 2 and 5 must go to one of the three remaining unspecified vertices. We will have to process some other blocks in order to update this PBG later.

Continuing the process gives the following solution to the puzzle:



3	9	4	6	1	7	8	2	5
7	1	8	5	2	4	3	9	6
5	6	2	9	8	3	4	7	1
8	4	7	2	3	6	5	1	9
9	2	1	7	4	5	6	3	8
6	3	5	8	9	1	7	4	2
4	8	9	3	5	2	1	6	7
2	7	3	1	6	8	9	5	4
1	5	6	4	7	9	2	8	3

You will get more practice with these rules in the exercises.

Induction and Graph Theory Cheat Sheet

Induction

Often times we can use a powerful mathematical tool, ***induction***, to prove whether a game or puzzle has a solution or winner.

A proof by induction involves the following steps:

1. Make a proposition in terms of $P(n)$, the thing you are trying to prove (for example, after n moves the number of pegs left in a peg solitaire game is x).
2. Prove the proposition for a base case. For example, $P(0)$: after 0 moves, ...
3. Show that for any n, $P(n)$ implies $P(n+1)$.
4. If we know $P(n)$ is true for a base case, and for each $P(n) \rightarrow P(n+1)$, then our proposition must be true, and we're done.

Graph Theory Definitions

Here is a quick review of some of the terms that will come up today. For a more comprehensive review of graph theory, see the readings for the week.

- ***Degree*** The degree of a vertex of a graph is the number of edges containing that node.
- ***Complete Graph*** K_n A graph on n nodes in which every node is connected to every other node. Each node has degree $n-1$.
- ***Chromatic Number*** χ The number of colors needed to color the vertices of a graph so that no adjacent nodes (those sharing an edge) have the same color.
- λ***-coloring*** A coloring of a graph using λ distinct colors.
- ***Proper Coloring*** A coloring of a graph such that no adjacent edges share a color.

- ***Partial Coloring*** A proper coloring of some subgraph G' of a graph G.
- ***Regular Graph*** A graph G in which every vertex has the same degree
- ***Planar Graph*** A graph that can be drawn in the plane without any crossing edges. Note that it is possible to draw a planar graph *with* crossings, but it is always possible to draw it *without* crossings.
- ***Subdivision*** A subdivision G' of a graph G can be formed by subdividing the edges of G. Subdividing an edge (u, v) means adding one or more vertices to create a new path, say (u, w, v), with the same endpoints as the original edge. Graphs formed from subdivisions of one another are said to be ***homeomorphic***. Below is an example of two homeomorphic graphs: