Galerkin and minimal residual approximations
for linear noncoercive elliptic problems

1 Problem description

1.1 Problem statement

We consider the evaluation of a linear-functional output associated with a parametrized linear elliptic PDE: given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, we find

$$s(\mu) = \ell(u(\mu); \mu),$$

(1)

where $u(\mu)$ is the solution of

$$a(u(\mu), v; \mu) = f(v; \mu), \quad \forall v \in X.$$  

(2)

Here $\mathcal{D} \subset \mathbb{R}^P$ is the parameter space in which the parameter $\mu$ resides; $a(\cdot, \cdot; \mu)$ is a bilinear form; $f(\cdot; \mu)$ and $\ell(\cdot; \mu)$ are linear functionals; and $X$ is an appropriate Hilbert space. In actual practice, $X$ is a finite element (FE) approximation of dimension $N$.

We also introduce a dual, or adjoint, problem associated with $\ell$: find $\psi(\mu) \in Y$ such that

$$a(v, \psi(\mu); \mu) = -\ell(v; \mu), \quad \forall v \in X.$$  

(3)

The introduction of the dual problem aims to improve the output convergence. This will become clear in the forthcoming analysis. In the compliance case, symmetric $a$ and $\ell = f$, the dual problem coincides with the primal problem.

1.2 Stability and continuity

For a noncoercive operator $a$, the stability and continuity constants are defined as

$$\beta(\mu) = \inf \sup_{w \in X} \frac{a(w, v; \mu)}{\|w\|_X \|v\|_X}, \quad \gamma(\mu) = \sup \sup_{w \in X} \frac{a(w, v; \mu)}{\|w\|_X \|v\|_X}. \quad (4)$$

We shall assume that $\beta(\mu) > 0, \forall \mu \in \mathcal{D}$ and that $\gamma(\mu) < \infty, \forall \mu \in \mathcal{D}$.
We next introduce the supremizing operator $T_\mu : X \rightarrow X$ such that for any given $\mu \in \mathcal{D}$ and any $w \in X$,

$$(T_\mu w, v)_X = a(w, v; \mu), \quad \forall v \in X. \quad (5)$$

We can then show that

**Q1.** As defined above, $T_\mu w$ satisfies

$$T_\mu w = \arg \sup_{v \in X} \frac{a(w, v; \mu)}{\|v\|_X}, \quad (6)$$

We can further prove

**Q2.** As defined above, $\beta(\mu)$ and $\gamma(\mu)$ are given by

$$\beta(\mu) = \inf_{w \in X} \frac{\|T_\mu w\|_X}{\|w\|_X}, \quad \gamma(\mu) = \sup_{w \in X} \frac{\|T_\mu w\|_X}{\|w\|_X}. \quad (7)$$

We now introduce an eigenvalue problem: given $\mu \in \mathcal{D}$, find $\chi_i(\mu) \in X$ and $\lambda_i(\mu) \in \mathbb{R}$, $i = 1, \ldots, N$, such that

$$(T_\mu \chi_i(\mu), T_\mu v)_X = \lambda_i(\mu)(\chi_i(\mu), v)_X, \quad \forall v \in X,$$

$$\|\chi_i(\mu)\|_X = 1. \quad (8)$$

Here the eigenvalues and eigenvectors are ordered such a way that $\lambda_1(\mu) \leq \lambda_2(\mu) \leq \ldots \leq \lambda_N(\mu)$. We show that

**Q3.** As defined above, $\beta(\mu)$ and $\gamma(\mu)$ can be computed as

$$\beta(\mu) = \sqrt{\lambda_1(\mu)}, \quad \gamma(\mu) = \sqrt{\lambda_N(\mu)}. \quad (9)$$

In essence, $\beta(\mu)$ and $\gamma(\mu)$ are respectively the square root of the smallest and largest eigenvalues.
2 Galerkin approximation

2.1 Approximation

We assume that we are given an orthonormal primal RB space $W^\text{pr}_N = \text{span}\{\zeta^\text{pr}_n, 1 \leq n \leq N\}$ and an orthonormal dual RB space $W^\text{du}_N = \text{span}\{\zeta^\text{du}_n, 1 \leq n \leq N\}$ such that

\begin{align}
(\zeta^\text{pr}_i, \zeta^\text{pr}_j)_X &= \delta_{ij}, \quad 1 \leq i, j \leq N, \\
(\zeta^\text{du}_i, \zeta^\text{du}_j)_X &= \delta_{ij}, \quad 1 \leq i, j \leq N.
\end{align}

Typically, the bases $W^\text{pr}_N$ and $W^\text{du}_N$ are constructed by using the POD-greedy sampling procedure.

We next apply Galerkin projection to obtain: For the primal problem (2), $u_N(\mu) \in W^\text{pr}_N$ satisfies

\begin{equation}
a(u_N(\mu), v; \mu) = f(v; \mu), \quad \forall v \in W^\text{pr}_N;
\end{equation}

and for the dual problem (3), $\psi_N(\mu) \in W^\text{du}_N$ satisfies

\begin{equation}
a(v, \psi_N(\mu); \mu) = -\ell(v; \mu), \quad \forall v \in W^\text{du}_N.
\end{equation}

We finally evaluate the output estimate as

\begin{equation}
s_N(\mu) = \ell(u_N(\mu); \mu) - r^\text{pr}(\psi_N(\mu); u_N(\mu); \mu),
\end{equation}

where, for $w_N \in W^\text{pr}_N$, the primal residual $r^\text{pr}(\cdot; w_N; \mu)$ is given by

\begin{equation}
r^\text{pr}(v; w_N; \mu) = f(v; \mu) - a(w_N, v; \mu), \quad \forall v \in X.
\end{equation}

We proceed to derive the discrete equations.

2.2 Discrete equations

Inserting $u_N(\mu) = \sum_{j=1}^N u_{Nj}(\mu)\zeta^\text{pr}_j$ and choosing $v = \zeta^\text{pr}_i, 1 \leq i \leq N$, in (12) we obtain

\begin{equation}
A^\text{pr}_N(\mu)u_N(\mu) = F^\text{pr}_N(\mu)
\end{equation}

where $A^\text{pr}_N(\mu) \in \mathbb{R}^{N \times N}$ and $F^\text{pr}_N(\mu) \in \mathbb{R}^N$

\begin{equation}
A^\text{pr}_{Nij}(\mu) = a(\zeta^\text{pr}_i, \zeta^\text{pr}_j; \mu), \quad F^\text{pr}_{Ni}(\mu) = f(\zeta^\text{pr}_i; \mu), \quad 1 \leq i, j \leq N.
\end{equation}
Similarly, expressing \( \psi_N(\mu) = \sum_{j=1}^{N} \psi_{Nj}(\mu) \zeta_j^{du} \) and choosing \( v = \zeta_i^{du}, 1 \leq i \leq N \), in (13) we arrive at
\[
A_{N}^{du}(\mu) \psi_N(\mu) = -L_{N}^{du}(\mu) \tag{18}
\]
where \( A_{N}^{du}(\mu) \in \mathbb{R}^{N \times N} \) and \( L_{N}^{du}(\mu) \in \mathbb{R}^{N} \) are given by
\[
A_{Nij}^{du}(\mu) = a(\zeta_i^{du}, \zeta_j^{du}; \mu), \quad L_{Ni}^{du}(\mu) = \ell(\zeta_i^{du}; \mu), \quad 1 \leq i, j \leq N. \tag{19}
\]
In what follows, we shall give a detailed analysis of this reduced-basis (RB) approximation and develop associated a posteriori error estimation.

Let us use \( A(\mu) \in \mathbb{R}^{N \times N}, F(\mu) \in \mathbb{R}^{N}, L(\mu) \in \mathbb{R}^{N} \), and \( C \in \mathbb{R}^{N \times N} \) to represent the discrete form of \( a(\cdot, \cdot; \mu), f(\cdot; \mu), \ell(\cdot; \mu) \), and \( (\cdot, \cdot)_X \), respectively. Let further \( Z_{pr}^N = [\zeta_1^{pr} \ldots \zeta_N^{pr}] \in \mathbb{R}^{N \times N} \) and \( Z_{du}^N = [\zeta_1^{du} \ldots \zeta_N^{du}] \in \mathbb{R}^{N \times N} \), where \( \zeta_n^{pr} \in \mathbb{R}^{N} \) and \( \zeta_n^{du} \in \mathbb{R}^{N} \) are the nodal vectors of \( \zeta_n^{pr} \) and \( \zeta_n^{du} \), respectively.

The RB primal system
\[
A_N(\mu) \tag{20}
\]
and for the dual problem (3), \( \psi_N(\mu) \in W_{du}^N \) satisfies
\[
a(v, \psi_N(\mu); \mu) = \ell(v; \mu), \quad \forall v \in W_{du}^N. \tag{21}
\]

2.3 Conditioning number

We show that the both primal and dual RB approximations inherit the conditioning properties of the underlying PDE. We make a proof for the primal problem. The result for the dual problem is similarly obtained. We first note that for any \( w_N \in W_{pr}^N \), we can write \( w_n = \sum_{i=1}^{N} w_{Ni} \zeta_i^{pr} \). It then follows from (10) and (11) that
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} w_N w_{Ni} w_{Nj} a(\zeta_i^{pr}, \zeta_j^{pr}; \mu) \geq \alpha(\mu) \sum_{i=1}^{N} \sum_{j=1}^{N} w_{Ni} w_{Nj} (\zeta_i^{pr}, \zeta_j^{pr})_X
\]
\[
= \alpha(\mu) \sum_{i=1}^{N} \sum_{j=1}^{N} w_{Ni} w_{Nj} \delta_{ij}
\]
\[
= \alpha(\mu) \sum_{i=1}^{N} w_{Ni}^2. \tag{22}
\]
Similarly, we have
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} w_{Ni} w_{Nj} a(\zeta_{i}^{pr}, \zeta_{j}^{pr}; \mu) \leq \gamma(\mu) \sum_{i=1}^{N} \sum_{j=1}^{N} w_{Ni} w_{Nj} (\zeta_{i}^{pr}, \zeta_{j}^{pr}) X
\]
\[
= \gamma(\mu) \sum_{i=1}^{N} \sum_{j=1}^{N} w_{Ni} w_{Nj} \delta_{ij}
\]
\[
= \gamma(\mu) \sum_{i=1}^{N} w_{Ni}^{2} .
\]

Combining (74) and (75) yields
\[
\alpha(\mu) \leq \sum_{i=1}^{N} \sum_{j=1}^{N} w_{Ni} w_{Nj} a(\zeta_{i}^{pr}, \zeta_{j}^{pr}; \mu) \leq \gamma(\mu), \quad \forall w_{N} \in \mathbb{R}^{N} .
\]

Clearly, our RB system has the same conditioning properties as the underlying PDE. In the worst case, the condition number is bounded by the ratio \(\gamma(\mu)/\alpha(\mu)\), which is independent of \(N\).

### 2.4 A priori convergence

To begin, we introduce the operator \(T_{\mu}^{N}: W_{N} \to W_{N}\) such that, for any \(w_{N} \in W_{N}\),
\[
(T_{\mu}^{N} w_{N}, v_{N})_{X} = a(w_{N}, v_{N}; \mu), \quad \forall v_{N} \in W_{N} .
\]

We then define \(\beta_{N}(\mu) \in \mathbb{R}\) as
\[
\beta_{N}(\mu) \equiv \inf_{w_{N} \in W_{N}} \sup_{v_{N} \in W_{N}} \frac{a(w_{N}, v_{N}; \mu)}{\|w_{N}\|_{X} \|v_{N}\|_{X}} ,
\]
and note that
\[
\beta_{N}(\mu) = \inf_{w_{N} \in W_{N}} \frac{\|T_{\mu}^{N} w_{N}\|_{X}}{\|w_{N}\|_{X}} .
\]

It thus follows that
\[
\beta_{N}(\mu)\|w_{N}\|_{X} \|T_{\mu}^{N} w_{N}\|_{X} \leq a(w_{N}, T_{\mu}^{N} w_{N}; \mu), \quad \forall w_{N} \in W_{N} .
\]

We assume that \(\beta_{N}(\mu) > 0, \forall \mu \in \mathcal{D}\). It can be shown that

**Q4.** We have \(\beta_{N}(\mu) \leq \beta(\mu), \forall N, \forall \mu \in \mathcal{D}\).

Furthermore, show that
Q5. The condition number of both $A^{pr}_N(\mu)$ and $A^{du}_N(\mu)$ is bounded by $\gamma(\mu)/\beta_N(\mu)$.

It is a standard result of the Galerkin approximation to demonstrate that

Q6. The primal RB solution $u_N(\mu)$ and the dual RB solution $\psi_N(\mu)$ are optimal in the $X$-norm in the following sense

$$
\|u(\mu) - u_N(\mu)\|_X \leq \left(1 + \frac{\gamma(\mu)}{\beta_N(\mu)}\right) \min_{w_N \in W^{pr}_N} \|u(\mu) - w_N(\mu)\|_X ,
$$

(29)

$$
\|\psi(\mu) - \psi_N(\mu)\|_X \leq \left(1 + \frac{\gamma(\mu)}{\beta_N(\mu)}\right) \min_{w_N \in W^{du}_N} \|\psi(\mu) - w_N(\mu)\|_X .
$$

(30)

As regards the output, we may prove

Q7. The error in the output is a priori bounded by

$$
|s(\mu) - s_N(\mu)| \leq \gamma(\mu) \left(1 + \frac{\gamma(\mu)}{\beta_N(\mu)}\right) \min_{w_N \in W^{pr}_N} \|u(\mu) - w_N\|_X \min_{\varphi_N \in W^{du}_N} \|\psi(\mu) - \varphi_N\|_X .
$$

(31)

$$
|s(\mu) - s_N(\mu)| = |\ell(u(\mu) - u_N(\mu)) - r^{pr}(\psi_N(\mu); \mu)|
$$

$$
= |a(u(\mu) - u_N(\mu), \psi(\mu); \mu) - f(\psi_N(\mu); \mu) + a(u_N(\mu), \psi_N(\mu); \mu)|
$$

$$
= |a(u(\mu) - u_N(\mu), \psi(\mu) - \psi_N(\mu); \mu)|
$$

$$
\leq \gamma(\mu) \|u(\mu) - u_N(\mu)\|_X \|\psi(\mu) - \psi_N(\mu)\|_X .
$$

(32)

from the definition of the primal and the adjoint problems, and the continuity condition.

2.5 A posteriori error estimation

We assume we are given a positive $\mu$-dependent lower bound $\hat{\beta}(\mu)$ for the stability constant $\beta(\mu)$: $\beta(\mu) \geq \hat{\beta}(\mu) \geq \beta_0 > 0, \forall \mu \in D$. We then introduce the dual norm of the primal residual

$$
\varepsilon^{pr}_N(\mu) = \sup_{v \in X} \frac{r^{pr}(v; u_N(\mu); \mu)}{\|v\|_X} ,
$$

(33)

and the dual norm of the dual residual

$$
\varepsilon^{du}_N(\mu) = \sup_{v \in X} \frac{r^{du}(v; \psi_N(\mu); \mu)}{\|v\|_X} ,
$$

(34)
where
\[ r_{du}^{\mu}(v; \varphi_N; \mu) = -\ell(v) - a(v, \varphi_N; \mu), \quad \forall v \in X \]  
(35)
is the dual residual associated with \( \varphi_N \in W_N^{du} \).

We may now introduce our energy error bounds
\[ \Delta_{pr}^N(\mu) = \frac{\varepsilon_{pr}^N(\mu)}{\hat{\beta}(\mu)}, \quad \Delta_{du}^N(\mu) = \frac{\varepsilon_{du}^N(\mu)}{\hat{\beta}(\mu)} \]  
(36)
The associated effectivities are defined as
\[ \eta_{pr}^N(\mu) \equiv \frac{\Delta_{pr}^N(\mu)}{\| u(\mu) - u_N(\mu) \|_X}, \quad \eta_{du}^N(\mu) \equiv \frac{\Delta_{du}^N(\mu)}{\| \psi(\mu) - \psi_N(\mu) \|_X}. \]  
(37)
We may also develop error bounds for the error in the output. In particular, we define the error estimator for the output error as
\[ \Delta_{s}^N(\mu) \equiv \frac{\varepsilon_{pr}^N(\mu)\varepsilon_{du}^N(\mu)}{\hat{\beta}(\mu)}, \]  
(38)
and the corresponding effectivity as
\[ \eta_{s}^N(\mu) \equiv \frac{\Delta_{s}^N(\mu)}{|s(\mu) - s_N(\mu)|}. \]  
(39)
Note that \( \Delta_{s}^N(\mu) \) scales as the product of the dual norm of the primal residual and the dual norm of the dual residual.

We can demonstrate that

**Q8.** The effectivities are bounded by
\[ 1 \leq \eta_{pr}^N(\mu) \leq \gamma(\mu)/\hat{\beta}(\mu), \quad \forall N, \forall \mu \in \mathcal{D}, \]  
(40)
\[ 1 \leq \eta_{du}^N(\mu) \leq \gamma(\mu)/\hat{\beta}(\mu), \quad \forall N, \forall \mu \in \mathcal{D}, \]  
(41)
\[ 1 \leq \eta_{s}^N(\mu) \leq \gamma(\mu)/\hat{\beta}(\mu), \quad \forall N, \forall \mu \in \mathcal{D}. \]  
(42)

It is important to observe that our effectivity upper bounds are independent of \( N \), and hence stable with respect to RB refinement.
3 Minimal residual approximation

3.1 Approximation

Given the RB spaces $W_N^{pr}$ and $W_N^{du}$, for all $w_N \in W_N^{pr}$ and $\varphi_N \in W_N^{du}$, we define the associated primal residual and dual residual as

$$
r_{pr}(v; w_N; \mu) \equiv f(v; \mu) - a(w_N, v; \mu), \quad \forall v \in X,
$$

(43)

$$
r_{du}(v; \varphi_N; \mu) \equiv -\ell(v; \mu) - a(v, \varphi_N; \mu), \quad \forall v \in X.
$$

(44)

It follows from our primal and dual problem statements that

$$
r_{pr}(v; w_N; \mu) = a(u(\mu) - w_N, v; \mu), \quad \forall v \in X,
$$

(45)

$$
r_{du}(v; \varphi_N; \mu) = a(v, \psi(\mu) - \varphi_N; \mu), \quad \forall v \in X
$$

which is the standard residual-error relation evoked in most a posteriori frameworks.

The minimal residual approach finds $u_{N}^{MR}(\mu) \in W_N^{pr}$ such that

$$
u_{N}^{MR}(\mu) = \arg \inf_{w_N \in W_N^{pr}} \sup_{v \in X} \frac{r_{pr}(v; w_N; \mu)}{\|v\|_X},
$$

(46)

and $\psi_{N}^{MR}(\mu) \in W_N^{du}$ such that

$$\psi_{N}^{MR}(\mu) = \arg \inf_{\varphi_N \in W_N^{du}} \sup_{v \in X} \frac{r_{du}(v; \varphi_N; \mu)}{\|v\|_X},
$$

(47)

which is a minimum-residual (or least-squares) projection. Invoking the standard duality argument, we obtain

$$
u_{N}^{MR}(\mu) = \arg \inf_{w_N \in W_N^{pr}} \|P^{pr}_\mu w_N\|_X,
$$

(48)

$$\psi_{N}^{MR}(\mu) = \arg \inf_{\varphi_N \in W_N^{du}} \|P^{du}_\mu \varphi_N\|_X,
$$

(49)

where $P^{pr}_\mu : w \in X \to X$ and $P^{du}_\mu : w \in X \to X$ are given by

$$
(P^{pr}_\mu w, v)_X = r_{pr}(v; w; \mu), \quad \forall v \in X,
$$

(50)

$$
(P^{du}_\mu w, v)_X = r_{du}(v; w; \mu), \quad \forall v \in X.
$$

(51)

Our output approximation is then given by

$$s_{N}^{MR}(\mu) = \ell(u_{N}^{MR}(\mu); \mu) - r_{pr}(\psi_{N}^{MR}(\mu); u_{N}^{MR}(\mu); \mu).
$$

(52)

The additional adjoint term will improve the accuracy.
3.2 Relation to Petrov-Galerkin approximation

We introduce the supremizing operator $T_\mu : X \to X$ such that for any given $\mu \in \mathcal{D}$ and any $w \in X$,

$$(T_\mu w, v)_X = a(w, v; \mu), \quad \forall v \in X. \quad (53)$$

We then define supremizing spaces

$$V_{pr}^N(\mu) = \text{span}\{T_\mu \zeta_{pr}^n, 1 \leq n \leq N\}, \quad (54)$$

$$V_{du}^N(\mu) = \text{span}\{T_\mu \zeta_{du}^n, 1 \leq n \leq N\}. \quad (55)$$

Unlike earlier definitions of reduced-basis spaces, the supremizing spaces are parameter-dependent.

The Petrov-Galerkin approach consists in finding $u_{PG}^N(\mu) \in W_{pr}^N$ from

$$a(u_{PG}^N(\mu), v; \mu) = f(v; \mu), \quad \forall v \in V_{pr}^N(\mu), \quad (56)$$

and $\psi_{PG}^N(\mu) \in W_{du}^N$ from

$$a(v, \psi_{PG}^N(\mu); \mu) = -\ell(v; \mu), \quad \forall v \in V_{du}^N(\mu). \quad (57)$$

The output estimate is evaluated as

$$s_{PG}^N(\mu) = \ell(u_{PG}^N(\mu); \mu) - r_{pr}^N(\psi_{PG}^N(\mu); u_{PG}^N(\mu); \mu). \quad (58)$$

We can demonstrate that

**Q9.** The minimum-residual approximation is equivalent to the standard Petrov-Galerkin approximation: $u_{MR}^N(\mu) = u_{PG}^N(\mu)$, $\psi_{MR}^N(\mu) = \psi_{PG}^N(\mu)$, and $s_{MR}^N(\mu) = s_{PG}^N(\mu)$.

**Lemma 1.** The solution $u_N(\mu)$ of (?) satisfies

$$a(u_N(\mu), v; \mu) = f(v; \mu), \quad \forall v \in V_{pr}^N(\mu). \quad (59)$$

An analogous result applies for the dual problem and dual solution.

**Proof.** We express $u_N(\mu) = \sum_{n=1}^N u_{Nn}(\mu)\zeta_{pr}^n$ and $w_N = \sum_{n=1}^N w_{Nn}(\mu)\zeta_{pr}^n$. It follows from (?) that

$$u_N(\mu) = \arg \inf_{w_N \in \mathbb{R}^N} G(w_N; \mu), \quad (60)$$
where

\[
G(w_N; \mu) = \| P_\mu^{pr} \left( \sum_{i=1}^{N} w_{N_i} \zeta_i^{pr} \right) \|^2_X
\]

\[
= r^{pr} (P_\mu^{pr} \left( \sum_{i=1}^{N} w_{N_i} \zeta_i^{pr} \right); \sum_{j=1}^{N} w_{N_j} \zeta_j^{pr}; \mu)
\]

\[
= f(P_\mu^{pr} \left( \sum_{i=1}^{N} w_{N_i} \zeta_i^{pr} \right); \mu) - a(\sum_{j=1}^{N} w_{N_j} \zeta_j^{pr}, P_\mu^{pr} \left( \sum_{i=1}^{N} w_{N_i} \zeta_i^{pr} \right); \mu). \quad (61)
\]

Taking the first derivatives of \( G(w_N; \mu) \) and using \( \partial P_\mu^{pr} w/\partial w = -T_\mu \), we obtain

\[
\frac{\partial G(w_N; \mu)}{\partial w_{N_i}} = -f(T_\mu \zeta_i^{pr}; \mu) - a(\zeta_i^{pr}, P_\mu^{pr} \left( \sum_{j=1}^{N} w_{N_j} \zeta_j^{pr} \right)) + \sum_{j=1}^{N} w_{N_j} a(\zeta_j^{pr}, T_\mu \zeta_i^{pr}; \mu). \quad (62)
\]

Furthermore, from (\ref{eq:62}), we have

\[
a(\zeta_i^{pr}, P_\mu^{pr} \left( \sum_{j=1}^{N} w_{N_j} \zeta_j^{pr} \right)) = (P_\mu^{pr} \left( \sum_{j=1}^{N} w_{N_j} \zeta_j^{pr} \right), T_\mu \zeta_i^{pr})_X
\]

\[
= r^{pr} (T_\mu \zeta_i^{pr}, \sum_{j=1}^{N} w_{N_j} \zeta_j^{pr}; \mu)
\]

\[
= f(T_\mu \zeta_i^{pr}; \mu) - a(\sum_{j=1}^{N} w_{N_j} \zeta_j^{pr}, T_\mu \zeta_i^{pr}; \mu) \quad (63)
\]

Combining yields

\[
\frac{\partial G(w_N; \mu)}{\partial w_{N_i}} = -2f(T_\mu \zeta_i^{pr}; \mu) + 2 \sum_{j=1}^{N} w_{N_j} a(\zeta_j^{pr}, T_\mu \zeta_i^{pr}; \mu). \quad (64)
\]

Therefore, \( u_N(\mu) \) is the solution of

\[
\sum_{j=1}^{N} a(\zeta_j^{pr}, T_\mu \zeta_i^{pr}; \mu) u_{N_j}(\mu) = f(T_\mu \zeta_i^{pr}; \mu), \quad i = 1, \ldots, N. \quad (65)
\]

This equation is the desired result.

We further note from (\ref{eq:66}) that

\[
\| P_\mu^{pr} \left( \sum_{n=1}^{N} w_{N_n}(\mu) \zeta_n^{pr} \right) \|^2_X = f(P_\mu^{pr} \left( \sum_{n=1}^{N} w_{N_n}(\mu) \zeta_n^{pr} \right); \mu) - a(w_N, P_\mu^{pr} \left( \sum_{n=1}^{N} w_{N_n}(\mu) \zeta_n^{pr} \right); \mu) \quad (67)
\]
3.3 Discrete equations

Inserting \( u^{\text{MR}}_N(\mu) = \sum_{j=1}^N u^{\text{MR}}_{Nj}(\mu) \zeta^\text{pr}_j \) and choosing \( v(\mu) = T_\mu c^\text{pr}_i, 1 \leq i \leq N \), in (56) we obtain

\[
A^{\text{pr},\text{MR}}_N(\mu)u^{\text{MR}}_N(\mu) = F^{\text{pr},\text{MR}}_N(\mu)
\]  

(68)

where \( A^{\text{pr},\text{MR}}_N(\mu) \in \mathbb{R}^{N \times N} \) and \( F^{\text{pr},\text{MR}}_N(\mu) \in \mathbb{R}^N \) are given by

\[
A^{\text{pr},\text{MR}}_{Nij}(\mu) = a(\zeta^\text{pr}_j, T_\mu c^\text{pr}_i; \mu), \quad F^{\text{pr},\text{MR}}_{Ni}(\mu) = f(T_\mu c^\text{pr}_i; \mu), \quad 1 \leq i, j \leq N .
\]  

(69)

Similarly, expressing \( \psi^{\text{MR}}_N(\mu) = \sum_{j=1}^N \psi^{\text{MR}}_{Nj}(\mu) \zeta^\text{du}_j \) and choosing \( v(\mu) = T_\mu c^\text{du}_i, 1 \leq i \leq N \), in (57) we arrive at

\[
A^{\text{du},\text{MR}}_N(\mu)\psi^{\text{MR}}_N(\mu) = -L^{\text{du},\text{MR}}_N(\mu)
\]  

(70)

where \( A^{\text{du},\text{MR}}_N(\mu) \in \mathbb{R}^{N \times N} \) and \( L^{\text{du},\text{MR}}_N(\mu) \in \mathbb{R}^N \) are given by

\[
A^{\text{du},\text{MR}}_{Nij}(\mu) = a(T_\mu c^\text{du}_i, \zeta^\text{du}_j; \mu), \quad L^{\text{du},\text{MR}}_{Ni}(\mu) = \ell(T_\mu c^\text{du}_i; \mu), \quad 1 \leq i, j \leq N .
\]  

(71)

In what follows, we shall give a detailed analysis of this reduced-basis (RB) approximation and develop associated a posteriori error estimation.

**Lemma 2.** The condition number of (7) is bounded by \( \gamma(\mu)/\beta(\mu) \) with

\[
\beta(\mu) = \inf_{v \in X} \frac{(T_\mu v, T_\mu v)_X}{\|v\|^2_X},
\]

(72)

\[
\gamma(\mu) = \sup_{v \in X} \frac{(T_\mu v, T_\mu v)_X}{\|v\|^2_X}.
\]

(73)

Proof.

\[
\sum_{i=1}^N \sum_{j=1}^N w_N i w_N j a(\zeta^\text{pr}_j, T_\mu c^\text{pr}_i; \mu) = \sum_{i=1}^N \sum_{j=1}^N w_N i w_N j (T_\mu c^\text{pr}_j, T_\mu c^\text{pr}_i)_X
\]

\[
\geq \beta(\mu) \sum_{i=1}^N \sum_{j=1}^N w_N i w_N j (c^\text{pr}_j, c^\text{pr}_i)_X
\]

\[
= \beta(\mu) \sum_{i=1}^N \sum_{j=1}^N w_N i w_N j \delta_{ij}
\]

\[
= \beta(\mu) \sum_{i=1}^N w^2_{Ni}.
\]

(74)
Similarly, we have

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} w_{Ni} w_{Nj} a(\zeta_i^{pr}, T_{\mu} \zeta_j^{pr}; \mu) \leq \gamma(\mu) \sum_{i=1}^{N} \sum_{j=1}^{N} w_{Ni} w_{Nj} (\zeta_i^{pr}, \zeta_j^{pr}) X
\]

\[
= \gamma(\mu) \sum_{i=1}^{N} \sum_{j=1}^{N} w_{Ni} w_{Nj} \delta_{ij}
\]

\[
= \gamma(\mu) \sum_{i=1}^{N} w_{Ni}^2 .
\]

(75)

Combining (74) and (75) yields

\[
\beta(\mu) \leq \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} w_{Ni} w_{Nj} a(\zeta_i^{pr}, T_{\mu} \zeta_j^{pr}; \mu)}{\sum_{i=1}^{N} w_{Ni}^2} \leq \gamma(\mu), \quad \forall w_N \in \mathbb{R}^N .
\]

(76)

Clearly, our RB system has the same conditioning properties as the underlying PDE. In the worst case, the condition number is bounded by the ratio \( \gamma(\mu)/\beta(\mu) \), which is independent of \( N \).

3.4 A priori convergence

Q9. Show that \( A^{pr, MR}_N(\mu) \) and \( A^{du, MR}_N(\mu) \) are symmetric positive definite. Furthermore, the condition number of both \( A^{pr, MR}_N(\mu) \) and \( A^{du, MR}_N(\mu) \) is bounded by \( \gamma(\mu)/\beta(\mu) \).

Q10. The primal RB solution \( u_{MR}^N(\mu) \) and the dual RB solution \( \psi_{MR}^N(\mu) \) are optimal in the \( X \)-norm in the following sense

\[
\| u(\mu) - u_{MR}^N(\mu) \|_X \leq \left( 1 + \frac{\gamma(\mu)}{\beta(\mu)} \right) \min_{w_N \in W^{pr}_N} \| u(\mu) - w_N(\mu) \|_X ,
\]

(77)

\[
\| \psi(\mu) - \psi_{MR}^N(\mu) \|_X \leq \left( 1 + \frac{\gamma(\mu)}{\beta(\mu)} \right) \min_{w_N \in W^{du}_N} \| \psi(\mu) - w_N(\mu) \|_X .
\]

(78)

Q11. The error in the output is a priori bounded by

\[
| s(\mu) - s_{MR}^N(\mu) | \leq \gamma(\mu) \left( 1 + \frac{\gamma(\mu)}{\beta(\mu)} \right) \min_{w_N \in W^{pr}_N} \| u(\mu) - w_N \|_X \min_{\varphi_N \in W^{du}_N} \| \psi(\mu) - \varphi_N \|_X .
\]

(79)
3.5 A posteriori error estimation

We introduce the dual norm of the primal residual

\[ \varepsilon_{pr,MR}^N(\mu) = \sup_{v \in X} \frac{r_{pr}(v; u_{MR}^N(\mu); \mu)}{\|v\|_X}, \quad (80) \]

and the dual norm of the dual residual

\[ \varepsilon_{du,MR}^N(\mu) = \sup_{v \in X} \frac{r_{du}(v; \psi_{MR}^N(\mu); \mu)}{\|v\|_X}, \quad (81) \]

where We may now introduce our energy error bounds

\[ \Delta_{pr,MR}^N(\mu) = \varepsilon_{pr,MR}^N(\mu) \hat{\beta}(\mu), \quad \Delta_{du,MR}^N(\mu) = \frac{\varepsilon_{du,MR}^N(\mu)}{\hat{\beta}(\mu)} \quad (82) \]

The associated effectivities are defined as

\[ \eta_{pr,MR}^N(\mu) \equiv \frac{\Delta_{pr,MR}^N(\mu)}{\|u(\mu) - u_{MR}^N(\mu)\|_X}, \quad \eta_{du,MR}^N(\mu) \equiv \frac{\Delta_{du,MR}^N(\mu)}{\|\psi(\mu) - \psi_{MR}^N(\mu)\|_X}. \quad (83) \]

We may also develop error bounds for the error in the output. In particular, we define

\[ \Delta_{s,MR}^N(\mu) \equiv \frac{\varepsilon_{pr,MR}^N(\mu)\varepsilon_{du,MR}^N(\mu)}{\hat{\beta}(\mu)}, \quad (84) \]

and the corresponding effectivity as

\[ \eta_{s,MR}^N(\mu) \equiv \frac{\Delta_{s,MR}^N(\mu)}{|s(\mu) - s_{MR}^N(\mu)|}. \quad (85) \]

Note that \( \Delta_{s,MR}^N(\mu) \) scales as the product of the dual norm of the primal residual and the dual norm of the dual residual.

We can demonstrate that

**Q12.** We have \( \Delta_{pr,MR}^N(\mu) \leq \Delta_{pr}^N(\mu), \Delta_{du,MR}^N(\mu) \leq \Delta_{du}^N(\mu), \) and \( \Delta_{s,MR}^N(\mu) \leq \Delta_{s}^N(\mu). \)

Furthermore, it can be shown that

\[ 1 \leq \eta_{pr,MR}^N(\mu) \leq \frac{\gamma(\mu)}{\hat{\beta}(\mu)}, \quad \forall N, \forall \mu \in \mathcal{D}, \quad (86) \]

\[ 1 \leq \eta_{du,MR}^N(\mu) \leq \frac{\gamma(\mu)}{\hat{\beta}(\mu)}, \quad \forall N, \forall \mu \in \mathcal{D}, \quad (87) \]

\[ 1 \leq \eta_{s,MR}^N(\mu) \leq \frac{\gamma(\mu)}{\hat{\beta}(\mu)}, \quad \forall N, \forall \mu \in \mathcal{D}. \quad (88) \]