On the Stability of Wholesale Electricity Markets under Real-Time Pricing

Mardavij Roozbehani †   Munther Dahleh ‡   Sanjoy Mitter ♯

Abstract— The paper proposes a mathematical model for the dynamic evolution of supply, demand, and clearing prices under a class of real-time pricing mechanisms characterized by passing on the real-time wholesale prices to the end consumers. The effects that such mechanisms could pose on the stability and efficiency of the entire system is investigated and several stability criteria are presented. It is shown that relaying the real-time wholesale electricity prices to the end consumers creates a closed loop feedback system which could be unstable or lack robustness, leading to extreme price volatility. Finally, a result is presented which characterizes the efficiency losses incurred when, in order to achieve stability, the wholesale prices are adjusted by a static pricing function before they are passed on to the retail consumers.

I. INTRODUCTION AND MOTIVATION

The increasing demand for energy and growing environmental concerns have created the need for a more efficient, modern power grid that will accommodate distributed and renewable energy resources, storage, and real-time demand response technologies. In this paper, we are concerned with the analysis, and to a lesser extent, the design of a particular class of dynamic pricing mechanisms for real-time retail pricing of electricity in modern power grids. Our focus will be almost entirely on the stability and efficiency properties of the ensuing closed loop feedback system.

There is an existing body of literature on dynamic pricing in communication or transportation networks. See for instance [5], [2], [7] and the references therein. However, the specific characteristics of power systems arising from the close interaction between physics and economics, along with the safety-critical nature of the system and the uncertainty in consumer behavior, raise very unique challenges that need to be addressed.

In [1], Borenstein et. al. study both the theoretical and the practical implications of various forms of dynamic pricing such as *Critical Peak Pricing*, *Time-of-Use Pricing*, and *Real-Time Pricing*. They argue in favor of real-time pricing, characterized by passing on a price, that best reflects the wholesale market prices, to the end consumers. They conclude that real-time pricing delivers the most benefits in the

sense of reducing the peak and flattening the load curve. A similar conclusion is reached in a study conducted by Energy Futures Australia (EFA-IEADSM) [3].

The appeal of dynamic retail pricing is not limited to theoretical research and academic studies. In California, the state's Public Utility Commission (CPUC) has enacted a series of new energy regulations which set a deadline of 2011 for the state utility PG&E to propose a new *dynamic pricing* rate structure, specifically defined as an electric rate structure that reflects the actual wholesale market conditions, such as critical peak pricing or real-time pricing. CPUC defines the real-time price as a rate linked to the actual hourly wholesale energy price [13]. In this paper, we show that directly linking the consumer prices to the wholesale market prices creates a close-loop feedback system in which the prices may oscillate or diverge to unacceptable limits. We do not argue against the idea of real-time pricing in power grids in its entirety, nor do we suggest that it cannot fulfill its purposes. The message that we intend to deliver is that the design of a real-time pricing mechanism must take system stability issues into consideration, and that successful design and implementation of such a mechanism entails careful analysis of consumer behavior in response to price signals. Whether or not directly linking consumer prices to the wholesale market prices will cause instability depends on the utility functions of consumers and producers.

The organization of this paper is as follows. In Section II, we present a simple model for dynamic evolution of real-time wholesale electricity prices. It is assumed that the consumers are price-taking agents who respond to price signals by adjusting their consumption, so as to maximize a quasi-linear concave utility function. It is also assumed that supply always follows demand, in the sense that at each instant of time, the amount of electricity demanded by the consumers must be matched by the producers, and the per-unit price associated with this exchange is the marginal cost of supplying the demand. The consumers then adjust their usage (myopically) by maximizing their utility functions for the next time period based on the new given or predicted price. Their adjusted demand is then a feedback signal to the wholesale market and affects the prices for the next time step. Section III contains the main theoretical contributions of this paper. We analyze the stability and efficiency of the closed loop system arising from the setup and present several stability criteria based on the cost functions of the producers and the value functions of the consumers.

II. PRELIMINARIES

A. Notation

The set of positive real numbers (integers) is denoted by $\mathbb{R}_+$ ($\mathbb{Z}_+$), nonnegative real numbers (integers) by $\overline{\mathbb{R}}_+$ ($\overline{\mathbb{Z}}_+$). The class of real-valued functions with a continuous n-th derivative on $X \subset \mathbb{R}$ is denoted by $\mathcal{C}^n X$. For a differentiable function f, we use $\dot{f}$ to denote the derivative of f with respect to its argument: $\dot{f}(x) = df(x)/dx$. Since throughout the paper time is a discrete variable, this notation would not be confused with derivative with respect to time. Finally, for a measurable set $X \subset \mathbb{R}$, $\mu_L(X)$ is the Lebesgue measure of X.

B. Market Participants

We begin with developing a simple electricity market model with three participants: 1. The suppliers, 2. The consumers, and 3. An independent system operator (ISO). The suppliers and the consumers are price-taking, profit-maximizing agents. The ISO is an independent, profit-neutral player in charge of clearing the market, that is, matching supply and demand subject to the network constraints with the objective of maximizing the social welfare, i.e., the aggregate surplus of consumers and producers. When the demand is fixed, the objective is to minimize the total cost of production. Below, we describe the characteristics of each agent in detail.

1) The Consumers and the Producers: Let $D = \{1, ..., n_d\}$ and $S = \{1, ..., n_s\}$ denote the index sets of consumers and producers respectively. Each consumer $j \in D$ is associated with a value function $v_j(\cdot) : \overline{\mathbb{R}}_+ \to \mathbb{R}$, where $v_j(x)$ which can be thought of as the dollar value that consumer j derives from consuming x units of the resource, electricity in this case. Similarly, each producer $i \in S$, is associated with a cost function $c_i(\cdot) : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ representing the dollar cost per unit production of the resource.

Assumption I: For all $i \in S$, the cost functions $c_i(\cdot)$ are in $\mathcal{C}^2(0, \infty)$, strictly increasing, and strictly convex. For all $j \in D$, the value functions $v_j(\cdot)$ are in $\mathcal{C}^2(0, \infty)$, strictly increasing, and strictly concave.

Let $d_j : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$, $j \in D$, and $s_i : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$, $i \in S$ denote $\mathcal{C}^1$ functions mapping price to consumption and production respectively. In the framework of price-taking, utility-maximizing agents, each agent maximizes the net benefit that they can derive from the market. Therefore,

$$d_j(\lambda) = \arg\max_{x \in \overline{\mathbb{R}}_+} v_j(x) - \lambda x, \qquad j \in D, \quad (1)$$

$$s_i(\lambda) = \arg\max_{x \in \overline{\mathbb{R}}_+} \lambda x - c_i(x), \qquad i \in S. \quad (2)$$

Remark 1: Under Assumption I, when $\lambda \in (0, \infty)$, the maximization problems defined in (1) and (2) have a unique solution in $\overline{\mathbb{R}}_+$ and the functions $d_j(\cdot)$ and $s_i(\cdot)$ are well-defined. Furthermore,

$$d_j(\lambda) = \max\{0, \{x \mid \dot{v}_j(x) = \lambda\}\}$$
$$s_i(\lambda) = \max\{0, \{x \mid \dot{c}_i(x) = \lambda\}\}$$

In the interest of simplicity and in order to avoid distracting details, for the rest of this paper, we assume that $d_j(\lambda) = \dot{v}^{-1}(\lambda)$ and $s_i(\lambda) = \dot{c}^{-1}(\lambda)$. This can be mathematically justified by adding the assumptions $\dot{v}(0) = \infty$, and $\dot{c}(0) = 0$ to Assumption I, or by assuming that $\lambda \in [\dot{c}(0), \dot{v}(0)]$.

Definition 1: The social welfare $\mathcal{S}$ is the aggregate benefit of the producers and the consumers:

$$\mathcal{S} = \sum_{j \in D}(v_j(d_j) - \lambda_j d_j) - \sum_{i \in S}(\lambda_i s_i - c_i(s_i))$$

When the system is at the equilibrium in the sense that the total supply equals the total demand and there is a unique clearing price λ for the entire system, then:

$$\mathcal{S} = \sum_{j \in D} v_j(d_j) - \sum_{i \in S} c_i(s_i)$$

2) The Independent System Operator (ISO): The ISO is a non-for-profit entity whose primary function is to optimally match supply and demand (adjusted for reserve requirements) subject to network constraints. The network constraints include power flow constraints (Kirchhoff's laws), transmission line constraints, generator capacity constraints, local and system wide-reserve capacity requirements and potentially some other ISO-specific constraints [12], [10], [11]. For real-time market operation, the constraints are linearized near the steady state operating point and the ISO optimization problem is reduced to a linear program, often referred to as the *Economic Dispatch Problem (EDP)*. A set of Locational Marginal Prices emerge as the dual variables corresponding to the nodal power balance constraints of this optimization problem. These prices vary from location to location as they represent the marginal cost of supplying electricity to a particular location. We refer the interested reader to [9], [11], [10] for more details. However, we would like to point out that the spatial variation in the LMPs is a consequence of congestion in the transmission lines. When there is sufficient transmission capacity in the network, a uniform price will materialize for the entire system. With this observation in sight and in order to develop a tractable mathematical model, we make the following simplifying assumptions:

1) Resistive losses in the transmission and distribution lines are negligible.
2) The line capacities are high enough, so, congestion will not occur.
3) There are no generator capacity constraints.
4) There are no reserve capacity requirements.

Under the first two assumptions, the network parameters become irrelevant in the supply-demand optimal matching problem. The third and fourth assumptions are made in the interest of keeping the development in this paper focused. They could, otherwise, be relaxed at the expense of a more involved technical analysis. See [8] for results on dynamic pricing in electricity networks with transmission line and generator capacity constraints.

The following problem then characterizes the ISO's optimization problem:

$$\max \quad \sum_{j \in D} v_j(d_j) - \sum_{i \in S} c_i(s_i)$$
$$\text{s.t.} \quad \sum_{j \in D} d_j = \sum_{i \in S} s_i \tag{3}$$

The following lemma which is adopted from [6], provides the justification for defining the LMPs as the Lagrangian multipliers corresponding to the balance constraints.

Lemma 1: Let $d^* = \left[d_1^*, \cdots, d_{n_d}^*\right]$, and $s^* = \left[s_1^*, \cdots, s_{n_s}^*\right]$ where d_j^*, $j \in D$ and s_i^*, $i \in S$, solve (3). There exists a price $\lambda^* \in (0, \infty)$, such that d^* and s^* solve (1) and (2). Furthermore, λ^* is the Lagrangian multiplier corresponding to the balance constraint in (3).

Proof: The proof is based on Lagrangian duality and is omitted for brevity. The proof in [6] would be applicable here with some minor adjustments. ∎

The implication of Lemma 1 is that by setting the market price to λ^*, the system operator creates a competitive environment in which, the collective selfish behavior of the participants results in a system-wide optimal condition. In other words, the aggregate surplus is maximized while each agent maximizes her own net benefit.

Consider the special case where the consumers do not bid in the market, that is, they do not provide their value functions to the system operator. In this case, which is most relevant to real-time system operation, the demand is taken as fixed, and (3) is reduced to meeting the fixed demand at minimum cost:

$$\min \quad \sum_{i \in S} c_i(s_i)$$
$$\text{s.t.} \quad \sum_{j \in D} d_j = \sum_{i \in S} s_i \tag{4}$$

The case of fixed demand can be associated with a non-differentiable value function and in theory, the completely inelastic consumers would be willing to pay any price $\lambda \in [0, \infty)$. However, we assume that the system operator solves (4) and sets the price to the marginal cost of production at the minimum cost solution. This is the model adopted in this paper. We assume that the retail consumers do not bid in the market and hence, the demand is taken as fixed over each pricing interval. The specific details are presented in Subsection II-C.

3) Representative Agent Model: A representative agent is a fictitious agent whose response to a signal or an event is mathematically equivalent to the aggregate response of a group of agents [4]. In this subsection, we develop an abstract model of (3)−(4) with only one producer agent and one consumer agent representing the entire group of producers and consumers respectively. The rationale is that it is the aggregate supply or demand that influences the macroscopic properties of the system. For the purpose of theoretical analysis of the system of interest in this paper,

such construction is always possible, though, explicit formulae for the representative agent may sometimes be very complicated or impossible to find. The following lemma presents a construction for the representative agent model, applicable to the development in this paper.

Lemma 2: Let functions v_j, $j \in D$, and c_i, $i \in S$, satisfy Assumption I, $\dot{v}_j(0) = \infty$, $\forall j$, and $\dot{c}_i(0) = 0$, $\forall i$. Suppose that there exists functions v and c satisfying Assumption I, and

$$\lambda = \dot{v}\left(\sum_{i=1}^{n_d} \dot{v}_i^{-1}(\lambda)\right), \quad \forall \lambda \in \mathbb{R}_+ \tag{5}$$

and

$$\lambda = \dot{c}\left(\sum_{i=1}^{n_s} \dot{c}_i^{-1}(\lambda)\right), \quad \forall \lambda \in \mathbb{R}_+. \tag{6}$$

Then:

1) If (d^*, s^*) solves (3), then $\bar{d}^* \overset{\text{def}}{=} \sum d_j^*$ and $\bar{s}^* \overset{\text{def}}{=} \sum_i^* s_i^*$ satisfy:

$$\bar{d}^* = \bar{s}^* = x^*$$

where x^* solves:

$$\max_x \quad v(x) - c(x) \tag{7}$$

2) If λ^* and $\bar{\lambda}^*$ are the optimal clearing prices corresponding to (3) and (7) respectively, then $\lambda^* = \bar{\lambda}^* = \dot{v}(x^*) = \dot{c}(x^*)$.

Example 1: Consider the case where all agents are identical: $v_{i_1} = v_{i_2}$, $\forall i_1, i_2$. Then $v(x) = n_d v_1\left(n_d^{-1} x\right)$ satisfies (5). As another example, consider $v_i(x) = \alpha_i \log(1 + x)$, and define $v(x) = \bar{\alpha} \log(n_d + x)$, where $\bar{\alpha} = \sum \alpha_i$. Then v satisfies (5). However, since $\dot{v}_i(0) = \alpha_i < \infty$, the response of the representative is equal the sum of the responses of the individual agents only when $\lambda \leq \min_i \alpha_i$.

C. Dynamic Supply-Demand Model

In this section we develop a dynamical system model for the interaction of wholesale supply and retail demand in electricity markets. The model is consistent the current practice in real-time balancing markets in the United States, with the exception that it assumes that the consumers adjust their usage based on the real-time wholesale market prices.
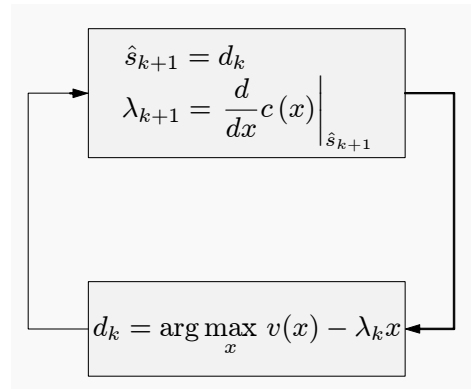

Fig. 1. Exanté Priced Supply/Demand Feedback

In a power grid, the aggregate supply has to match the aggregate demand at every instant of time. Therefore, in real-time, supply always follows demand. The real-time market is cleared at discrete time intervals and the prices are calculated and announced for each interval[1]. When the price announced at time k is calculated based on the actual demand during the time interval $[k-1, k]$, it is called the *ex-post price*. In ex-post pricing the demand is subject to some price uncertainty as the actual price will be revealed after the consumption has materialized. When the price announced at time k is calculated based on the predicted demand for the interval $[k, k+1]$, it is called the *exanté price*. In exanté pricing without ex-post adjustment, the ISO faces price uncertainty as it will have to reimburse the generators based on the actual marginal cost of production (that is, the ex-post price), while it can charge the demand only based on the exanté price, which is only a prediction of the actual price.

1) Price Dynamics under Exanté Pricing: We will use representative agent models with cost and value functions $c(\cdot)$ and $v(\cdot)$ to represent supply and demand respectively. Let λ_k denote the exanté price corresponding to the consumption of one unit of electricity in the time interval $[k, k+1]$. Let d_k be the actual consumption during this interval:

$$d_k = \arg\max_{x \in \overline{\mathbb{R}}_+} \; v(x) - \lambda_k x.$$

Since $v(\cdot)$ is known only to the consumer, at time k, only an estimate of d_k is available to the ISO, based on which the price λ_k has been calculated. At time $k+1$, the ISO needs to announce λ_{k+1}, which will be the marginal cost of predicted production during the next time interval. We assume that the ISO's predicted production for each time interval is equal to the demand at the previous time interval: $\hat{s}_{k+1} = d_k$. The following equations describe the dynamics of the market:

$$\begin{aligned}
\lambda_{k+1} &= \dot{c}(\hat{s}_{k+1}) \\
\hat{s}_{k+1} &= d_k \\
d_k &= \arg\max_{x \in \overline{\mathbb{R}}_+} \; v(x) - \lambda_k x
\end{aligned}$$

The price dynamics can be obtained from the above equations and is as follows:

$$\lambda_{k+1} = \dot{c}\left(\dot{v}^{-1}(\lambda_k)\right). \tag{8}$$

Remark 2: ISO's Risk: The system operator commits to a price of $\lambda_k = \dot{c}(d_{k-1})$ for the consumers, while the generators demand the price $\lambda_{k+1} = \dot{c}(d_k)$. The ISO's revenue differential (either excess or shortfall) is:

$$\Delta_k = [\dot{c}(d_k) - \dot{c}(d_{k-1})] d_k \simeq \ddot{c}(d_k) d_k (d_k - d_{k-1}).$$

2) Price Dynamics under Ex-post Pricing: Under ex-post pricing, the price charged for consumption of one unit of electricity during the interval $[k, k+1]$ is declared at the end of the interval, when the total consumption has materialized. In this case, the price uncertainty and the associated risks
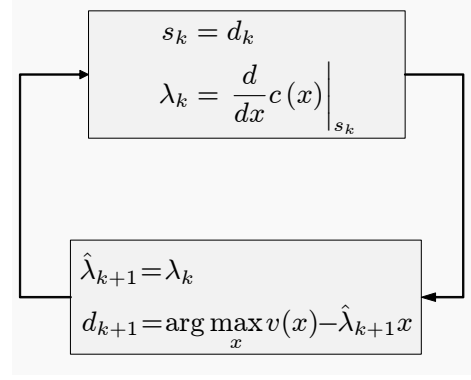
Fig. 2. Expost Priced Supply/Demand Feedback

are bore by the consumer. In order to decide on the amount to consume during $[k, k+1]$ the consumer needs to make a prediction about the price for this interval. We assume that the consumer's predicted price is equal to the price at the previous interval: $\hat{\lambda}_{k+1} = \lambda_k$. Therefore,

$$\begin{aligned}
\hat{\lambda}_{k+1} &= \lambda_k \\
d_{k+1} &= \arg\max_x v(x) - \hat{\lambda}_{k+1} x \\
\lambda_{k+1} &= \dot{c}(d_{k+1})
\end{aligned}$$

It is observed that the price dynamics is identical to the case with exanté pricing (8). The difference is that the price uncertainty affects the consumer.

III. THEORETICAL STATEMENTS

A. Stability Analysis

In this section we present several stability criteria based on Lyapunov techniques and examine stability properties of the clearing price dynamics (8).

Theorem 1: Consider a sequence $\{x_k\} \in \mathbb{R}_+$ satisfying

$$\begin{aligned}
x_0 &\in X_0 \subset \mathbb{R}_+ \\
x_{k+1} &= \psi(x_k)
\end{aligned} \tag{9}$$

for some function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. There exists a function $x^* : X_0 \to \overline{\mathbb{R}}_+$, satisfying

$$\lim_{k \to \infty} x_k = x^*(x_0) \tag{10}$$

if either of the following three conditions hold:

1) $\psi(x) \leq x, \; \forall x \in \mathbb{R}_+$.

2) $\psi \in \mathcal{C}^1(0, \infty)$, and the following two conditions hold:

(i) $\left|\dot{\psi}(x)\right| \leq 1.$ $\quad \mu_L\left(\{x \mid \dot{\psi}(x) = 1\}\right) = 0$

(ii) $\lim_{x \to \infty} \{\psi(x) - x\} < 0$

3) There exist continuously differentiable functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying

$$g(x_{k+1}) = f(x_k) \tag{11}$$

and

(i) $\left|\dot{f}(x)\right| \leq |\dot{g}(x)|, \quad \mu_L(\{x \mid \dot{f}(x) = \dot{g}(x)\}) = 0$

(ii) $\lim_{x \to \infty} \{f(x) - g(x)\} < 0$

(iii) either $\dot{g}(x) \geq 0$, or $\dot{g}(x) \leq 0$, $\forall x \in \mathbb{R}_+$

(12)

Proof: To prove 1, let $V(x) = \psi(x)$, and note that V is a Lyapunov function for (9) in the sense that $V(x_{k+1}) \leq V(x_k)$. Consider the iterative equation (9) and define

$$\underline{k} = \inf \left\{ k \mid V\left(\psi^{k+1}(x_0)\right) = V\left(\psi^k(x_0)\right) \right\}$$

If $\underline{k} < \infty$, then $x^*(x_0) \stackrel{\text{def}}{=} \lim_{k \to \infty} x_k = \psi^{\underline{k}+1}(x_0)$. If $\underline{k} = \infty$, then $\{V(x_k)\}$ is a strictly decreasing bounded sequence and must converge to a limit $\psi^* \in \overline{\mathbb{R}}_+$. Then, $x^*(x_0) \stackrel{\text{def}}{=} \lim_{k \to \infty} x_k = \psi^*$. Only 3 needs to be proven as 2 is a special case of 3 with $g(x) = x$, and $f(x) = \psi(x)$. Let $V(x) = |f(x) - g(x)|$. Then

$$\forall x, y \in \mathbb{R}_+, \ x \neq y:$$

$$|f(x) - f(y)| \leq \left| \int_y^x \left|\dot{f}(\tau)\right| d\tau \right|$$

$$< \left| \int_y^x |\dot{g}(\tau)| d\tau \right| = |g(x) - g(y)| \quad (13)$$

(The last equality holds under assumption 3-(iii), though, the sign-invariance assumption could be relaxed at the expense of a more involved analysis and the inequality would hold as long as $\dot{g}(\cdot)$ does not change sign infinitely often over a finite interval). We have

$$
\begin{aligned}
V(x_{k+1}) &- V(x_k) \\
&= |f(x_{k+1}) - g(x_{k+1})| - |f(x_k) - g(x_k)| \\
&= |f(x_{k+1}) - f(x_k)| - |g(x_{k+1}) - g(x_k)| \\
&< 0. \quad (14)
\end{aligned}
$$

Therefore, $\{V(x_k)\}$ is a strictly decreasing bounded sequence and converges to a limit $c \geq 0$. We show that $c > 0$ is not possible. Note that the sequence x_k is bounded from below as $\psi(x_k) > 0$, $\forall k$. Furthermore, it can be shown—using an argument similar to the one in the proof of statement 1—that the condition $\lim_{x \to \infty} \{f(x) - g(x)\} < 0$ implies that

$$\forall x_0 : \exists\, M < \sup_{x \in \mathbb{R}_+} g(x),\ N \in \mathbb{Z}_+ : g(x_k) \leq M,\ \forall k \geq N.$$

(15)

Subsequently, (15) along with $x_k > 0$ and continuity of $g(\cdot)$ imply that $\{x_k\}$ is a bounded sequence. Therefore, either $\lim_{k \to \infty} x_k = 0$ (in which case $x^*(x_0) = 0$) or $\{x_k\}$ has a subsequence $\{x_{k_i}\}$ which converges to a limit $x^* \in \mathbb{R}_+$. In the latter case we have

$$
\begin{aligned}
\lim_{k \to \infty} V(x_k) &= \lim_{i \to \infty} V(x_{k_i}) = \left| \lim_{i \to \infty} \{f(x_{k_i}) - g(x_{k_i})\} \right| \\
&= |f(x^*) - g(x^*)|
\end{aligned}
$$

If $g(x^*) = g(\psi(x^*))$ then $c = |f(x^*) - g(\psi(x^*))| = 0$ (due to (11)). If $g(x^*) \neq g(\psi(x^*))$ then

$$\exists \delta, \varepsilon > 0, \ \text{s.t.}\ |g(\psi(x)) - g(x)| \geq \varepsilon,\ \forall x \in \mathcal{B}(x^*, \delta)$$

Consider the function $\theta : \mathcal{B}(x^*, \delta) \to \overline{\mathbb{R}}_+$, where

$$\theta : x \mapsto \frac{|f(\psi(x)) - f(x)|}{|g(\psi(x)) - g(x)|}$$

Then $\theta(x) < 1$, $\forall x \in \mathcal{B}(x^*, \delta)$ (cf. 14). Furthermore, the function is continuous over the compact set $\mathcal{B}(x^*, \delta)$ and achieves its supremum $\bar{\theta}$, where $\bar{\theta} < 1$. Since x_{k_i} converges to x^* there exists $\hat{k} \in \mathbb{N}$, such that $x_{\hat{k}} \in \mathcal{B}(x^*, \delta)$. Then

$$
\begin{aligned}
V(x_{k+1}) &- \bar{\theta} V(x_k) = \\
|f(x_{k+1}) - f(x_k)| &- \bar{\theta} |g(x_{k+1}) - g(x_k)| \leq 0,\ \forall k \geq \hat{k}
\end{aligned}
$$

Since $\bar{\theta} < 1$, this proves that $c = 0$. Finally,

$$\lim_{k \to \infty} f(x_k) = \lim_{k \to \infty} g(x_k) = g(x^*) = f(x^*)$$

$$x^* = g^{-1}\big(\lim_{k \to \infty} f(x_k)\big) = \lim_{k \to \infty} g^{-1} \circ f(x_k) = \lim_{k \to \infty} x_k$$

This completes the proof. ∎

Remark 3: If condition 3-(iii) of Theorem 1 is relaxed, then the core of the proof remains valid. Only the final step would need to change as $g(\cdot)$ might not be invertible and the conclusion would be that $\{g(x_k)\}$ is convergent. Also, if condition 3-(i) is changed to the more conservative condition:

$$\exists \theta \in (0, 1) : \forall x \in \mathbb{R}_+ : \left|\dot{f}(x)\right| \leq \theta |\dot{g}(x)|$$

then 3-(ii) is not needed, and if it is changed to:

$$\exists \theta \in (-1, 1) : \forall x \in \mathbb{R}_+ : \left|\dot{f}(x)\right| \leq \theta \dot{g}(x) \quad (16)$$

then 3-(iii) is automatically satisfied. In order to present our results more concisely and with less technical details, we will present the rest of our results based on (16).

There are situations in which, a natural decomposition of system (9) via functions f and g satisfying (11) (or (16)) is readily available. As it was already mentioned, this is the case for the price dynamics (8), where $\psi = \dot{c} \circ \dot{v}^{-1}$, and the decomposition is obtained with $g = \dot{c}^{-1}$, and $f = \dot{v}^{-1}$. However, f and g obtained in this way may not readily satisfy (12). We present the following corollaries.

Corollary 1: Consider system (9) and suppose that continuously differentiable functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (11) are given. If there exists $\theta \in (-1, 1)$, and a continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\left| \rho(f(x))\, \dot{f}(x) \right| \leq \theta \rho(g(x))\, \dot{g}(x),\ \forall x \in \mathbb{R}_+,$$

then there exists functions $x^* : X_0 \to \overline{\mathbb{R}}_+$, and $r : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ satisfying

$$
\begin{aligned}
\lim_{k \to \infty} r(g(x_k)) &= x^*(x_0), \\
\dot{r}(x) &= \rho(x),\ \forall x \in \mathbb{R}_+.
\end{aligned}
$$

Furthermore, if $r \circ g$ is invertible, then (10) holds.

Proof: If f and g satisfy (11) then so do $r \circ f$ and $r \circ g$ for any $r \in \mathcal{C}(0, \infty)$. The result then follows from Theorem 1 and the discussion in Remark 3. ∎

Corollary 2: The system (8) is stable in the sense defined in Theorem 1, if there exists $\theta \in (0, 1)$ and a continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$|\rho(\eta)\dot{\eta}| \leq \theta\rho(\sigma)\dot{\sigma} \tag{17}$$

where

$$\eta = \dot{v}^{-1} \text{ and } \sigma = \dot{c}^{-1} \tag{18}$$

Furthermore, if

$$|\ddot{c}| \leq \theta\ddot{v} \tag{19}$$

then (8) is stable.

Proof: The first statement follows from Corollary 1, and the monotonicity properties of σ and r (r is the integral of the positive function ρ) which guarantee invertibility of $r \circ \sigma$. The second statement is proven by taking $\rho = \ddot{c}$ and using the inverse derivative formulae

$$\frac{d}{dx}h^{-1}(x) = \frac{1}{\dot{h}(h^{-1}(x))}$$

∎

Stability and convergence analysis of the wholesale market prices can be as well done using the model of demand (or supply) dynamics. It can be verified that the price dynamics (8) is stable if and only if the demand dynamics

$$d_{k+1} = \dot{v}^{-1}(\dot{c}(d_k)) \tag{20}$$

is stable. The advantage of using (20) instead of (8) is that application of Theorem 1 to (20) leads to simpler conditions. We have the following Corollary.

Corollary 3: The system (8) is stable in the sense defined in Theorem 1, if there exists $\theta \in (-1, 0)$ and a continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\rho(\dot{c})\ddot{c} \leq \theta\rho(\dot{v})\ddot{v} \tag{21}$$

Proof: It is sufficient to prove that (20) is stable, which in turn follows from (21) and Corollary 1. ∎

Remark 4: It is observed that condition (19), which was obtained by the choice of $\rho = \ddot{c}$ in (17), can be obtained from (21) by choosing $\rho(x) = 1$, for all $x \in \mathbb{R}_+$. Though the criteria in Corollary 2 and 3 are mathematically equivalent, it is a matter of availability of explicit expressions that determines which of the two is more convenient to apply. When the cost and value functions are explicitly available, condition (21) is more convenient to check, whereas, when explicit expressions are available for the supply and demand functions (σ and η), it is more convenient to work with (17).

Example 2: Consider (8) with $c(x) = x^\beta$, and $v(x) = x^{1/\alpha}$, where $\alpha, \beta > 1$. Then

$$\lambda_{k+1} = \beta(\alpha\lambda_k)^{\frac{\alpha\beta-\alpha}{1-\alpha}}$$

$$\dot{v}(x) = \alpha^{-1}x^{\frac{1-\alpha}{\alpha}}, \quad \ddot{v}(x) = (1-\alpha)\alpha^{-2}x^{\frac{1-2\alpha}{\alpha}}$$

$$\dot{c}(x) = \beta x^{\beta-1}, \quad \ddot{c}(x) = \beta(\beta-1)x^{\beta-2}$$

It can be verified that there does not exists a constant $\theta \in \mathbb{R}$ for which $|\ddot{c}(x)| \leq \theta\ddot{v}(c), \forall x \in \mathbb{R}$. However, with $\rho(x) = 1/x$, we have:

$$\rho(\dot{c}(x))\ddot{c}(x) = (\beta-1)x^{-1}, \quad \rho(\dot{v})\ddot{v} = (1-\alpha)\alpha^{-1}x^{-1}$$

Therefore, (21) is satisfied with

$$\theta = \frac{\alpha(\beta-1)}{1-\alpha}, \qquad \theta > -1 \text{ if } \beta < 2 - \alpha^{-1}$$

Hence, the system is stable for $\beta < 2 - \alpha^{-1}$. It can be shown that the condition is also necessary and the system diverges for $\beta > 2 - \alpha^{-1}$. Moreover, application of Corollary 2 with the same function $\rho(x) = 1/x$ yields exactly the same result, though, this need not be the case in general.

Remark 5: Example 2 uses Theorem 1 for analysis of System (8), which was in turn obtained based on specific assumptions on demand prediction by the ISO (in the exanté pricing case), or price prediction by the consumers (in the ex-post case). As a result, the finding that $\beta < 2 - \alpha^{-1}$ is necessary for stability is valid only under these assumptions. A natural question arises here regarding the effects of more sophisticated prediction strategies, e.g., time-series analysis, on system stability. Not surprisingly, time-series analysis of demand or price does have a stabilizing effect. However, our simulations show that the ratio

$$\theta = \frac{\alpha(\beta-1)}{1-\alpha}$$

is an indicator for hardness of stabilization. For very large values of θ, the system could not be stabilized using simple time-series prediction of price and/or demand.

B. Periodic Demand with an Elastic Component

The model that we have used so far in this paper assumes that the entire demand makes adjustments in response to price signals, and that the response is completely characterized by the value function of the consumer. In this section, we examine a more generic model in which, the demand is comprised of inelastic and elastic components. We assume that the inelastic component is a periodic function of time, and is insensitive to price variations. As before, the elastic component is characterized by a concave value function. More specifically, we have

$$\begin{aligned} d_k &= (1-\mu)p_k + \mu\dot{v}^{-1}(\lambda_k) \tag{22} \\ \lambda_{k+1} &= \dot{c}(d_k), \end{aligned}$$

where $p : \overline{\mathbb{Z}}_+ \to [\omega, \infty)$ is a periodic function representing the natural fluctuations of demand, $\omega \geq 0$ is the minimum demand, and $\mu \in [0, 1]$ is a parameter. An interpretation of (22) is as follows. The periodic function p represents the total population's demand in the absence of dynamic pricing and $\dot{v}^{-1}(\cdot)$ represents the demand when the entire population is responsive. The parameter μ in this case represents the percentage of population that has subscribed to real-time pricing and $(1-\mu)p_k + \mu\dot{v}^{-1}(\lambda_k)$ is the entire demand at time k.

Definition 2: Given a periodic function $p : \overline{\mathbb{Z}}_+ \to [\omega, \infty)$ satisfying $p_{k+T} = p_k$, $\forall k$, a periodic orbit of (22) is a function $\bar{\lambda} : \overline{\mathbb{Z}}_+ \to \mathbb{R}_+$, satisfying

$$\dot{c}^{-1}\left(\bar{\lambda}_{k+1}\right) = (1-\mu)\, p_k + \mu \dot{v}^{-1}\left(\bar{\lambda}_k\right), \qquad \forall k \in \overline{\mathbb{Z}}_+$$
$$\bar{\lambda}_k = \bar{\lambda}_{k+T}, \qquad \forall k \in \overline{\mathbb{Z}}_+.$$

Theorem 2: Consider system (22), and suppose that the function $p : \overline{\mathbb{Z}}_+ \to [\omega, \infty)$ satisfies $p_{k+T} = p_k$ for some $T \in \overline{\mathbb{Z}}_+$. Let $\gamma = (1-\mu)\, \omega \geq 0$. If there exists $\theta \in (-1, 1)$, and a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\mu \rho\left(\dot{c}\left(x\right)\right) \ddot{c}\left(x\right) \leq \theta \rho\left(\dot{v}\left(\frac{x-\gamma}{\mu}\right)\right) \ddot{v}\left(\frac{x-\gamma}{\mu}\right), \ \forall x \geq \gamma \tag{23}$$

or

$$\mu \left|\rho\left(\mu\eta\left(\lambda\right)+\gamma\right) \dot{\eta}\left(\lambda\right)\right| \leq \theta \rho\left(\sigma\left(\lambda\right)\right) \dot{\sigma}\left(\lambda\right), \ \forall \lambda > 0 \tag{24}$$

where, $\dot{\eta}$ and $\dot{\sigma}$ are defined as in (18), then (22) has a periodic orbit $\bar{\lambda}$ with period T. Furthermore, all solutions converge to the periodic orbit in the sense that

$$\lim_{k \to \infty} \left|\lambda_k - \bar{\lambda}_k\right| = 0$$

for all functions λ satisfying (22).

Proof: Omitted for brevity. ∎

The implication of Theorem 2 is that participation of a small portion of the population in real-time pricing will not have a severe destabilizing effect on the system as satisfying either (23) or (24) is typically easier for smaller μ and larger ω. System stability concerns should arise when a large portion of the population is exposed to real-time pricing.

Example 3: Consider (8) with $c\left(x\right) = x^\beta$, $\beta > 1$, and $v\left(x\right) = \log\left(x\right)$. Let $\rho\left(x\right) = 1/x$. Then

$$\left|\frac{\rho\left(\dot{c}\left(x\right)\right) \ddot{c}\left(x\right)}{\rho\left(\dot{v}\left(x\right)\right) \ddot{v}\left(x\right)}\right| = \beta - 1.$$

Hence, (8) is stable for all $\beta < 2$. It can be verified that $\beta < 2$ is also necessary for stability, and (8) diverges for all $\beta < 2$. Now, consider (22), with $\mu \leq 0.5$, $\omega = 2$. and apply criteria (23) with $\rho\left(x\right) = 1/x^2$. Then

$$\left|\frac{\mu\rho\left(\dot{c}\left(x\right)\right) \ddot{c}\left(x\right)}{\rho\left(\dot{v}\left(\frac{x-\gamma}{\mu}\right)\right) \ddot{v}\left(\frac{x-\gamma}{\mu}\right)}\right| = \frac{\mu\left(\beta-1\right) x^{-\beta}}{\beta}$$
$$\leq \frac{\mu\left(\beta-1\right)\gamma^{-\beta}}{\beta}$$
$$< \frac{\beta-1}{\beta} < 1$$

Therefore, for all cost functions $c\left(x\right) = x^\beta$, $\beta > 1$, all solutions of (22) converge to a periodic orbit when $\mu \leq 0.5$, $\omega \geq 2$.

C. Pricing for Stabilization and Loss of Efficiency

In this section we examine a pricing mechanism in which, the retail price is a static function of the wholesale price. Recent results on stabilization of electricity markets via dynamic pricing functions can be found in [8]. The results of are applicable to equilibrium analysis of dynamic pricing mechanisms as well. If the retail market prices are allowed to be different than the wholesale market prices then achieving stability is not difficult. For instance, a constant retail market price is always stabilizing. We are interested in examining the effects of this type of pricing on the efficiency of the system. Suppose that the system has reached an equilibrium state with $\bar{\lambda}^r$ and $\bar{\lambda}^w$ as the retail and wholesale market prices respectively. Then:

$$\mathcal{S} = v\left(x\right) - s\left(x\right)$$
$$= v\left(\dot{v}^{-1}\left(\bar{\lambda}^r\right)\right) - c\left(\dot{c}^{-1}\left(\bar{\lambda}^w\right)\right)$$

where $\mathcal{S}$ is the aggregate surplus. Let us denote by S_ϕ the surplus function corresponding to the case where $\lambda_k^r = \phi\left(\lambda_k^w\right)$ for some function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$. We present the following Theorem.

Theorem 3: Suppose that at any given time k, the wholesale price λ_k^w, and the consumer price λ_k^r satisfy $\lambda_k^w = \phi\left(\lambda_k^r\right)$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a $C^1(0, \infty)$ function. Then the wholesale market price dynamics is give by

$$\lambda_{k+1}^w = \dot{c}\left(\dot{v}^{-1}\left(\phi\left(\lambda_k^w\right)\right)\right) \tag{25}$$

and converges to an equilibrium price $\bar{\lambda}^w$ satisfying $\phi\left(\bar{\lambda}^w\right) = \bar{\lambda}^r$ provided that there exists a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying:

$$\left|\rho\left(\eta\left(\phi\right)\right) \dot{\eta}\left(\phi\right) \dot{\phi}\right| \leq \theta \rho\left(\sigma\right) \dot{\sigma}$$

where $\sigma = \dot{c}^{-1}$, and $\eta = \dot{v}^{-1}$. Furthermore, if for functions ϕ_1 and ϕ_2, either

$$0 < -\bar{\lambda}_1^w + \phi_1\left(\bar{\lambda}_1^w\right) < -\bar{\lambda}_2^w + \phi_2\left(\bar{\lambda}_2^w\right)$$

or

$$0 > -\bar{\lambda}_1^w + \phi_1\left(\bar{\lambda}_1^w\right) > -\bar{\lambda}_2^w + \phi_2\left(\bar{\lambda}_2^w\right)$$

Then

$$S_{\phi_2} < S_{\phi_1}$$

Proof: The first statement is a corollary of Theorem 1. We present a proof for the second statement. Let $x_{\bar{\lambda}^w}$ denote the equilibrated supply and demand. Then:

$$x_{\bar{\lambda}^w} = \dot{c}^{-1}\left(\bar{\lambda}^w\right) = \dot{v}^{-1}\left(\phi\left(\bar{\lambda}^w\right)\right)$$

$$S\left(\bar{\lambda}^w\right) = v\left(x_{\bar{\lambda}^w}\right) - c\left(x_{\bar{\lambda}^w}\right) = v\left(\dot{c}^{-1}\left(\bar{\lambda}^w\right)\right) - c\left(\dot{c}^{-1}\left(\bar{\lambda}^w\right)\right)$$

$$\frac{dS\left(\bar{\lambda}^w\right)}{d\bar{\lambda}^w} = \dot{v}\left(\dot{c}^{-1}\left(\bar{\lambda}^w\right)\right) \dot{\sigma}\left(\bar{\lambda}^w\right) - \bar{\lambda}^w \dot{\sigma}\left(\bar{\lambda}^w\right)$$
$$= \left(\phi\left(\bar{\lambda}^w\right) - \bar{\lambda}^w\right) \dot{\sigma}\left(\bar{\lambda}^w\right)$$

Since by assumption $c\left(\cdot\right)$ is convex, $\dot{c}^{-1}\left(\cdot\right)$ is increasing and $\dot{\sigma}\left(\bar{\lambda}^w\right) > 0$. Therefore, $dS\left(\bar{\lambda}^w\right)/d\bar{\lambda}^w$ is zero only when $\bar{\lambda}^w = \phi\left(\bar{\lambda}^w\right)$, which immediately implies that there is a loss of efficiency when the wholesale price and the consumer
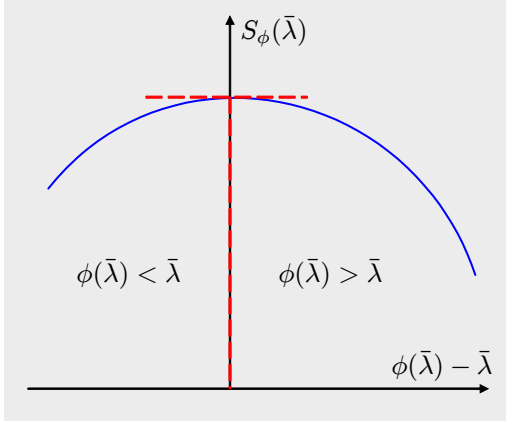
Fig. 3. The aggregate surplus as a function of the difference between the wholesale and the retail price.

price at the equilibrium are not identical. Furthermore,

$$
\begin{aligned}
\frac{d\left(S\left(\bar{\lambda}^w\right)\right)}{d\left(\phi\left(\bar{\lambda}^w\right)-\bar{\lambda}^w\right)} &= \frac{d\left(S\left(\bar{\lambda}^w\right)\right)/d\bar{\lambda}^w}{d\left(\bar{\lambda}^r-\bar{\lambda}^w\right)/d\bar{\lambda}^w} \\
&= \frac{\left(\phi\left(\bar{\lambda}^w\right)-\bar{\lambda}^w\right)\dot{\sigma}\left(\bar{\lambda}^w\right)}{d\bar{\lambda}^r/d\bar{\lambda}^w - 1} \\
&= \frac{\left(\phi\left(\bar{\lambda}^w\right)-\bar{\lambda}^w\right)\dot{\sigma}\left(\bar{\lambda}^w\right)}{\ddot{v}\left(\sigma\left(\bar{\lambda}^w\right)\right)\dot{\sigma}\left(\bar{\lambda}^w\right)-1}
\end{aligned}
$$

where the last equality follows from $\bar{\lambda}^r = \dot{v}\left(\dot{c}^{-1}\left(\bar{\lambda}^w\right)\right)$ and taking the derivative. The above derivation shows that $S\left(\bar{\lambda}^w\right)$ is in increasing function of $\phi\left(\bar{\lambda}^w\right)-\bar{\lambda}^w$ as long as $\phi\left(\bar{\lambda}^w\right)-\bar{\lambda}^w \leq 0$ (since $\ddot{v}\left(\sigma\left(\bar{\lambda}^w\right)\right)\dot{\sigma}\left(\bar{\lambda}^w\right)-1 < 0$ and $\dot{\sigma}\left(\bar{\lambda}^w\right) > 0$), and a decreasing function of $\phi\left(\bar{\lambda}^w\right)-\bar{\lambda}^w$ as long as $\phi\left(\bar{\lambda}^w\right)-\bar{\lambda}^w \geq 0$. ∎

The above Theorem indicates that when the consumer price is a (non-identity) function of the wholesale market price there is generally a loss of efficiency, and furthermore, the greater the discrepancy between the consumer price and the wholesale price, the greater the efficiency loss. Since the system is at the optimum if and only if $\phi\left(\bar{\lambda}^w\right) = \bar{\lambda}^w$, any function ϕ that results in an equilibrium with this property necessarily satisfies:

$$
\bar{\lambda}^w = \dot{c}\left(\dot{v}^{-1}\left(\phi\left(\bar{\lambda}^w\right)\right)\right) = \dot{c}\left(\dot{v}^{-1}\left(\bar{\lambda}^w\right)\right)
$$

Hence, any such $\bar{\lambda}^w$ should necessarily be the equilibrium of the original system under direct pricing.

Remark 6: System (25) could as well represent the dynamics of an ex-post priced market with $\phi\left(\cdot\right)$ representing the consumer price prediction function: $\hat{\lambda}_k = \phi\left(\lambda_k\right)$. Even when the price prediction function has memory, the results of Theorem 3 can be applied for equilibrium analysis. A larger discrepancy between the wholesale and retail prices, indicates a more inefficient equilibrium, which could arise from consumer's specific price prediction strategy in an ex-post priced market. A similar analogy can be made for ISO's demand prediction function in an exanté priced market.

IV. Conclusions and Future Work

We investigated the effects of real-time pricing on the stability and efficiency of electricity markets and showed that exposing the consumers to the real-time wholesale market prices could create an unstable closed loop feedback system. In practice, this instability could manifest itself as extreme price volatility. We presented several criteria characterizing convergence of the prices based on the relation between the cost functions of the producers and the value functions of the consumers. The criteria were extended to the case where the demand is combination of an inelastic periodic function and an elastic component. We established that existence of a positive inelastic component in demand has a strong stabilizing effect. Our results are consistent with the intuition that system instability concerns should be greater when larger portions of the population participate in real-time pricing. It was further shown that when the consumer prices are a static function of the wholesale market prices, there can be a loss of efficiency. The larger the discrepancy between the wholesale market price and the retail price, the farther is the system from an optimal equilibrium. Although this result was obtained for a static pricing function, it is applicable to equilibrium analysis of other pricing mechanisms. A discrepancy between wholesale and retail equilibrium prices indicates inefficiency, regardless of how the equilibrium was reached. Analysis of the case of time-varying or stochastically fluctuating cost functions is an important direction for future research.

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