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Trajectory tracking control design for autonomous helicopters using a backstepping algorithm

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Abstract

In this paper we present a tracking controller for a class of underactuated mechanical systems, based on a backstepping procedure. This class includes an approximation of small helicopter dynamics. The need to avoid artificial singularities due to the attitude representation is the main driver behind the control design presented in this paper: to achieve this goal, we will operate directly in the configuration manifold of the vehicle. The control design provides asymptotic tracking for an approximate model of small helicopters, and bounded tracking when more complete models are considered. Simulation examples, including both point stabilization and aggressive maneuver tracking, are presented and discussed.

1 Introduction

In the recent past, the design and implementation of control algorithms for autonomous helicopters has been the object of a relevant body of research, due to an identified need for maneuverable autonomous aerial vehicles, for both military and civil applications. While slower and less fuel-efficient than airplanes, helicopters are capable of vertical take off and landing, hover, and in general are more maneuverable in tight spaces than airplanes. As a consequence, helicopters are one of the best platforms for operations in urban or otherwise cluttered environments. However, in many respects the dynamics of a helicopter are more complicated than the dynamics of a fixed wing aircraft: a helicopter is inherently unstable at hover, and the flight characteristics change dramatically over the entire flight envelope.

In recent papers, feedback linearization techniques have been applied to helicopter models. The main difficulty in the application of such techniques is the fact that, for any meaningful selection of outputs, the helicopter dynamics are non-minimum phase, and hence are not directly input-output linearizable. However, it is possible to find good approximations to the helicopter dynamics such that the approximate system is input-output linearizable, and bounded tracking can be achieved [1, 2] The feedback linearization approach suffers from the fact that, since the attitude is parameterized using Euler angles, singularities arise when performing some maneuvers, such as loops, barrel rolls, split-s's etc. A possible solution to the singularity problem is represented by chart switching when approaching a singularity. However, this can be cumbersome in implementation, and can lead to excessively high gains in the proximity of singularities.

On the other hand, the singularities arising in these model are artifacts due to the choice of the attitude parameterization (Euler angles), and do not reflect any intrinsic characteristic of the helicopter dynamics. The need to avoid artificial singularities due to the attitude representation is the main driver behind the control design presented in this paper: to achieve this goal, we will operate directly in the configuration manifold of the helicopter. The "tracking on manifolds" problem was solved in [3] for fully actuated mechanical systems: in this paper we present an extension, for achieving asymptotic (locally exponential) tracking of trajectories for a particular class of underactuated mechanical systems. An approximate model of helicopter dynamics can be shown to be in this class: the approximation that will be used in the paper is the same one that leads to feedback linearizability, or differential flatness of the model. However, the method presented here can deal without any modification with more accurate models, including for example simple aerodynamic forces.

The control design will be based on a non-trivial extension of backstepping ideas [4] to dynamic systems on manifolds. In its basic form the backstepping procedure is carried out on a chain of integrators (integrator backstepping); in our case the backstepping procedure is implemented on a dynamic system evolving on the group of rotations in the three-dimensional space $SO(3)$. A backstepping approach for control of underactuated, non-minimum phase nonlinear systems was used in [5] for control of surface vessels: our problem is more difficult since we need to control the rigid body motion in the three-dimensional space, as opposed to the plane. At the time of writing this paper, the authors also became aware of the work in [6], where a backstepping procedure was implemented to design a controller for a helicopter close to hover. However, the approach in the above paper is still partially based on

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a coordinate representation, and Euler angles are used in the expression of the control law. As a consequence, geometric singularities are not eliminated, and the system is not able to track trajectories in which the helicopter "turns upside down"[6]. In addition to providing a more rigorous approach to the "backstepping on manifolds" design procedure, our formulation avoids the introduction of artificial singularities, and results in a controller that is capable of tracking any feasible trajectory (within the limitations of the model).

The paper is structured as follows: first we describe the class of nonlinear systems that we are considering; next we will explain the controller design. The controller design can be split into three parts: control of the translational dynamics, control of the attitude, and the backstepping procedure to construct a stabilizing (or tracking) controller for the overall system. The controller design will be carried out on an approximate model, as a consequence we will have to assess the impact of the neglected dynamics: this will be done in the following section. Finally we present and discuss some simulation examples.

2 Helicopter dynamics

The helicopter model that we will use here is based on the model presented in [2]. A very similar model has been widely used in the nonlinear control literature, in the three degrees of freedom case, as a VTOL aircraft model [1, 7]. More details on helicopter dynamics can be found in [8, 9]. The dynamics of small model helicopter can be adequately described by the rigid body equations [10]. The configuration of the vehicle will be described by an element g of the Special Euclidean group in the three-dimensional space, usually denoted by $SE(3)$. Using homogeneous coordinates, a matrix representation of $g \in SE(3)$ is the following: $g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$ where $R \in SO(3)$ is a rotation matrix and $p \in \mathbb{R}^3$ is a translation vector. The kinematics of the rigid body are determined by $\dot{g} = g\hat{\xi}$ where $\hat{\xi}$, denoted as *twist*, is an element of the Lie algebra $\mathfrak{se}(3)$ associated with $SE(3)$. A matrix representation of an element $\hat{\xi} \in \mathfrak{se}(3)$ is $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$ where ω and v are respectively the angular and translational velocities in body axes, and the skew matrix $\hat{\omega}$ is the unique matrix such that $\hat{\omega}u = \omega \times u$, for all $u \in \mathbb{R}^3$ The full state of the vehicle as a rigid body will then be represented by $x = (g, \hat{\xi})$, with $X = SE(3) \times \mathfrak{se}(3)$. The dynamics equations will be given by:

$$J_b\dot{\omega} = -\omega \times J_b\omega + M_b(g, \hat{\xi}, u, w) \qquad (1)$$

$$m\dot{v} = -\omega \times mv + F_b(g, \hat{\xi}, u, w) \qquad (2)$$

where J_b and m are the vehicle's inertia tensor and mass, and M_b and F_b represent the torques and forces in body axes, which are in general a function of the vehicle state $x = (g, \hat{\xi})$, of the control inputs u, and of the disturbances w. We can use the following form for the body force and moment in eq. (1):

$$\left\{ \begin{array}{lcl} F_b & = & mR^{-1}\tilde{g} + u_4\,[0, 0, -1]^T + \mathcal{E}(u_1, u_2, u_3) \\ M_b & = & = [u_1, u_2, u_3]^T \end{array} \right. \qquad (3)$$

In the above, $\tilde{g}$ is the gravity acceleration, and we have defined:

$$\mathcal{E}(u_1, u_2, u_3) = \begin{bmatrix} 0 & -\epsilon_2 & 0 \\ \epsilon_1 & 0 & -\epsilon_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 - \epsilon_4(u_4) \end{bmatrix}$$

where:

$$\begin{array}{lcl@{\qquad}rcl} u_1 & := & -z_{mr}Tb_1 & \epsilon_1 & := & -1/z_{mr} \\ u_2 & := & -z_{mr}Ta_1 & \epsilon_2 & := & -1/z_{mr} \\ u_3 & := & k_0 + k_T T^{1.5} + x_{tr}F_t & \epsilon_3 & := & -1/x_{tr} \\ u_4 & := & T & \epsilon_4(T) & := & k_0 + k_T T^{1.5} \end{array}$$

where T is the main rotor thrust, a_1, b_1 are respectively the pitch and roll rotor flapping angles, F_t is the tail rotor thrust, $-z_{mr}$ and $-x_{tr}$ are the moment arms of the main and tail rotor with respect to the helicopter center of mass, and k_0, k_T are the coefficient in the approximate expression of the main rotor reaction torque. As suggested and motivated in [2], we will use a dynamic extension procedure to ensure that the augmented system has constant relative degree. This is done by appending two integrators to the thrust control input u_4. We will thus consider as the control inputs the vector $\mathbf{u} := [u_1, u_2, u_3]^T$ and the scalar $\ddot{u}_4$.

3 Control Design

The objective of the feedback control law will be to track a smooth, feasible reference trajectory $(g_{ref}(t), \hat{\xi}_{ref}(t))$. We will start by designing a controller for the approximate system obtained by setting $\epsilon_i = 0, i = 1 \dots 4$. This approximation simplifies the control design considerably: it can be shown that the resulting approximate system is differentially flat, and hence feedback linearizable [2]. Even though we will not use a feedback linearization technique, the absence of unstable zero dynamics gives an insight on the nature and the advantages of the approximate system.

3.1 Translational dynamics

If we consider as translational coordinates the position and velocity in the inertial frame $(p, \dot{p}) = (p, Rv)$, the translational dynamics is composed of three double integrators in parallel, driven by the force input $\alpha(R, u) := RF_b(u)$. Since the translational dynamics block is essentially linear, it is easy to design a Lyapunov function V_p and a control policy $\mathcal{K}_p$ such that if

$$\alpha(R, u_4) = \bar{\alpha}(p, \dot{p}) := \mathcal{K}_p(p, \dot{p}) - m\tilde{g} \qquad (4)$$

then the translational dynamics is stable, that is $\dot{V}_p(p,\dot{p}) \leq -W(p,\dot{p})$, where W is a positive definite function. The above can be easily extended for tracking of a reference trajectory $p_{ref}(t)$, by adding the appropriate feed-forward terms. For simplicity, we can use a quadratic Lyapunov function and a proportional-derivative (PD) form for the translational dynamics control law.

If the function $\alpha(R, u_4)$ were invertible, then we would be able to use the attitude R as a control input to the translational block. This is not the case, however, we can select the desired attitude R_d as the "closest" (in the sense explained below) element to reference attitude R_{ref} for which we can find a u_4 such that $\alpha(R_d, u_4) = \bar{\alpha}(p, \dot{p})$. A measure of the distance between two elements R_1, R_2 of $SO(3)$ can be derived from the relative rotation $\delta R := R_1 R_2^{-1}$ (group error), that is still an element of $SO(3)$ [3]. All the elements of $SO(3)$ can be described by a fixed axis $\tilde{r}$, corresponding to the single real eigenvector, and an angle of rotation $\tilde{\theta}$, which can be derived from the complex conjugate eigenvalues. As a measure of the magnitude of the group error δR, that is the distance between the rotations R_1 and R_2, we can consider the following function:

$$\Theta(\delta R) := 1 - \cos(\tilde{\theta}) = 2 \sin^2 \frac{\tilde{\theta}}{2} = \frac{1}{2} \mathrm{Tr}(I - \delta R) \quad (5)$$

At this point, we can define the desired attitude and thrust as the solution of the following optimization problem:

$$\begin{aligned} (R_d, u_{4d}) \;=\; & \underset{(R,u) \in SO(3) \times \mathbb{R}}{\arg\min} \quad \Theta(R R_{ref}^{-1}) \\ & \qquad s.t. \quad \alpha(R, u) = \bar{\alpha}(p, \dot{p}) \end{aligned} \quad (6)$$

It can be verified (see the following) that a unique solution exists, and that the dependence of (R_d, u_{4d}) on $(p, \dot{p})$ is smooth, excluding the sets over which $R_{ref} e_3 \cdot \bar{\alpha}(p, \dot{p}) = 0$. This includes the case in which the commanded acceleration of the helicopter is equal to the gravity acceleration. This singularity is inherent to the physics of the problem, and as such cannot be avoided: it corresponds to the fact that if $u_4 = 0$ the helicopters enters a free fall, regardless of the attitude. Moreover, in the case in which the commanded acceleration requires a rotation of $\pi/2$ radians of amplitude, there are two equivalent solutions to the problem (6), corresponding to $u_{4d} = \pm \bar{u}$. Having defined the vector $r := R_{ref} e_3 \times \bar{\alpha}(p, \dot{p}) / ||\bar{\alpha}(p, \dot{p})||_2$, simple geometric reasoning provides the following solution to the above minimization problem:

$$(R_d, u_{4d}) = \begin{cases} \left(\mathrm{Rot}(-r, \sin^{-1} ||r||_2) R_{ref},\; ||\bar{\alpha}(p, \dot{p})||_2\right), \\ \qquad\qquad \text{if } R_{ref} e_3 \cdot \bar{\alpha}(p, \dot{p}) \leq 0 \\ \left(\mathrm{Rot}(r, \sin^{-1} ||r||_2) R_{ref},\; -||\bar{\alpha}(p, \dot{p})||_2\right), \\ \qquad\qquad \text{if } R_{ref} e_3 \cdot \bar{\alpha}(p, \dot{p}) > 0 \end{cases} \quad (7)$$

where $e_3 := [0, 0, 1]^T$, $\mathrm{Rot}(r, \theta)$ is the rotation about the fixed axis r through an angle θ, and $\cdot$ and $\times$ represent the scalar and cross products of vectors in $\mathbb{R}^3$. The rotation $\mathrm{Rot}(r, \theta)$ is given by Rodrigues' formula:

$$\mathrm{Rot}(r, \theta) = I_3 + \sin\theta \frac{\hat{r}}{||r||_2} + (1 - \cos\theta) \frac{\hat{r}^2}{||r||_2^2} \quad (8)$$

3.2 Attitude dynamics and backstepping control design

Once we have the desired attitude, using ideas derived from the backstepping concept [4], we want to track R_d in such a way to stabilize the overall system. However, before we can go on with the control design, we have to take a look at the rate of change of the objects introduced in the previous section (see also the derivations in [3]). First of all, we have that $\dot{R}_{ref} = R_{ref} \hat{\omega}_{ref}$; accordingly, we can define $\hat{\omega}_d := R_d^T \dot{R}_d$. Furthermore, the following equalities hold:

$$\begin{aligned} \tfrac{d}{dt} \Theta(R R_d^T) &= \sin\tilde{\theta} \tfrac{d\tilde{\theta}}{dt} = \mathrm{Skew}(R_d^T R)^\vee \cdot (\omega - \omega_d) = \\ &= \mathrm{Skew}(R R_d^T)^\vee \cdot R_d (\omega - \omega_d) = \\ &= \sin\tilde{\theta}\, \tilde{r} \cdot R_d (\omega - \omega_d) = \sin\tilde{\theta}\, \tilde{r}_d \cdot (\omega - \omega_d) \end{aligned} \quad (9)$$

where $\mathrm{Skew}(M) = \frac{1}{2}(M - M^T)$, the operator $(\cdot)^\vee$ is the inverse of the "hat" operator (i.e. $S^\vee \times u = S u$, $\forall u \in \mathbb{R}^3$, and S is a skew 3×3 matrix), and $\tilde{r}, \tilde{\theta}$ are respectively the fixed axis and rotation angle of the attitude error $R R_d^T$, obtained by:

$$\cos\tilde{\theta} = \frac{\mathrm{Tr}(R R_d^T) - 1}{2}; \quad \sin\tilde{\theta} \tilde{r} = \mathrm{Skew}(R R_d^T)^\vee$$

We can rewrite eq.(9) as: $\frac{d}{dt} \Theta(R R_d^T) = \nabla\Theta \cdot (\omega - \omega_d)$ having defined: $\nabla\Theta := \sin(\tilde{\theta}) \tilde{r}_d = \mathrm{Skew}(R_d^T R)^\vee$. Now we are ready to state the following result (for the definition of asymptotic tracking on manifolds, see [3]):

Theorem 1 (Tracking for the approx. system) *Given a smooth, feasible state trajectory $x_{ref}(t) = (g_{ref}(t), \xi_{ref}(t))$ for a rigid body under the action of the forces in eq. (3), with $\epsilon_i = 0, i = 1 \ldots 4$, there exists an (almost everywhere) smooth control law under which the system state $x(t)$ globally asymptotically, and locally exponentially tracks the reference trajectory $x_{ref}(t)$.*

Proof: We will prove the above statement by actually building a tracking control law. Define a candidate Lyapunov function by adding to V_p terms that opportunely penalize the attitude configuration and velocity errors. Such a candidate Lyapunov function is the following:

$$\begin{aligned} V = & V_p(p, \dot{p}) + k_\theta \Theta(R R_d^T) + \\ & + \frac{1}{2} \left(||\eta||_2^2 + k_u ||u_4 - u_{4d}||_2^2 + ||\zeta||_2^2 \right) \end{aligned} \quad (10)$$

where:

$$\eta \;:=\; \omega - \omega_d - \frac{u_{4d}}{k_\theta \cos\frac{\tilde{\theta}}{2}} R_d^T \left(\tilde{t} \times \nabla_{\dot{p}} V_p\right) \qquad (11)$$

$$\zeta \;:=\; \dot{u}_4 - \dot{u}_{4d} - \frac{1}{k_u} \nabla_{\dot{p}} V_p \cdot R e_3 \qquad (12)$$

and:

$$\tilde{t} := \frac{(R + R_d) e_3}{||(R + R_d) e_3||_2} \qquad (13)$$

Computing the time derivative of V, with the definition of R_d and u_4 given in the previous section, we get:

$$\frac{dV}{dt} \leq -W_p(p, \dot{p}) + \nabla_{\dot{p}} V_p(p, \dot{p})(u_{4d} R_d e_3 - u_4 R e_3) + \\ + k_\theta \nabla\Theta \cdot (\omega - \omega_d) + \eta\dot{\eta} + k_u (u_4 - u_{4d})(\dot{u}_4 - \dot{u}_{4d}) + \zeta\dot{\zeta} \qquad (14)$$

We can make the above negative semidefinite by imposing:

$$\dot{\eta} \;=\; -k_\eta \eta - k_\theta \nabla\Theta \qquad (15)$$

$$\dot{\zeta} \;=\; -k_\zeta \zeta - k_u (u_4 - u_{4d}) \qquad (16)$$

where k_η, k_θ, k_ζ, k_u, are all positive constants. Noting that:

$$\nabla_{\dot{p}} V_p \cdot (u_{4d} R_d e_3 - u_4 R e_3) = \\ = -\nabla_{\dot{p}} V_p \cdot \left[2 u_{4d} \sin\frac{\tilde{\theta}}{2} (\tilde{r} \times \tilde{t}) + (u_4 - u_{4d}) R e_3 \right] = \\ = k_\theta \nabla\Theta \cdot (\eta - \omega + \omega_d) + k_u (u_4 - u_{4d})(\zeta - \dot{u}_4 + \dot{u}_{4d}) \qquad (17)$$

we get:

$$\frac{dV}{dt} \leq -W_p(p, \dot{p}) - k_\eta ||\eta||_2^2 - k_\zeta ||\zeta||_2^2 \leq 0 \qquad (18)$$

The time derivative along system trajectory of the Lyapunov function V is hence negative-semidefinite: asymptotic stability can be inferred from LaSalle's principle. To prove local exponential stability, augment the Lyapunov function (10) with a cross term:

$$V_{cross} = \chi \left[\nabla\Theta \cdot \eta + (u_4 - u_{4d}) \zeta\right] \qquad (19)$$

where χ is a small positive constant. The time derivative of the cross term, under the control law (15) can be computed as:

$$\frac{d}{dt} V_{cross} = \chi \left[-k_\theta ||\nabla\Theta||_2^2 - k_\eta \nabla\Theta \eta + \frac{d\nabla\Theta}{dt} \cdot \eta + \right. \\ \left. -k_u (u_4 - u_{4d})^2 - k_\zeta (u_4 - u_{4d}) \zeta + (\dot{u}_4 - \dot{u}_{4d}) \zeta \right]$$

Moreover, it is possible to find positive constants c_1, c_2 so that we have the following bounds:

$$\frac{d\nabla\Theta}{dt} \cdot \eta \leq (1 + ||\nabla\Theta||_2) ||\eta||_2^2 + c_1 ||\nabla\Theta||_2^2 \, ||\eta||_2 \, ||P||_2^2 \qquad (20)$$

and:

$$(\dot{u}_4 - \dot{u}_{4d}) \zeta \leq \zeta^2 + c_2 ||P||_2 \zeta \qquad (21)$$

where $P := [p - p_{ref}, \dot{p} - \dot{p}_{ref}]^T$. For sufficiently small χ, and error vector $\delta := [P, \nabla\Theta, \eta, u_4 - u_{4d}, \zeta]^T$, the derivative of the augmented Lyapunov function $V_{total} := V + V_{cross}$ is negative definite, and it is be possible to find a $\lambda > 0$ such that $\dot{V}_{total} < -\lambda V_{total}$, which proves local exponential stability. The control law is smooth almost everywhere, that is for all conditions for which $\tilde{\theta} \neq \pi/2$. The explicit expression for the control torques $\mathbf{u} = [u_1, u_2, u_3]^T$, and the control force second derivative $\ddot{u}_4$ are given by:

$$\ddot{u}_4 \;=\; \ddot{u}_{4d} - k_\zeta \zeta - k_u (u_4 - u_{4d}) + \tfrac{1}{k_u} \tfrac{d}{dt} \left(\nabla_{\dot{p}} V_p \cdot R e_3\right)$$

$$\mathbf{u} \;=\; \omega \times \mathbb{J}\omega + \mathbb{J} \left[-k_\eta \eta - k_\theta \nabla\Theta + \frac{d\omega_d}{dt} + \right. \\ \left. + \tfrac{d}{dt} \left(\tfrac{u_{4d}}{k_\theta \cos\frac{\theta}{2}} R_d^T \left(\tilde{t} \times \nabla_{\dot{p}} V_p\right) \right) \right]$$

$\blacksquare$

To the authors' knowledge, the above control law is a new result, providing asymptotic tracking for a class of underactuated mechanical systems on SE(3). While based on the control design framework presented in [3], the control law we presented provides asymptotic tracking for a broader class of systems. The class of systems for which the control law is applicable comprises vehicles modeled as rigid bodies subject to one force in a body-fixed direction, and three independent torque components. The main advantage of (3.2) is the absence of artificial singularities, deriving from attitude parameterizations, like Euler angles. The elimination of geometric singularities has been accomplished through an over-parameterization of the outputs: we need to fully specify the reference attitude that we want to track. In order to achieve asymptotic tracking, we need the full trajectory to be feasible, and the reference attitude has to satisfy the constraints represented by the system dynamics. On the other hand we can also specify an unfeasible attitude reference trajectory: in that case we will not be able to achieve asymptotic tracking for the whole state. However, we can guarantee that the system trajectory will be such to asymptotically track the position reference, and the attitude will be the *closest* (in the sense of eq. (5)) element in $SO(3)$ to the reference attitude that ensures the feasibility of the trajectory.

3.3 Tracking for the actual model

So far, we have been able to design a controller to achieve asymptotic tracking of a reference trajectory for the approximate system. The terms neglected in the approximation appear as perturbations in the nominal model. A first question that arises is how will the controller designed for the approximate system behave for the actual system, and in the presence of bounded external forces $||F_e||_2 \leq \Delta_e$, like for example those ensuing from uncertainties in the aerodynamics, or from wind gusts. Assume that the reference trajectory $x_{ref}(t) = (g_{ref}(t), \xi_{ref}(t))$ is feasible for a rigid body under the forces in (3). Since the unmodeled forces F_u are a function of the control, given in eq. (3), that is a smooth function of the states and the reference trajectory, we have that, in a compact set $\mathcal{R} := x \in X, V(x, x_{ref}) \leq \bar{V}$, we can characterize the effect of the neglected coupling as: $||F_u||_2 \leq \Delta_u + \epsilon||\delta||_2$ where δ is the state error vector $\delta = [p - p_{ref}, \dot{p} - \dot{p}_{ref}, \nabla\Theta, \eta, u_4 - u_{4d}, \zeta]^T$. We have the following:

Theorem 2 (Bounded tracking for the full model) *Given a reference trajectory* $x_{ref}(t) = (g_{ref}(t)), \xi_{ref}(t))$ *for a rigid body under the action of the forces in eq. (3), there exist sufficiently small* $\epsilon, \Delta = \Delta_u + \Delta_e$ *such the control law defined in section (3.2) is such that for all initial conditions in* $\mathcal{R}$ *the state* $x(t)$ *achieves bounded tracking of the reference trajectory* $x_{ref}(t)$.

Proof: If we compute the time derivative of the Lyapunov function V_{total} under the effect of the disturbance forces, we get that:

$$\begin{array}{rcl} \dot{V}_{total} & \leq & -\lambda V_{total} + \nabla V_{\dot{p}}(F_e + F_u) \leq \\ & \leq & -\lambda V_{total} + \beta||\delta||_2 \, (\Delta + \epsilon||\delta||_2) \leq \\ & \leq & -(\lambda - \beta\epsilon\mu)V_{total} + \beta\Delta\sqrt{\mu V_{total}} \end{array} \tag{22}$$

where μ is such that $||\delta||_2^2 \leq \mu V_{total}$. Define the set:

$$\Omega = \left\{ x \in X \mid V_{total}(x, x_{ref}) < \mu \left(\frac{\beta\Delta}{\lambda - \beta\epsilon\mu} \right)^2 \right\} \tag{23}$$

If $\epsilon < \lambda/(\beta\mu)$, $\dot{V}_{total}$ is negative semi-definite in the set $\mathcal{R} \setminus \Omega$. Note that this, to be of any significance, requires that Δ be small enough that $\Omega \subset \mathcal{R}$, that is $\mu \left(\frac{\beta\Delta}{\lambda - \beta\epsilon\mu} \right)^2 < \bar{V}$. Finally, notice that if $\Delta = 0$, the control law for the approximate system is asymptotically stabilizing for the actual system. ■

It turns out that the values of ϵ for small helicopters are such that the conditions in the above theorem are satisfied (see simulation results in section 4).

4 Simulation examples

In this section we show some simulations obtained for the tracking control law for point stabilization, trim trajectory tracking, and for tracking of "aggressive" maneuvers that cannot be handled in a straightforward manner by coordinate-based controllers because of singularities in the attitude parameterization.

The first example is a stabilization problem. The helicopter starts at hover with non-zero position coordinates, and we want to hover at the origin, heading due North. In Fig. 1,2 we show the response of the approximate system and of the actual system in the two cases of "normal" attitude and inverted flight. Note that in order to obtain discernible plots, we had to multiply the ϵ_i values for a typical model helicopter by a factor of 10. As we can see, the application of the controller designed for the approximate system to the actual system (with exaggerated coupling terms) gives very good results. As expected, the response of the approximate system is identical in the two cases (up to a rotation of 180 degrees). However, we notice the typical "undershoot" of non-minimum phase systems in the response of the actual system in the inverted flight case . The helicopter model used in this paper is known in the literature as being non-minimum phase, where the zero dynamics can be represented as undamped oscillators, of the form $\ddot{\phi} = -k \sin\phi$. We notice the non-minimum phase behavior when the attitude of the helicopter is such that the zero dynamics evolve close to the unstable equilibrium point.

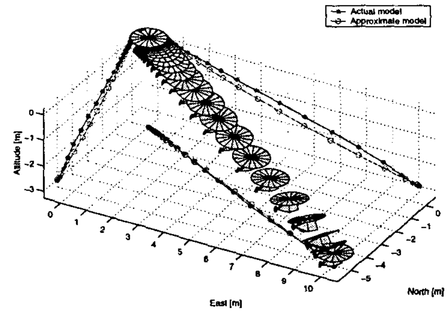

Figure 1: Point stabilization example

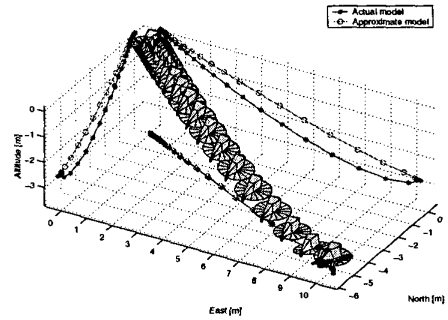

Figure 2: Point stabilization example - inverted flight

A third example, in fig. (3), is about tracking of a trim trajectory, in this case a climbing turn. Again we see satisfactory performance, even though we notice that tracking a time-parameterized trajectory could require excessive control effort and flying aggressiveness. Maneuver tracking techniques [11, 12] could be profitably used in this case, but we will leave this to future work.

As a fourth example, in Fig. (4) we consider tracking a trajectory that performs a transition to inverted flight. This maneuver, in the case in which continuous controls are required, has to go through the singularity at

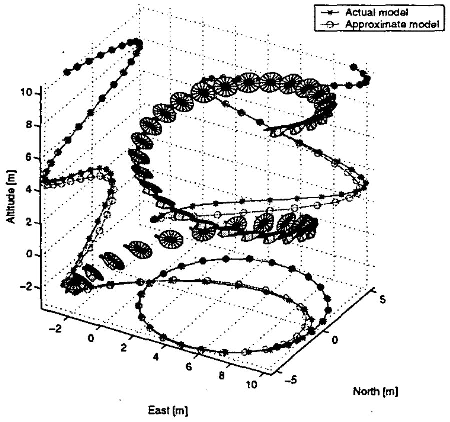

Figure 3: Trim trajectory tracking

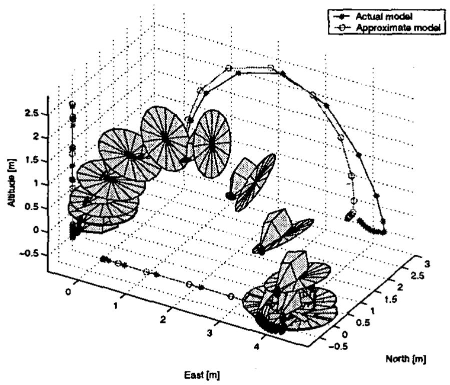

Figure 4: Maneuver tracking example

$T = u_4 = 0$, since the thrust will be positive (upwards in the body frame) in the initial condition, and negative (downwards in the body frame) in the inverted flight condition. In the example, attitude control was shut off when $|u_{4_d}|$ was smaller than a preset value T_{min}. In the transition to inverted flight maneuver that we are considering, we are close to the singular condition only for a short period of time, during which we cannot guarantee that $\dot{V} < 0$. This is the reason for the relatively large deviations from the reference trajectory in the second half of the maneuver (following the singularity). However, the increment ΔV between before and after the singularity can be made arbitrarily small, for example by reducing T_{min}, at the expense of a larger required control authority.

5 Conclusions and future work

In this paper we have presented a control design methodology for a class of underactuated mechanical systems, which includes an approximate model of helicopter dynamics. The controller is free from any artificial singularities that derive from the attitude parameterization choice. This is obtained via a coordinate-independent approach to the control design problem. The only singularity that is retained is due to the physics of the system, and is as such unavoidable if continuity of the control inputs is required. The controller has been implemented on a helicopter simulation, and has given very satisfactory results. Other issues associated with control input saturation, more accurate helicopter dynamics modeling, and the singularity at zero thrust have yet to be explored, and will be the object of further investigation.

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