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A Control Lyapunov Function Approach to Robust Stabilization of Nonlinear Systems[1]

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Abstract

We propose an alternative to gain scheduling for stabilizing a class of nonlinear systems. The computation times required to find stability regions for a given control Lyapunov function vary polynomially with the state dimension for a fixed number of scheduling variables. Control Lyapunov functions to various trim points are used to expand the stability region, and a Lyapunov based synthesis formula yields a control law guaranteeing stability over this region. Robustness to bounded disturbances is easily handled, and the optimal stability margin, defined as a Lyapunov derivative, is recovered asymptotically. We apply the procedure to an example.

1. Introduction

Control of nonlinear systems has been a topic of intense research for some time. Progress on this problem is difficult because of the inherent complexity of general methods which apply to arbitrary systems. An approach which has recently come into favor is the method of *control Lyapunov functions (CLFs)*, which guarantees closed loop stability whenever a CLF can be found. Since no systematic procedure exists for finding CLFs of arbitrary nonlinear systems, research in this direction tends to fall into one of three categories.

- The solution to an optimal control problem based on the solution to a Hamilton-Jacobi equation [1, 5, 13]. It is possible to find approximate solutions to these equations, but this problem is intractable for high order systems.
- The construction of a stabilizing control law based on a *known* CLF [4, 7]. This does not require the intractable computations of the previous category, but it depends on the existence of a CLF, the construction of which is a separate problem.
- The design of a stabilizing control law for a specific class of nonlinear systems [3, 8, 12]. These approaches can be applied successfully in practical situations, but only for very restricted classes of nonlinear systems.

The main contribution of this paper is a nonlinear control procedure with the following properties.

- CLFs are computed systematically.
- The computations are tractable for high order systems.
- Optimal performance (in the sense of a Lyapunov derivative satisfying a desired stability margin) is recovered asymptotically.
- Robustness to bounded disturbances is easily handled.
- The method applies to a useful class of systems.

We seek to stabilize a nonlinear system in the following sense.

Definition 1 Given a system $\dot{x} = f(x, w)$ with $w(t) \in \mathcal{W} \subseteq \mathbf{R}^l$ for all $t \geq 0$, a positively invariant set $\mathcal{X} \subseteq \mathbf{R}^n$, and a compact subset $\Omega \subset \mathcal{X}$, the system is *robustly uniformly asymptotically stable over* $\mathcal{X}$ *with respect to* Ω, or RUAS($\mathcal{X}$,Ω), if it is uniformly asymptotically stable with respect to Ω (see [7]) whenever $x(0) \in \mathcal{X}$ and $w(t) \in \mathcal{W}$ for all $t \geq 0$. We call the set $\mathcal{X}$ a *region of stability (RS)* for the system.

In many applications, the engineer knows that only a few states affect the system dynamics in a nonlinear way. In this paper, we consider the following control synthesis problem, where the system dynamics are assumed to depend nonlinearly on only the first k states.

Problem 1 Consider a system with control $u(t) \in \mathbf{R}^m$ and disturbance $w(t) \in \mathcal{W} \subseteq \mathbf{R}^l$. The state $x(t) \in \mathbf{R}^n$ is partitioned into "nonlinear states" $x_N \in \mathbf{R}^k$ and "linear states"

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$x_L \in \mathbf{R}^{n-k}$. The dynamics are

$$\begin{bmatrix} \dot{x}_N \\ \dot{x}_L \end{bmatrix} = \begin{bmatrix} f_N(x_N) \\ f_L(x_N) \end{bmatrix} + \begin{bmatrix} A_N(x_N) \\ A_L(x_N) \end{bmatrix} x_L + \begin{bmatrix} g_w^N(x_N) \\ g_w^L(x_N) \end{bmatrix} w + \begin{bmatrix} g_u^N(x_N) \\ g_u^L(x_N) \end{bmatrix} u, \tag{1}$$

where all functions of x_N are C^1. Construct sets $\Omega \subset \mathcal{X} \subseteq \mathbf{R}^n$ containing a desired equilibrium point $x_0 \in \mathbf{R}^n$, and a static state feedback control law $\mu : \mathbf{R}^n \to \mathbf{R}^m$, such that the closed loop system with $u = \mu(x)$ is RUAS($\mathcal{X}$,Ω).

We would like $\mathcal{X}$ to be as large as possible and Ω as small as possible. In this paper, we develop a computationally tractable procedure for computing the RS of the system (1), and we use this fact to design a controller to expand the RS.

Gain scheduling is a common control design approach for systems of the form (1). In this method, linear controllers are designed for the linearized system at trim points corresponding to fixed values of the "nonlinear" states. The control gains are then interpolated based on these *scheduling variables*. Unfortunately, the system is not guaranteed to be stable when the scheduling variables are changing. The stability properties of such systems are analyzed in [6, 10], but the design of a gain scheduled controller to guarantee stability remains an open problem. In this paper, we use *robust control Lyapunov functions* for the system linearized about various trim points to guarantee stability over a range of operating conditions.

2. Main Solution Procedure

Before proceeding to develop the main solution procedure, we define some relevant terms pertaining to systems of the general form given below, where all functions of x are C^1.

$$\dot{x} = f(x) + g_w(x)w + g_u(x)u. \tag{2}$$

Definition 2 A function $V(x)$ is a *positive definite function centered at* x_0 if $V(x_0) = 0$ and $V(x) > 0$ for all $x \neq x_0$.

Definition 3 A *level set* of a proper, positive definite function $V(x)$ is defined by a real number $c \geq 0$ (the corresponding *level value*) via $\Omega \doteq V^{-1}[0, c] = \{x \in \mathbf{R}^n \mid V(x) \leq c\}$.

Definition 4 A point $x_0 \in \mathbf{R}^n$ is a *trim point* of the system (2) if there exists a control $u_0 \in \mathbf{R}^m$ such that $f(x_0)+g_u(x_0)u_0 = 0$. A connected set of trim points is a *trim surface*.

Definition 5 ([4, 11]) Consider a subset $\mathcal{W} \subseteq \mathbf{R}^l$, a point $x_0 \in \mathbf{R}^n$, a positive definite function $W(x)$ centered at x_0, and real numbers $c_2 > c_1 \geq 0$. A locally Lipschitz, proper, positive definite function $V(x)$ centered at x_0 is a *robust control Lyapunov function (RCLF) to* x_0 *with stability margin (SM)* $W(x)$ *over* $V^{-1}[c_1, c_2]$ for the system (2) if

$$\sup_{w \in \mathcal{W}} L_f V(x) + L_{g_w} V(x) w + W(x) \leq 0, \tag{3}$$

for all $x \in V^{-1}[c_1, c_2] \cap \ker(L_{g_u} V)$, where $L_h V(x)$ is the Lie derivative of $V(x)$ along $h(x)$.

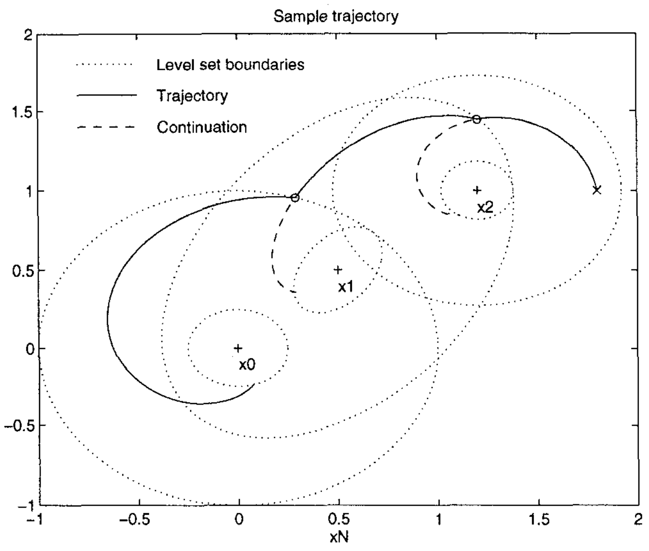

Figure 1: Sample trajectory to illustrate switching between level sets. The solid curve is a trajectory beginning at the point marked by an X; the circles are switching points. The dashed curves are continuations of the trajectory toward intermediate trim points.

A given RCLF decreases along trajectories of the closed loop system under *inverse optimal control* [4] whenever $w(t) \in \mathcal{W}$. Hence, the system is RUAS(Ω_2,Ω_1) for $\Omega_2 \doteq V^{-1}[0, c_2]$ and $\Omega_1 \doteq V^{-1}[0, c_1]$, and Problem 1 reduces to finding an RCLF. First, we design an RCLF to the equilibrium point x_0. We can always do this if the linearized dynamics about x_0 are stabilizable and the disturbances are sufficiently small. Computing an RS based on this RCLF is tractable for the system (1). We design RCLFs to other trim points in the same manner. Trajectories starting in a level set Ω_2^i about x_i converge to a smaller level set $\Omega_1^i \subset \Omega_2^i$. The union of these sets is the new RS, as shown in Figure 1. To ensure convergence to x_0, we introduce a positive definite function U with $U(0) = 0$. For every $i > 0$, we require that $\Omega_1^i \subseteq \Omega_2^j$ and $U(j) < U(i)$ for some $j \neq i$, so that a trajectory starting in Ω_2^i eventually reaches Ω_2^j. We compute the control using the RCLF indexed by

$$i^*(x) \doteq \arg \min_{\{j \mid x \in \Omega_2^j\}} U(j). \tag{4}$$

When a trajectory in a given level set intersects a level set with a lower index, the control law switches to the lower index.

Algorithm 1 We propose the following complete solution procedure for Problem 1.

I. Choose trim points and design RCLFs. On each iteration $i \geq 0$, do the following.

   A. If $i > 0$, find a trim point $x_i \in \operatorname{int}(\Omega_2^j)$ for some $j < i$.

   B. Select a quadratic RCLF $V_i(x)$ and SM $W_i(x)$ based on the linearized dynamics about x_i and a local optimization problem.

C. Find $\Omega_1^i \subset \Omega_2^i$ such that the *nonlinear* system is RUAS(Ω_2^i,Ω_1^i). If $i > 0$, also require $\Omega_1^i \subseteq \Omega_2^j$; else, repeat step I.A with a new trim point.

D. If $i > 0$, set $U(i) = U(j) + 1$; else, set $U(0) = 0$.

E. Repeat until the region $\mathcal{X} \doteq \cup_{j=0}^{i} \Omega_2^j$ covered by level sets is satisfactory.

II. Implement the control law.

A. Select one of the level sets in which the state x lies.

B. Apply inverse optimal control [4] to get $u = \mu(x)$.

The following two results show that Algorithm 1 solves Problem 1 and that we can construct $\mathcal{X}$ to contain any compact trim surface if $\mathcal{W}$ is sufficiently small.

Theorem 1 Suppose there exist trim points $x_0, \ldots, x_N$ of the system (2); functions $V_0(x), \ldots, V_N(x)$, $W_0(x), \ldots, W_N(x)$; constants $c_2^0 > c_1^0 \geq 0, c_2^1 > c_1^1 > 0, \ldots, c_2^N > c_1^N > 0$; and a positive definite function $U : \{0, \ldots, N\} \to \mathbf{R}$; such that:

1. For each i, $V_i(x)$ is an RCLF to x_i with SM $W_i(x)$ over $V_i^{-1}[c_1^i, c_2^i]$.
2. For each $i \neq 0$, there exists $j \neq i$ such that $V_i^{-1}[0, c_1^i] \subseteq V_j^{-1}[0, c_2^j]$ and $U(j) < U(i)$.

Then a control law exists which makes the system RUAS($\mathcal{X}$,Ω) with $\mathcal{X} \doteq \cup_{i=0}^{N} V_i^{-1}[0, c_2^i]$ and $\Omega \doteq V_0^{-1}[0, c_1^0]$.

Theorem 2 If the linearized dynamics about each point in a compact trim surface $\mathcal{M}$ of the system (2) are stabilizable, then there exist $K > 0$; points $x_0, \ldots, x_N \in \mathcal{M}$; symmetric positive definite matrices $P_0, \ldots, P_N$, $Q_0, \ldots, Q_N$; constants $c_2^0 > c_1^0 \geq 0, \ldots, c_2^N > c_1^N \geq 0$; and a positive definite function $U : \{0, \ldots, N\} \to \mathbf{R}$; such that:

1. For each i, $V_i(x) \doteq (x-x_i)^T P_i (x-x_i)$ is an RCLF to x_i with SM $W_i(x) \doteq (x-x_i)^T Q_i (x-x_i)$ over $V_i^{-1}[c_1^i, c_2^i]$ when $\mathcal{W} = \{w \mid \|w\|_2 \leq K\}$.
2. $\mathcal{M} \subseteq \cup_{i=0}^{N} V_i^{-1}[0, c_2^i)$.
3. For each $i \neq 0$, there exists $j \neq i$ such that $V_i^{-1}[0, c_1^i] \subset V_j^{-1}[0, c_2^j)$ and $U(j) < U(i)$.

If $\mathcal{W} = \{0\}$, this statement holds with $c_1^i = 0$ for all i.

2.1. Trim Point Selection

Given a desired scheduling variable value, x_{iN}, it is straightforward to find a corresponding trim point x_i for the system (1) by solving the following for x_{iL} and u_i.

$$\begin{bmatrix} f_N(x_{iN}) \\ f_L(x_{iN}) \end{bmatrix} + \begin{bmatrix} A_N(x_{iN}) \\ A_L(x_{iN}) \end{bmatrix} x_{iL} + \begin{bmatrix} g_u^N(x_{iN}) \\ g_u^L(x_{iN}) \end{bmatrix} u_i = 0. \quad (5)$$

At least one solution to (5) exists as long as the matrix $[A(x_{iN}) \mid g_u(x_{iN})]$ has full row rank. The desired values of x_{iN} are found by iterating over grid points in the scheduling variable space to find, at each iteration i, a trim point such that $V_j(x_i) < c_2^j$ for some $j < i$. Detailed algorithms appear in [9].

2.2. Local Control Lyapunov Functions

Next we construct an RCLF to x_i. Lyapunov linearization yields a natural quadratic Lyapunov function based on the linearized dynamics. Hence, we propose a quadratic Lyapunov function $V_i(x) = (x - x_i)^T P_i (x - x_i)$ and SM $W_i(x) = (x - x_i)^T Q_i (x - x_i)$, where $P_i = P_i^T > 0$ and $Q_i = Q_i^T > 0$.

One way to select an RCLF and SM is to require the closed loop system locally to approximate the solution to an optimal control problem with some cost

$$J(x_0) = \int_0^\infty [(x - x_i)^T Q_c (x - x_i) + (u - u_i)^T R_c (u - u_i)] \, dt$$

based on $Q_c > 0$ and $R_c > 0$ for the linearized system

$$\dot{x} = A(x - x_i) + B_w w + B_u (u - u_i).$$

Therefore, we find P_i to be the stabilizing solution to the algebraic Riccati equation.

$$A^T P_i + P_i A + Q_c - P_i B_u R_c^{-1} B_u^T P_i = 0. \quad (6)$$

For the closed loop system with the optimal control (LQR), the SM is given by $Q_i = Q_c + P_i B_u R_c^{-1} B_u^T P_i$ when $w \equiv 0$. In the nonlinear system, it may only be possible to achieve a SM $\alpha W_i(x)$, for some $\alpha \in (0, 1)$, over some set $V_i^{-1}[c_1^i, c_2^i]$.

2.3. Level Set in Known Stability Region

If $i > 0$, Theorem 1 requires $V_i^{-1}[0, c_1^i] \subseteq V_j^{-1}[0, c_2^j]$ for some j. Given c_2^j, we can use the $\mathcal{S}$-procedure [2] to compute

$$c_1^i \doteq \min_{V_j(x) \geq c_2^j} V_i(x).$$

2.4. Computing the Region of Stability

To check if $V_i(x)$ is an RCLF to x_i with SM $\alpha W_i(x)$ over $V_i^{-1}[c_1^i, c_2^i]$ for given α and $c_2^i > c_1^i \geq 0$, we parameterize the set $V_i^{-1}[c_1^i, c_2^i] \cap \ker(L_{g_u} V_i)$ and evaluate condition (3). We translate x_i to the origin and partition P_i and Q_i to obtain

$$\begin{aligned} V_i(x) &= x_N^T P_{NN} x_N + 2x_N^T P_{NL} x_L + x_L^T P_{LL} x_L, \quad (7) \\ W_i(x) &= x_N^T Q_{NN} x_N + 2x_N^T Q_{NL} x_L + x_L^T Q_{LL} x_L. \quad (8) \end{aligned}$$

To check that $V_i(x)$ is an RCLF with a given SM $\alpha W_i(x)$, it is useful first to parameterize the set $\ker(L_{g_u} V_i)$. For the system (1), the condition $L_{g_u} V_i(x) = 0$ is equivalent to

$$\begin{aligned} Y(x_N)^T x_L &= -[P_{NN} g_u^N(x_N) + P_{NL} g_u^L(x_N)]^T x_N, \\ Y(x_N) &\doteq P_{NL}^T g_u^N(x_N) + P_{LL} g_u^L(x_N). \end{aligned}$$

To simplify the algebra, we assume that $Y(x_N)$ has rank m for all $x_N \in \mathbf{R}^k$. Theorem 3 shows how to parameterize $\ker(L_{g_u} V_i)$ by x_N and a parameter $\lambda \in \mathbf{R}^{n-k-m}$.

Theorem 3 Given a function $V_i(x)$ of the form (7),

$$\begin{aligned} \ker(L_{g_u} V_i) &= \{ \begin{bmatrix} x_N \\ G(x_N)\lambda - P_{LL}^{-1} P_{NL}^T x_N - \xi(x_N) \end{bmatrix}, \\ &\qquad x_N \in \mathbf{R}^k, \lambda \in \mathbf{R}^{n-k-m} \}, \\ \xi &\doteq P_{LL}^{-1} Y [Y^T P_{LL}^{-1} Y]^{-1} [g_u^N]^T R x_N, \\ R &\doteq P_{NN} - P_{NL} P_{LL}^{-1} P_{NL}^T, \end{aligned}$$

where G is any matrix of full rank such that $Y^T G \equiv 0$.

To analyze stability over the level set $V_i^{-1}[c_1^i, c_2^i]$, we need the following definitions.

$$\begin{aligned} V_i(x_N, \lambda) &= x_N^T R x_N + \xi^T P_{LL} \xi + \lambda^T G^T P_{LL} G \lambda, \\ \mathcal{Y}_i(c_2^i) &\doteq \{x_N \in \mathbf{R}^k \mid V_i(x_N, 0) \leq c_2^i\}, \\ \mathcal{Z}_i(c_1^i, c_2^i, x_N) &\doteq \{\lambda \in \mathbf{R}^{n-k-m} \mid c_1^i \leq V_i(x_N, \lambda) \leq c_2^i\}, \\ \Gamma_i(c_1^i, c_2^i, x_N) &\doteq \max_{\lambda \in \mathcal{Z}_i(c_1^i, c_2^i, x_N)} \sup_{w \in \mathcal{W}} \\ & L_f V_i(x) + L_{g_w} V_i(x) w + \alpha W_i(x). \end{aligned}$$

Proposition 1 $V_i(x)$ is an RCLF with SM $\alpha W_i(x)$ over the set $V_i^{-1}[c_1^i, c_2^i]$ iff $\Gamma_i(c_1^i, c_2^i, x_N) \leq 0$ for all $x_N \in \mathcal{Y}_i(c_2^i)$.

To test this condition, we check $\Gamma_i(c_1^i, c_2^i, x_N) \leq 0$ at grid points $x_N \in \mathcal{Y}_i(c_2^i)$. For the system (1), we can do this with substantially less computation time than is required to determine the RS for a general nonlinear system.

We apply Theorem 3 to get the following.

$$\begin{aligned} \Gamma_i(c_1^i, c_2^i, x_N) = & \max_{c_1^i \leq V_i(x_N,\lambda) \leq c_2^i} \sup_{w \in \mathcal{W}} \\ & a_0(x_N) + b_0(x_N)^T \lambda + \lambda^T C_0(x_N) \lambda + \\ & w^T [s(x_N) + T(x_N)\lambda]. \end{aligned}$$

The coefficients are found by simple algebra. For $\mathcal{W} = \{0\}$,

$$\Gamma_i(c_1^i, c_2^i, x_N) = \max_{c_1^i \leq V_i(x_N,\lambda) \leq c_2^i} a_0 + b_0^T \lambda + \lambda^T C_0 \lambda. \quad (9)$$

This quadratic optimization problem with quadratic constraints can be solved using the $\mathcal{S}$-procedure as discussed in [2]. Theorem 4 shows that the two constraints are never simultaneously active; therefore, the $\mathcal{S}$-procedure is nonconservative [2].

Theorem 4 In the problem (9) with $x_N \in \mathcal{Y}_i(c_2^i)$, if $C_0 \not< 0$, then $\Gamma_i(c_1^i, c_2^i, x_N) = \Gamma_i(0, c_2^i, x_N)$. If $C_0 < 0$, one of the following holds for $\lambda^* = -\frac{1}{2} C_0^{-1} b_0$.

1. If $V_i(x_N, \lambda^*) < c_1^i$, $\Gamma_i(c_1^i, c_2^i, x_N) = \Gamma_i(c_1^i, \infty, x_N)$.
2. If $V_i(x_N, \lambda^*) > c_2^i$, $\Gamma_i(c_1^i, c_2^i, x_N) = \Gamma_i(0, c_2^i, x_N)$.
3. Otherwise, $\Gamma_i(c_1^i, c_2^i, x_N) = a_0 - \frac{1}{4} b_0^T C_0^{-1} b_0$.

If $\mathcal{W}$ is a convex polytope, robust stability can be analyzed using Theorem 4 with each of the extreme points of $\mathcal{W}$ substituted for w because the condition is affine in w.

2.5. Iteration over Level Values

Since $\Gamma_i(c_1^i, c_2^i, x_N)$ is nondecreasing in c_2^i, a bisection can be used to find the largest c_2^i satisfying $\Gamma_i(c_1^i, c_2^i, x_N) \leq 0$ over $x_N \in \mathcal{Y}_i(c_2^i)$. At x_0, we need both c_1^0 and c_2^0 so that $\Gamma_0(c_1^0, c_2^0, x_N) \leq 0$ over $x_N \in \mathcal{Y}_0(c_2^0)$. If $\mathcal{W} = \{0\}$, then $c_1^0 = 0$, and we find c_2^0 by bisection. Otherwise, a procedure such as the one outlined in [9] is required.

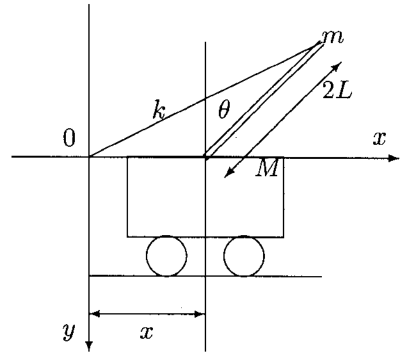

Figure 2: Cart with inverted pendulum and spring.

2.6. Control Law Implementation

The correct RCLF to use in the control input computation is the one corresponding to the index i, given by (4), with the smallest value of $U(i)$ such that $V_i(x) \leq c_2^i$. The largest SM achieved over a given level set depends on the corresponding level value, so the SM should be scaled by a continuous positive function $\alpha(V_i(x))$. In this way, the control can achieve the maximum guaranteed SM over any level set up to the RS. The optimal $\alpha^*(V_i(x))$ is hard to determine, but we can compute level values at finitely many α and interpolate so that $\alpha \leq \alpha^*$, with $\alpha = \alpha^*$ at the interpolation points. As the partition of $\alpha \in (0, 1]$ is refined, the optimal SM is recovered asymptotically.

3. Computational Complexity

The computation times for each step in Algorithm 1 are polynomial in n for fixed k. This is a significant savings over the related problem of gridding the state space to find an RS for a general nonlinear system, which is exponential in n. Trim point selection involves gridding over x_N and solving (5) at each grid point. If there are N_g grid points for each dimension in x_N, the computation time is approximately $N_g^k \mathcal{O}(n^4)$. The time required to solve the Riccati equation (6) for $V_i(x)$ and $W_i(x)$ is $\mathcal{O}(n^2)$. To analyze stability over a given level set for the system (1), we grid $x_N \in \mathcal{Y}_i(c_2^i)$ and solve an LMI problem to compute $\Gamma_i(c_1^i, c_2^i, x_N)$ at each grid point. In the case $\|w\|_\infty \leq 1$, the computation time to solve this problem using an ellipsoid algorithm [2] is roughly $N_c^k 2^l \mathcal{O}(n^3)$, where N_c is the number of grid points in $\mathcal{Y}_i(c_2^i)$ for each dimension.

4. Cart-pole Example

In the system shown in Figure 2, a pole is hinged on a cart, and a spring joins the top of the pole to a fixed point on the wall behind the cart. The control is a force on the cart and the disturbance is a torque on the pole. We want a control design to drive the system to the origin from an initial condition. A simplified model of the dynamics has the form

$$\begin{bmatrix} \dot{\theta} \\ \dot{x}_L \end{bmatrix} = f(\theta) + A(\theta) x_L + g_w(\theta) w + g_u(\theta) u,$$

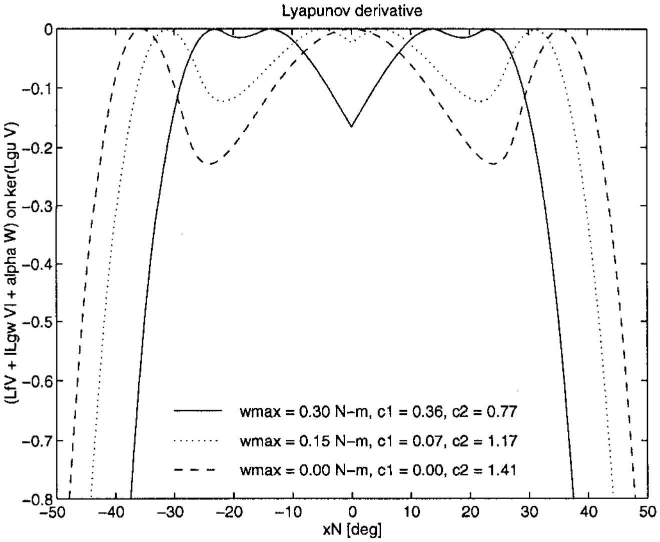

Figure 3: Variation of $\Gamma_0(c_1^0, c_2^0, x_N)$ with disturbance constraint.

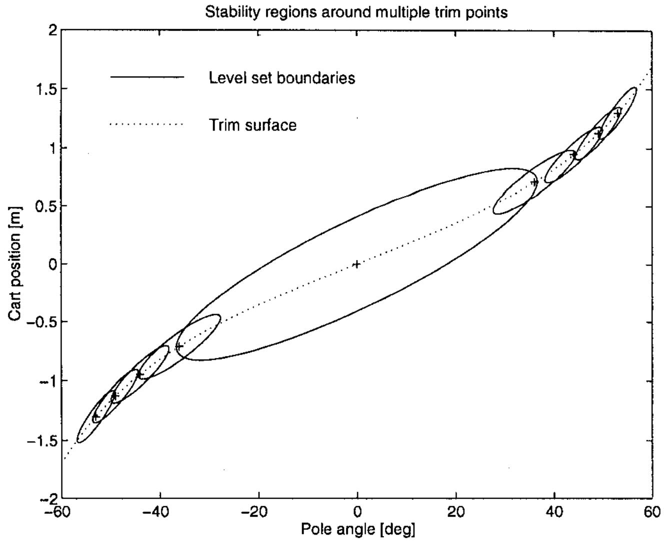

Figure 4: Regions of stability about multiple trim points.

where $x_N = \theta$ and $x_L = [q; x; v]$. After designing an RCLF $V_0(x)$ and SM $W_0(x)$ for the linearized dynamics about $x_0 = 0$ as in Section 2.2, we analyze stability with disturbances $|w| \leq w_{max}$ and a SM scaling $\alpha = 0.1$. The results are plotted in Figure 3 for several values of w_{max}. We can repeat this to get $c_1^0(\alpha)$ and $c_2^0(\alpha)$ over a range of α. We expand the RS using RCLFs to multiple trim points, as shown in Figure 4.

5. Conclusions

Progress in nonlinear control is difficult because arbitrary nonlinear systems are inherently complex. The new method is a stabilizing alternative to gain scheduling for the system (1), which is sufficiently restricted so that computations can be made tractable. The computation time is polynomial in the state dimension, and the control can be designed for robustness to bounded disturbances.

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