Stabilization of Linear Hybrid Systems in the Presence of Communication Constraints*

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Abstract

In this paper, we study the stabilizability of randomly switching linear systems in the presence of finite capacity feedback. Motivated by the structure of communication networks, a stochastic channel was considered that sends words whose size is governed by a random process. Such link is used to transmit state measurements between the plant and the controller. In accordance with previous works, stabilizability of unstable plants is possible if and only if the channel’s Shannon capacity is above given critical values. These will depend on the switching and the channel statistics as well as on the stability criteria. Almost sure and moment stability are considered. Our results show that moment stability needs critical rates which are, at least, as high as the ones required for almost sure stability.

1 Introduction

With a wide range of formulations [1]-[6], control in the presence of communication constraints has been the focus of intense research. In [4, 5, 6], such constraints appear due to the use of a finite Shannon capacity channel in the feedback loop (see Figure 1).

\[ \text{SYSTEM} \quad \rightarrow \quad \text{ENCODER} \]

\[ \text{CONTROLLER} \quad \leftarrow \quad \text{DECODER} \]

\[ \downarrow r_f(k) \text{ bits} \]

\[ \downarrow \text{CHANNEL} \]

Figure 1: Feedback loop with communication constraints.

If the plant is unstable and time-invariant, the construction of a stabilizing controller requires the Shannon capacity [4] of the channel to be above a non-zero critical value. If the

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channel is deterministic, such condition is also a sufficient. Tighter notions of stability, such as in the mean square sense, may lead to higher critical rates.

In this paper we extend the existing results to linear switching systems in the presence of stochastic channels. We derive conditions, on the Shannon capacity of the channel, that guarantee the existence of a stabilizing controller while not relying on the equimemory assumption [4].

The following notation is adopted:

Random variables are represented as boldface letters, such as \( \mathbf{w} \); if \( \mathbf{w}(k) \) is a stochastic process, then we use \( w(k) \) to indicate a specific realization; the expectation operator over a random variable \( \mathbb{E}[\mathbf{w}] \); if \( A \) is a probabilistic event, then its probability is indicated as \( \mathbb{P}(A) \); we write \( \log_2(.) \) simply as \( \log(.) \); \( \lim_{k \to \infty} \mathbb{E}[\mathbf{w}(k)] = \beta \) is used to indicate that a sequence of random variables \( \mathbf{w}(k) \) converges to \( \beta \) with probability 1 and that \( \lim_{k \to \infty} \mathbb{E}[\mathbf{w}(k)] = \beta \).

### 1.1 Problem Formulation

We wish to study the stabilization of randomly switching linear systems under communication constraints. Motivated by the type of constraints that arise in communication networks, we consider the following class of network channels:

**Definition 1.1 (Network Channel)** Consider a channel that, at every instant \( k \), transmits \( r_F(k) \) bits. We define it to be a network channel, provided that \( r_F(k) \) is a positive integer random process satisfying:

\[
\forall k_0 \in \mathbb{N}_+, \lim_{k \to \infty} \frac{\sum_{i=k_0}^{k} r_F(i)}{k} = C \geq 0 \tag{1}
\]

*From the properties of the convergence with probability 1* [9], (1) implies that \( C \) is the Shannon capacity of the channel [8].

In order to keep the paper concise, we restrict our analysis to scalar systems of the form:

\[
x(k + 1) = a(k)x(k) + b(k)u(k) \tag{2}
\]

where \( a(k) \) and \( b(k) \) are real valued stochastic processes. In addition, \( a(k) \) satisfies:

\[
\lim_{k \to \infty} \frac{\sum_{i=0}^{k} \log(|a(i)|)}{k} = \mathcal{R}_C \tag{3}
\]

We also assume that \( \mathbb{P}(a(k) = 0) = 0 \), otherwise the system is trivially stable. In order to avoid extra technicalities, we also assume that \( \mathbb{P}(b(k) = 0) = 0 \). The structure of the feedback loop imposes the following constraints:

- At every instant \( k \), the encoder has access to \( x(0), \ldots, x(k) \).
- The channel transmits, at every \( k \), a stream of \( r_F(k) \) bits \( (\alpha_{\eta(k-1)+1}, \ldots, \alpha_{\eta(k)}) \), where \( \alpha_i \in \{0, 1\} \), \( \eta(k) = \sum_{i=0}^{k} r_F(i) \) and \( \eta(-1) = 0 \).
- The control action exhibits the following functional dependence (for some \( f_k \)):

\[
u(k) = f_k(\alpha_1, \ldots, \alpha_{\eta(k)}) \tag{4}\]
Definition 1.2 (Feedback Scheme) We define a feedback scheme as being the collection of a controller, an encoder and a decoder.

Our problem consists in determining necessary and sufficient conditions, as a function of the Shannon capacity $\mathcal{C}$, that guarantee the existence of a stabilizing feedback scheme. The results must be derived for the following notions of stability:

Definition 1.3 (Almost Sure Stability) The system (2) is almost surely asymptotically stable, provided that the following holds:

$$
\mathcal{P}\left( \lim_{k \to \infty} \sup_{x(0) \in [0,1]} |x(k)| = 0 \right) = 1 \tag{5}
$$

Definition 1.4 (m-th Moment Stability) Given $m > 0$, the system (2) is m-th moment stable, provided that the following holds:

$$
\lim_{k \to \infty} \mathcal{E} \left[ \sup_{x(0) \in [0,1]} |x(k)|^m \right] = 0 \tag{6}
$$

We would like to clarify the abuse of notation in $\sup_{x(0) \in [0,1]} |x(k)|$. Consider that $g_k$ is a function such that:

$$
x(k) = g_k(x(0), a(0), \ldots, a(k-1), r_I(0), \ldots, r_I(k-1)) \tag{7}
$$

then the random variable $w(k) = \sup_{x(0) \in [0,1]} |x(k)|$ can be defined as:

$$
w(k) = \sup_{x(0) \in [0,1]} |g_k(x(0), a(0), \ldots, a(k-1), r_I(0), \ldots, r_I(k-1))| \tag{8}
$$

Note on the generality of the stability definitions: The definitions 1.3 and 1.4 are slightly different from most found in the literature [12, 11]. We start by arguing that they are not too strong. This can be illustrated by noticing that the forms of stability (9) and (10) imply, respectively, almost sure and moment stability according to our definitions.

$$
\mathcal{P}\left( \lim_{k \to \infty} \sup_{x(0) \in \mathbb{S} \setminus \{0\}} \frac{|x(k)|}{|x(0)|} = 0 \right) = 1, \ [0,1] \subset \mathbb{S} \subset \mathbb{R} \tag{9}
$$

$$
\lim_{k \to \infty} \mathcal{E} \left[ \sup_{x(0) \in \mathbb{S} \setminus \{0\}} \frac{|x(k)|^m}{|x(0)|^m} \right] = 0, \ [0,1] \subset \mathbb{S} \subset \mathbb{R} \tag{10}
$$

Also, note that the choice of the interval $[0,1]$ in (5) and (6) was done just to make the paper more readable. All results are valid if $[0,1]$ is replaced by any bounded interval.
1.2 Outline of the paper and summary of the results

The main results of the paper are organized in 2 sections:

Section 2 deals with stability in the almost sure sense. In theorem 2.1, we show that the following is a necessary condition for the existence of a stabilizing feedback scheme:

\[ C \geq R_C \]  

(11)

In section 2.1, we construct a channel, with \( C = R_C \), together with a stabilizing feedback scheme, thus showing that (11) is a non-conservative necessary condition. We stress that if we had required the channel to be stationary, or deterministic as in [4], then the necessary condition would be, instead, a strict inequality. We also show that \( C > R_C \) is a sufficient condition for the existence of a stabilizing feedback scheme.

The analysis, for stability in the m-th moment sense, is carried out in section 3. The result expressed in theorem 3.1 shows that \( C \geq R_C \) is still a non-conservative necessary condition for stabilization. Subsection 3.1 presents the case where \( r_f(k) \) is independent of \( a(k) \). Under this constraint, we compute new conditions for stabilization. Surprisingly, it follows from theorem 3.3 that if \( r_f(k) \) is constant, \( \log |a(k)| \) is i.i.d. and normally distributed, then \( C > R_C^m \) is a sufficient condition for stabilizability, where \( R_C^m \) is given by:

\[ R_C^m = R_C + \frac{m}{2} \log \left( 1 + \frac{\text{Var}(a(k))}{(E[a(k)])^2} \right) \]  

(12)

Note that the second term on the RHS resembles the Shannon capacity of a Gaussian channel [8]. For the feedback scheme used in the proof of the sufficient condition, \( C > R_C^m \) is also necessary.

We also would like to add that our results extend to second order switching systems, provided that the dynamic matrices have complex eigenvalues. Consequently, the extension to higher order systems can be achieved by means of real Jordan forms (such extension will soon be published elsewhere). Additional stabilizability and observability conditions have to be added.

2 Conditions for stabilization, in the almost sure sense

In this section we start by deriving a necessary condition, in terms of the Shannon capacity, that a network channel (definition 1.1) must satisfy for the existence of a stabilizing feedback scheme. The result is stated in theorem 2.1.

The following definitions are important in the quantification of how many bits flow through the channel.

**Definition 2.1 (Zero-state solution)** Consider the zero-state solution \( x_z(k) \) of (2), given by:

\[ x_z(k + 1) = a(k) x_z(k) + b(k) u(k), \quad k \geq 0, x_z(0) = 0 \]  

(13)

We define the initial condition estimate \( \hat{x}(k) \), based on the zero-state solution, as:

\[ \hat{x}(k) = - \left( \prod_{i=0}^{k-1} a(i) \right)^{-1} x_z(k) \]  

(14)
Definition 2.2 Let \( \delta, \beta \in \mathbb{R}_+ \) be given real numbers along with the following binary expansions:

\[
\delta = \sum_{i=-\infty}^{\infty} \alpha_i^\delta \frac{1}{2^i}, \quad \beta = \sum_{i=-\infty}^{\infty} \alpha_i^\beta \frac{1}{2^i}, \quad \alpha_i^\delta, \alpha_i^\beta \in \{0, 1\}
\]  

We define the first bit error function \( \mathcal{FBE} : \mathbb{R}_+^2 \to \mathbb{N} \) as:

\[
\mathcal{FBE}(\delta, \beta) = \min_i \{ i \in \mathbb{N} : \alpha_i^\delta \neq \alpha_i^\beta \}
\]

The following sets will allow us to infer \( \mathcal{FBE}(\delta, \beta) \) directly from \( |\delta - \beta| \).

Definition 2.3 (Information sets) For a given \( q \in \mathbb{N}_+ \), we define the following sets:

\[
\Pi_q = \{ \delta \in [0, 1] : \delta = \sum_{i=1}^{q} \alpha_i^\delta \frac{1}{2^i} + \frac{1}{2^q+1}, \alpha_i \in \{0, 1\} \}
\]

It is possible to prove that, if \( \gamma > 0 \) and \( q \in \mathbb{N}_+ \) are such that \( q + 1 < \gamma \), then the following is satisfied:

\[
\forall \delta \in \Pi_q, \forall \beta \in \mathbb{R}_+, \, |\delta - \beta| < 2^{-\gamma} \implies \mathcal{FBE}(\delta, \beta) \geq q + 1
\]

Also notice that \( \Pi_q = 2^q \).

Remark 2.1 Using (13) and (14), the solution of (2) can be expressed as:

\[
|x(k)| = 2^{\sum_{i=0}^{k-1} \log(|a(i)|)}|x(0) - \hat{x}(k)|
\]

This implies that, for any realization of \( a(k) \) and \( r_1(k) \), we can use (18) to show that:

\[
\sup_{x(0) \in [0, 1]} |x(k)| < 1 \implies \inf_{x(0) \in \Pi_q(k)} \mathcal{FBE}(|x(0)|, |\hat{x}(k)|) \geq q(k) + 1
\]

where \( q(k) = \lfloor \sum_{i=0}^{k-1} \log(|a(i)|) \rfloor - 2. \)

Remark 2.2 (Interpretation of \( \mathcal{FBE}(|x(0)|, |\hat{x}(k)|) \))

We start by noticing that \( \mathcal{FBE}(|x(0)|, |\hat{x}(k)|) - 1 \geq M \geq 0 \) implies that the binary expansions of \( |x(0)| \) and \( |\hat{x}(k)| \) will agree up to, at least, the term \( M \). It is a consequence of the feedback loop structure, that all information about \( x(0) \), that is conveyed to \( \hat{x}(k) \), has to go through the channel. Using counting arguments, that implies that for every realization of \( a(k) \) and \( r_1(k) \), the following must hold:

\[
\inf_{x(0) \in \Pi_q(k)} \mathcal{FBE}(|x(0)|, |\hat{x}(k)|) \geq q(k) + 1 \implies \frac{\sum_{i=0}^{k-1} r_1(i)}{k} \geq \frac{q(k)}{k}
\]
The following theorem shows that \( C \geq \mathcal{R}_C \) is a necessary condition for the stabilizability of (2). We provide a proof for a weaker notion of stability. The fact that convergence with probability 1 implies convergence in probability, shows that if (2) is almost surely stable then (22) must hold.

**Theorem 2.1 (Necessary condition for stability)** If \( x(k) \) evolves according to (2) then (22) implies (23).

\[
\lim_{k \to \infty} \mathcal{P} \left( \sup_{x(0) \in [0,1]} |x(k)| \geq 1 \right) = 0 \tag{22}
\]

\[
C \geq \mathcal{R}_C \tag{23}
\]

**Proof:**
From remark 2.1 and (22), we conclude that:

\[
\lim_{k \to \infty} \mathcal{P} \left( \inf_{x(0) \in \mathcal{E}} \mathcal{F} \mathcal{B} \mathcal{E}(|x(0)|, |\dot{x}(k)|) \geq q(k) + 1 \right) = 1 \tag{24}
\]

where \( q(k) = \lfloor \sum_{i=0}^{k-1} \log(|a(i)|) \rfloor - 2 \). According to remark 2.2, that implies:

\[
\lim_{k \to \infty} \mathcal{P} \left( \frac{\sum_{i=0}^{k-1} r_I(i)}{k} \geq \frac{q(k)}{k} \right) \tag{25}
\]

and, since \( l.i.m. \frac{q(k)}{k} = \mathcal{R}_C \) and \( l.i.m. \frac{\sum_{i=0}^{k-1} r_I(i)}{k} = C \), it allows us to conclude that \( C \geq \mathcal{R}_C \).

\[\square\]

### 2.1 Existence of stabilizing controllers

In this subsection, we construct a stabilizing feedback scheme in the presence of a network channel. The following defines a stabilizing feedback scheme.

**Definition 2.4 (Stabilizing feedback scheme)** Define a control sequence according to the following steps:

- **Encoder**: Measures \( x(0) \) and computes the binary expansion \( x(0) = \sum_{i=1}^{\infty} \alpha_i \frac{1}{2^i}, \alpha_i \in \{0, 1\} \)

- At every \( k \), send \( \alpha_{\eta(k-1)+1} \ldots \alpha_{\eta(k)} \) through the channel, where \( \eta(k) = \sum_{i=0}^{k} r_I(i) \) and \( \eta(-1) = 0 \).

- **Decoder and controller**: Compute \( u(k) \) as:

  \[
u(k) = -\prod_{i=0}^{k} \frac{a(i)}{b(k)} \sum_{i=\eta(k-1)+1}^{\eta(k)} \alpha_i \frac{1}{2^i} \tag{26}\]


Remark 2.3 According to the definition 2.4, if, for any initial condition \( x(0) \in [0, 1] \), the control constructed in (26) is applied to (2), then:

\[
|x(k)| \leq 2^{\sum_{i=0}^{k-1}[\log |a(i)| - r_l(i)]}
\] (27)

Subsequently, the following result shows that theorem 2.1 is a tight necessary condition.

Lemma 2.2 (Necessary condition is non-conservative) Let \( x(k) \) be the solution of (2), \( x(0) \in [0, 1] \) and \( C \geq 0 \). Then there exists a switching channel, with Shannon capacity \( C \), and a feedback scheme such that:

\[
|x(k)| \leq 2^{-\epsilon k + \sqrt{\epsilon}}
\] (28)

where \( \epsilon = C - \mathcal{R}_C \).

Proof: According to remark 2.3, there exists a control sequence such that:

\[
|x(k)| \leq 2^{\sum_{i=0}^{k-1}[\log |a(i)| - r_l(i)]}
\] (29)

The desired result follows once we select \( r_l(k) \) such that

\[
\sum_{i=0}^{k} r_l(i) = \max \left\{ \sum_{i=0}^{k} \log(|a(i)|) + \epsilon (k + 1) + \sqrt{k + 1}, 0 \right\}
\] (30)

Notice also that this choice implies that \( \lim_{k \to \infty} \frac{\sum_{i=0}^{k-1} r_l(i)}{k} = C \).

The following theorem provides a sufficient condition, on the channel Shannon capacity, that guarantees the existence of a stabilizing feedback scheme.

Theorem 2.3 (Sufficient condition) Let \( x(0) \in [0, 1] \) be an initial condition and \( u(k) \) be constructed according to (26). If \( C > \mathcal{R}_C \), then the following holds:

\[
\mathcal{P} \left( \lim_{k \to \infty} \sup_{x(0) \in [0, 1]} |x(k)| = 0 \right) = 1
\] (31)

Proof: Choose \( \gamma > 0 \) such that \( C - \mathcal{R}_C > \gamma \) and notice that, from remark 2.3, the following holds:

\[
|x(k)| \geq 2^{-\epsilon \gamma} \iff \frac{\sum_{i=0}^{k-1} r_l(i)}{k} - \frac{\sum_{i=1}^{k-1} \log |a(i)|}{k} \leq \gamma
\] (32)

At this point (31) follows from (32) and the convergence, with probability 1, of \( \frac{\sum_{i=0}^{k-1} \log |a(i)|}{k} \) and \( \frac{\sum_{i=0}^{k-1} r_l(i)}{k} \).
3 Stability in the m-th moment sense

We start by deriving a necessary condition by showing that a channel that enables stability, in the m-th moment sense, must satisfy $C \geq R_C$. In addition, we stress that lemma 2.2 guarantees that it is not conservative.

**Theorem 3.1 (Necessary condition for stability)** If $\mathbf{x}(k)$ evolves according to (2) then (33) implies (34).

$$\exists m \in \mathbb{R}, \ s.t. \ \lim_{k \to \infty} \mathbb{E}\left[ \sup_{x(0) \in [0,1]} |\mathbf{x}(k)|^m \right] = 0 \quad (33)$$

$$C \geq R_C \quad (34)$$

**Proof:**
From Markov’s inequality [9], we know that (33) implies:

$$\lim_{k \to \infty} \mathcal{P}\left( \sup_{x(0) \in [0,1]} |\mathbf{x}(k)| \geq 1 \right) = 0 \quad (35)$$

The final result is a consequence of theorem 2.1.

3.1 Analysis for $r_I$ independent of $a(k)$

In this subsection, our analysis is restricted to switching channels where $r_I(k)$ is independent of $a(k)$. We determine necessary and sufficient conditions, on the Shannon capacity of the channel, such that (26) is a stabilizing feedback scheme. Furthermore, we consider that $\log |a(k)|$ is normally distributed so that closed expressions can be obtained. As a consequence of the central limit theorem, such situation arises when certain continuous stochastic systems are discretized. It may also appear directly as the discrete equivalent of a continuous first order system whose time constant is a stochastic process normally distributed. If $\log |a(k)|$ is restricted to be i.i.d, but not necessarily Gaussian, then it is still possible to compute the critical rates on a case-by-case basis [11], [2].

**Theorem 3.2 (Necessary condition for stability)** Consider that $\log |a(k)|$ is normally distributed and that the following holds:

$$\lim_{k \to \infty} \frac{\text{Var} \left( \sum_{i=0}^{k-1} \ln |a(i)| \right)}{k} = \sigma_S^2 \quad (36)$$

If $\mathbf{x}(k)$ is the solution to (2), $u(k)$ is given by (26) and $r_I(k)$ is independent of $\log |a(k)|$, then (37) implies (38), where the critical capacity is given by (39).

$$\exists m > 0 \ s.t. \ \limsup_{k \to \infty} \mathbb{E}\left[ \sup_{x(0) \in [0,1]} |\mathbf{x}(k)|^m \right] < 1 \quad (37)$$
\[ C \geq \mathcal{R}_C^m \]  
\[ \mathcal{R}_C^m = \mathcal{R}_C + \frac{m\sigma_S^2}{2\ln 2} \]  

Furthermore, if \( a(k) \) is i.i.d., then:

\[ \frac{\sigma_S^2}{\ln 2} = \log \left( 1 + \frac{\text{Var}(a(k))}{\mathcal{E}[a(k)]^2} \right) \]  

**Proof:**

Note that the structure of (26) is such that for any realization of \( a(k) \) and \( r_f(k) \), we have:

\[ \sup_{x[0] \in [0,1]} |x(0) - \hat{x}(k)| = 2^{-\sum_{i=0}^{k-1} r_f(i)} \]  

Given that \( |x(k)| = 2^{\sum_{i=0}^{k-1} \log |a(i)||x(0) - \hat{x}(k)|} \), we have that the independence between \( a(k) \) and \( r_f(k) \) leads to:

\[ \mathcal{E} \left[ \sup_{x[0] \in [0,1]} |x(k)| \right] = \mathcal{E} \left[ 2^{\sum_{i=0}^{k-1} \log |a(i)|} \right] \mathcal{E} \left[ 2^{\sum_{i=0}^{k-1} r_f(i)} \right] \]  

Using Jensen’s inequality [8], we find that (42) and (37) imply:

\[ \forall \delta > 0, \exists k_0, \text{ s.t. } \forall k \geq k_0, \mathcal{E} \left[ \frac{\sum_{i=0}^{k-1} r_f(i)}{k} \right] > \frac{1}{m k} \frac{1}{k} \log \left( \mathcal{E} \left[ 2^{\sum_{i=0}^{k-1} \log |a(i)|} \right] \right) = \delta \]  

so that (38) follows after the computation of the following expectation:

\[ \lim_{m \to \infty} \frac{1}{k} \mathcal{E} \left[ 2^{m \sum_{i=0}^{k-1} \log |a(i)|} \right] = \left( \mathcal{R}_C + \frac{m\sigma_S^2}{2\ln 2} \right) \]  

Further algebraic manipulations will lead to (40).

\[ \square \]

By noticing that:

\[ \sup_{x[0] \in [0,1]} |x(k)| = 2^{\sum_{i=0}^{k-1} \log |a(i)|} \sup_{x[0] \in [0,1]} |x(0) - \hat{x}(k)| \]  

and by inspecting the previous proof, we conclude that theorem 3.2 is valid under more general conditions. The necessary condition holds if we consider that \( \sup_{x[0] \in [0,1]} |x(0) - \hat{x}(k)| \) is independent of \( r_f(k) \), instead of requiring that \( r_f(k) \) is independent of \( a(k) \) and \( u(k) \) is given by (26). That shows that, if we want to send information from \( x(0) \) to \( \hat{x}(k) \) at an instantaneous rate which is independent of the switching \( a(k) \), then \( C \geq \mathcal{R}_C^m \) must hold.

A similar approach leads to the following theorem which we state without proof. Notice that the condition bellow is different from the previous ones because the stationarity of \( r_f(k) \) implies that the inequality in (46) is strict. This result is consistent with the work in [4] since there the channel is deterministic and, as such, stationary.
Theorem 3.3 Under the conditions stated in theorem 3.2, if $r_i(k)$ is i.i.d, then the control sequence (26) is stabilizing if and only if

$$\mathcal{C} > R^m_C + \frac{1}{m} \log \mathcal{E} \left[ 2^{m[r_i(k)-C]} \right]$$

(46)

Notice that in (46), Jensen’s inequality guarantees that $\frac{1}{m} \log \mathcal{E} \left[ 2^{m[r_i(k)-C]} \right] \geq 0$.

References

[1] Elia, N., “Control-Oriented feedback communication schemes”, (not yet available)


