RESOLUTION OF A CONJECTURE OF ANDREWS AND LEWIS INVOLVING CRANKS OF PARTITIONS

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Abstract. In [1] Andrews and Lewis conjecture that the sign of the number of partitions of n with crank congruent to 0 mod 3, minus the number of partitions of n with crank congruent to 1 mod 3 is determined by the congruence class of n mod 3 apart from a finite number of specific exceptions. We prove this by using the "Circle Method" to approximate the value of this difference to great enough accuracy to determine its sign for all sufficiently large n.

1. Introduction and Statement of Results

We define a partition to be a (weakly) decreasing sequence of natural numbers $\pi = \{\pi_0, \pi_1, ..., \pi_{k-1}\}$. We define the weight of a partition $w(\pi) = \pi_0 + \pi_1 + ... + \pi_{k-1}$. Furthermore, we will use the definition of the crank of a partition used by Garvan and Andrews that

$$\text{crank}(\pi) = \begin{cases} \pi_0, & \text{if } \mu(\pi) = 0, \\ \nu(\pi) - \mu(\pi), & \text{if } \mu(\pi) > 0, \end{cases}$$

where $\mu(\pi)$ denoted the number of ones in π and $\nu(\pi)$ denotes the number of parts of π larger than $\mu(\pi)$. We also define

$$M(r, m, n) = \#\{\pi : w(\pi) = n, \text{crank}(\pi) \equiv r \mod m\}.$$

Lastly, we define

$$g(n) = M(0, 3, n) - M(1, 3, n).$$

In [1] it is shown that

$$\sum_{n=0}^{\infty} g(n) q^n = \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty} = 1 - 2q - q^2 + 3q^3 - q^4 + q^5 + 2q^6 + O\left(q^7\right),$$

where $(z;q)_\infty = \prod_{n=0}^{\infty}(1 - zq^n)$. In a recent paper [1], Andrews and Lewis make the following conjecture:

Conjecture (Andrews-Lewis). *If n is a positive integer, then*

$$\begin{aligned} g(3n) &> 0 \\ g(3n+1) &< 0 \\ g(3n+2) &< 0 \qquad \text{unless } n \in \{1, 4, 5\}. \end{aligned}$$

In this paper, we prove this conjecture by using the "Circle Method" to approximate the size of $g(n)$ from its generating function. In particular, we prove the following theorem:

Key words and phrases. Circle Method, Partitions, Cranks.

Theorem 1. *If n is a positive integer, then*

$$g(n) = \sum_{0<k^2<\frac{28n}{9}} A_k(n)k^{-3/2}\left[\frac{\cosh(x)}{x}\right]_{x=\frac{\pi}{3k}\sqrt{\frac{2(n-1/24)}{3}}} + E_n,$$

where $s(h,k)$ is the usual Dedekind sum and

$$A_k(n) = \sum_{(h,3k)=1} e^{\pi i(s(h,k)-2s(h,3k))}e^{-2\pi i n\frac{h}{3k}},$$

and where

$$|E_n| < 200n^{1/4}.$$

Corollary 2. *As $n \to \infty$ we have*

$$g(n) \sim \left(e^{-\pi i/9}e^{-2n\pi i/3} + e^{\pi i/9}e^{2n\pi i/3}\right)\sqrt{\frac{1}{6n}}e^{\frac{\pi}{3}\sqrt{2n/3}} = apr(n)$$

Moreover, the Andrews-Lewis Conjecture is true.

Example. Here we illustrate Corollary 2 by comparing values of $g(n)$ to the values of the asymptotic approximation $apr(n)$ for$n = 60, 61$ and 62.

$$n = 60,\ g(n) = 74,\ \ apr(n) = 74.511...$$
$$n = 61,\ g(n) = -66,\ apr(n) = -63.646...$$
$$n = 62,\ g(n) = -17,\ apr(n) = -15.112...$$

2. Definitions and Functional Equations

We let

$$F(q) = (q;q)_\infty^{-1}$$

We have the functional equation that if $k > 0, h, H$ are integers with $hH \equiv -1(\mod k)$ and

$$x = \exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}\right),$$

$$x' = \exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right),$$

then,

$$F(x) = e^{\pi is(h,k)}\left(\frac{z}{k}\right)^{1/2}\exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right)F(x').$$

The proof of this comes from the functional equation of the Dedekind eta-function and can be found in [2]. By substituting $y = \frac{2\pi z}{k^2}$, we find that

$$F\left(\exp\left\{\frac{2\pi ih}{k} - y\right\}\right) = e^{\pi is(h,k)}\left(\frac{yk}{2\pi}\right)^{1/2}\exp\left(\frac{\pi^2}{6k^2y} - \frac{y}{24}\right)F\left(\exp\left\{\frac{2\pi iH}{k} - \frac{4\pi^2}{k^2y}\right\}\right). \tag{2.1}$$

Let

$$f(q)=\sum_{n=0}^{\infty} g(n)q^n=\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}=\frac{F(q^3)}{F(q)^2}.$$

If $k>0$ is an integer not divisible by 3, and h,h',H,H' are integers so that $3h\equiv h' (\mod k)$, $hH\equiv h'H'\equiv -1(\mod k)$, then,

$$\begin{aligned} f\left(\exp\left\{\frac{2\pi ih}{k}-y\right\}\right) &= \frac{F\left(\exp\left\{\frac{2\pi ih'}{k}-3y\right\}\right)}{F^2\left(\exp\left\{\frac{2\pi ih}{k}-y\right\}\right)} \\ &= e^{\pi i(s(h',k)-2s(h,k))}\sqrt{\frac{6\pi}{ky}}\exp\left(-\frac{5\pi^2}{18k^2y}-\frac{y}{24}\right)\frac{F\left(\exp\left\{\frac{2\pi iH'}{k}-\frac{4\pi^2}{3k^2y}\right\}\right)}{F^2\left(\exp\left\{\frac{2\pi iH}{k}-\frac{4\pi^2}{k^2y}\right\}\right)}. \end{aligned}$$

Therefore, we have that

$$\left|f\left(\exp\left\{\frac{2\pi ih}{k}-y\right\}\right)\right|\le\sqrt{\frac{6\pi}{ky}}\exp\left(-\frac{5\pi^2}{18k^2y}-\frac{y}{24}\right)F^3\left(\exp\left\{-\frac{4\pi^2}{k^2}\Re(\frac{1}{y})\right\}\right). \tag{2.2}$$

Because the coefficients of $\log(F(q))$ are all positive.

Now, if $k>0$ and h,H are integers with $hH\equiv -1(\mod 3k)$, then

$$\begin{aligned} f\left(\exp\left\{\frac{2\pi ih}{3k}-y\right\}\right) &= \frac{F\left(\exp\left\{\frac{2\pi ih}{k}-3y\right\}\right)}{F^2\left(\exp\left\{\frac{2\pi ih}{3k}-y\right\}\right)} \\ &= e^{\pi i(s(h,k)-2s(h,3k))}\sqrt{\frac{2\pi}{3ky}}\exp\left(\frac{\pi^2}{54k^2y}-\frac{y}{24}\right)\frac{F\left(\exp\left\{\frac{2\pi iH}{k}-\frac{4\pi^2}{3k^2y}\right\}\right)}{F^2\left(\exp\left\{\frac{2\pi iH}{3k}-\frac{4\pi^2}{9k^2y}\right\}\right)} \\ &= e^{\pi i(s(h,k)-2s(h,3k))}\sqrt{\frac{2\pi}{3ky}}\exp\left(\frac{\pi^2}{54k^2y}-\frac{y}{24}\right)f\left(\exp\left\{\frac{2\pi iH}{3k}-\frac{4\pi^2}{9k^2y}\right\}\right). \end{aligned} \tag{2.3}$$

3. The integration Formula

We modify Andrews' treatment of the Circle Method. In particular, we assume that the reader is familiar with the results in Chapter 5 of [2]. Using Cauchy's integration formula, we have that

$$\begin{aligned} g(n) &= \int_0^1 f\left(\exp(-\rho+2\pi i\phi)\right)e^{n\rho-2\pi in\phi}d\phi \\ &= \sum_{k<N,0<h<k,(h,k)=1}\exp\left(-\frac{2\pi ih}{3k}\right)\int_{-\theta'_{h,k}}^{\theta''_{h,k}} f\left(\exp\left\{\frac{2\pi ih}{k}-(\rho-2\pi i\phi)\right\}\right)e^{n(\rho-2\pi i\phi)}d\phi \end{aligned}$$

where the θ are determined by the distance between the Farey number $\frac{h}{k}$ and the next Farey number. Therefore, $\theta_{h,k}$ satisfies the inequality $\frac{1}{kN}\ge\theta_{h,k}\ge\frac{1}{2kN}$ (see [3]). By making the substitution $w=\rho-2\pi i\phi$, we get that

$$g(n)=\sum_{k<N,0<h<k,(h,k)=1}\exp\left(-\frac{2\pi ih}{3k}\right)\frac{1}{2\pi i}\int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} f\left(\exp\left\{\frac{2\pi ih}{k}-w\right\}\right)e^{nw}dw. \tag{3.1}$$

This is to say that if

$$\sum{}' = \sum_{k<N, 0<h<k, (h,k)=1, \neg 3|k} \exp\left(-\frac{2\pi i h}{3k}\right) \frac{1}{2\pi i} \int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} f\left(\exp\left\{\frac{2\pi i h}{k} - w\right\}\right) e^{nw} dw,$$

and

$$\sum{}'' = \sum_{k<N, 0<h<k, (h,k)=1, 3|k} \exp\left(-\frac{2\pi i h}{3k}\right) \frac{1}{2\pi i} \int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} f\left(\exp\left\{\frac{2\pi i h}{k} - w\right\}\right) e^{nw} dw,$$

then

$$g(n) = \sum{}' + \sum{}''. \tag{3.2}$$

The rest of this paper is devoted to the study of $\sum'$ and $\sum''$, where n is assumed to be a given positive integer.

4. Bounds on $\sum'$

By (2.3) we have that

$$\left|\frac{1}{2\pi i} \int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} f\left(\exp\left\{\frac{2\pi i h}{k} - w\right\}\right) e^{nw} dw\right| \leq$$

$$\frac{1}{2\pi} \int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} \sqrt{\frac{6\pi}{k}} \exp\left(-\frac{5\pi^2}{18k^2}\Re(\frac{1}{w})\right) e^{(n-1/24)\rho} |w|^{-1/2} F^3\left(\exp\left\{-\frac{4\pi^2}{k^2}\Re(\frac{1}{w})\right\}\right) dw =$$

$$\sqrt{\frac{3}{2\pi k}} e^{n\rho} \int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} \exp\left(-\frac{5\pi^2}{18k^2}\Re(\frac{1}{w})\right) |w|^{-1/2} F^3\left(\exp\left\{-\frac{4\pi^2}{k^2}\Re(\frac{1}{w})\right\}\right) dw$$

Now, $\Re(\frac{1}{\rho+2\pi i\phi}) = \frac{\rho}{\rho^2+4\pi^2\phi^2}$. Therefore, $\Re(\frac{1}{w}) \geq \frac{\rho}{\rho^2+4\pi^2\theta^2} \geq \frac{k^2N^2\rho}{k^2N^2\rho^2+4\pi^2}$. Letting $a = N^2\rho$, we get that $\Re(\frac{1}{w}) \geq \frac{k^2 a}{a^2+4\pi^2}$. Therefore, $F\left(\exp\left\{-\frac{4\pi^2}{k^2}\Re(\frac{1}{w})\right\}\right) \leq F\left(\exp\left\{-\frac{4\pi a}{a^2+4\pi^2}\right\}\right)$. So if we insist on the condition

$$6 \leq a \leq 7, \tag{Condition 1}$$

then $\frac{4\pi^2 a}{a^2+4\pi^2} > 3$. Therefore,

$$F^3\left(\exp\left\{-\frac{4\pi^2}{k^2}\Re(\frac{1}{w})\right\}\right) \leq 1.2.$$

Now, since

$$\exp\left(-\frac{5\pi^2}{18k^2}\Re(\frac{1}{w})\right) |w|^{-1/2} \leq \rho^{-1/2}.$$

and, since the length of the integral is at most $\frac{4\pi}{kN}$, we have that

$$\left|\frac{1}{2\pi i} \int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} f\left(\exp\left\{\frac{2\pi i h}{k} - w\right\}\right) e^{nw} dw\right|$$
$$\leq 1.2\sqrt{\frac{3}{2\pi k}} e^{n\rho} \frac{4\pi}{kN} \rho^{-1/2} \leq 10.5 \frac{\rho^{-1/2} e^{n\rho}}{k^{3/2} N}.$$

If we sum over all $0 < h < k$, then it is at most $10.5\frac{\rho^{-1/2}e^{n\rho}}{k^{1/2}N}$. If we then sum over all $0 < k < N$ it is at most $5.3\rho^{-1/2}e^{n\rho}N^{-1/2} = 5.3a^{-1/4}\rho^{-1/4}e^{n\rho}$. Therefore,

$$\sum{}' \leq 3.4\rho^{-1/4}e^{n\rho}.$$

Now if we set

(Condition 2) $$\rho = \frac{1}{4n},$$

then

(4.1) $$\sum{}' \leq 7n^{1/4}.$$

5. Approximation of $\sum''$

We will look at

$$\frac{1}{2\pi i}\int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} f\left(\exp\left\{\frac{2\pi i h}{3k} - w\right\}\right) e^{nw}dw.$$

By (2.3), this is

$$\frac{1}{2\pi i}\int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} \omega_{h,k}\exp\left(\frac{2\pi i h}{3k}\right)\sqrt{\frac{2\pi}{3kw}}e^{(n-1/24)w}\exp\left(\frac{\pi^2}{54k^2w}\right) f\left(\exp\left\{\frac{2\pi i H}{3k} - \frac{4\pi^2}{9k^2w}\right\}\right) dw,$$

where

$$\omega_{h,k} = e^{\pi i(s(h,k)-2s(h,3k))}.$$

Now, we wish to approximate the f term by 1. We have that

(5.1) $$\sum{}'' = \sum{}''' + \sum{}''''$$

where

$$\sum{}''' =$$

$$\sum_{3k<N,0<h<3k,(h,3k)=1} \omega_{h,k}\exp\left(-\frac{2\pi i h}{3k}\right)\sqrt{\frac{2\pi}{3k}}\frac{1}{2\pi i}\int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} w^{-1/2}e^{(n-1/24)w}\exp\left(\frac{\pi^2}{54k^2w}\right) dw,$$

and

$$\left|\sum{}''''\right| \leq$$

$$\sum\sqrt{\frac{1}{6\pi k}}\int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} w^{-1/2}e^{nw}\exp\left(\frac{\pi^2}{54k^2w}\right)\left(f\left(\exp\left\{\frac{2\pi i H}{3k} - \frac{4\pi^2}{9k^2w}\right\}\right) - 1\right) dw.$$

(where the sum is over the same set)

6. Bounds on $\sum''''$

Notice that $g(n) \leq p(n)$ (where $p(n)$ is the number of partitions of n). Therefore, $|f(x)-1| \leq F(|x|)-1$. Therefore, $\left|\frac{f(x)-1}{x}\right| \leq \frac{F|x|-1}{|x|}$. By our discussion in section 4, $\Re\left(\frac{4\pi^2}{9k^2w}\right) \leq -\frac{1}{3}$ when w is between $\rho - 2\pi i\theta'_{h,k}$ and $\rho + 2\pi i\theta''_{h,k}$. Therefore, in this range,

$$\left|\left(f\left(\exp\left\{\frac{2\pi iH}{3k} - \frac{4\pi^2}{9k^2w}\right\}\right)-1\right)\exp\left(\frac{4\pi^2}{9k^2w}\right)\right| \leq 43.$$

Therefore,

$$\left|\int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} w^{-1/2}e^{nw}\exp\left(\frac{\pi^2}{54k^2w}\right)\left(f\left(\exp\left[\frac{2\pi iH}{3k}-\frac{4\pi^2}{9k^2w}\right]\right)-1\right)dw\right| \leq$$
$$43\rho^{-1/2}e^{n\rho}\left|\int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}}\exp\left(-\frac{23\pi^2}{54k^2w}\right)dw\right| \leq$$
$$541\rho^{-1/2}e^{n\rho}\frac{1}{kN}.$$

This means that

$$\begin{aligned}\left|\sum''''\right| &\leq \sum_{3k<N,0<h<3k,(h,3k)=1} 125\rho^{-1/2}e^{n\rho}\frac{1}{k^{3/2}N}\\ &\leq \sum_{3k<N} 250\rho^{-1/2}e^{n\rho}\frac{1}{k^{1/2}N}\\ &\leq 145\rho^{-1/2}e^{n\rho}N^{-1/2},\end{aligned}$$

Which by conditions 1 and 2 implies that,

$$\left|\sum''''\right| \leq 170n^{1/4}. \tag{6.1}$$

7. Approximation of $\sum'''$

We will consider

$$\frac{1}{2\pi i}\int_{\rho-2\pi i\theta'_{h,k}}^{\rho+2\pi i\theta''_{h,k}} w^{-1/2}e^{(n-1/24)w}\exp\left(\frac{\pi^2}{54k^2w}\right)dw.$$

This is

$$\frac{1}{2\pi i}\left(\int_{-\infty}^{(0+)} - \int_{-\infty+2\pi i\theta''_{h,k}}^{\rho+2\pi i\theta''_{h,k}} + \int_{-\infty-2\pi i\theta'_{h,k}}^{\rho-2\pi i\theta'_{h,k}}\right) w^{-1/2}e^{(n-1/24)w}\exp\left(\frac{\pi^2}{54k^2w}\right)dw,$$

where $\int_{-\infty}^{(0+)}$ denotes integration over the contour leading from one branch of $-\infty$ around 0, to the other branch (see [3]). Call the last two integrals $\int_1$ and $\int_2$ respectively. Now, if $z = x + 2\pi i\theta$, then

$$\begin{aligned}\Re\left(\frac{\pi^2}{54k^2z}\right) &= \frac{x\pi^2}{54k^2(x^2+4\pi^2\theta^2)}\\ &\leq \frac{x}{216k^2\theta^2}.\end{aligned}$$

Therefore,

$$\Re\left(\frac{\rho}{54k^2w}\right) \le \frac{a}{54} \le .13$$

Also on this line, $|w|^{-1/2} \le \sqrt{2kN}$. Therefore, the sum of the integrals over contours 1 and 2 is at most

$$2e^{.13}\sqrt{2kN}\frac{e^{n\rho}}{n-1/24} < 6.3\sqrt{kN}n^{-1}.$$

Therefore, the total error contributed by the integrals over 1 and 2 is at most

$$\begin{aligned}\sum_{0<3k<N,0<h<3k,(h,3k)=1} 1.5N^{1/2}n^{-1} &\le .25N^{5/2}n^{-1}\\ &\le 20n^{1/4}\end{aligned}$$

This leaves us to approximate

$$\frac{1}{2\pi i}\int_{-\infty}^{(0+)} w^{-1/2}e^{(n-1/24)w}\exp\left(\frac{\pi^2}{54k^2w}\right)dw.$$

This equals

$$\frac{1}{2\pi i}\int_{-\infty}^{(0+)} w^{-1/2}e^{(n-1/24)w}\sum_{s=0}^{\infty}\left(\frac{\pi^2}{54k^2}\right)^s\frac{w^{-s}}{s!}dw =$$

$$\sum_{s=0}^{\infty}\frac{\left(\frac{\pi^2}{54k^2}\right)^s}{s!}\frac{1}{2\pi i}\int_{-\infty}^{(0+)} w^{-s-1/2}e^{(n-1/24)w}dw =$$

$$(n-1/24)^{-1/2}\sum_{s=0}^{\infty}\frac{\left(\frac{\pi^2(n-1/24)}{54k^2}\right)^s}{s!}\frac{1}{2\pi i}\int_{-\infty}^{(0+)} z^{-s-1/2}e^{z}dz.$$

Now, letting $\alpha = \frac{\pi^2(n-1/24)}{54k^2}$ and using Hankel's loop integral formula (see [3]), we get that this equals

$$(n-1/24)^{-1/2}\sum_{s=0}^{\infty}\frac{\alpha^s}{s!\Gamma(s+1/2)}.$$

Using the functional equation for the factorial function, we get that this equals

$$\pi^{-1/2}(n-1/24)^{-1/2}\sum_{s=0}^{\infty}\frac{(4\alpha)^s}{(2s)!} = \pi^{-1/2}(n-1/24)^{-1/2}\cosh(2\sqrt{\alpha}).$$

Setting $x = \frac{\pi}{3k}\sqrt{2(n-1/24)/3}$, this is equal to

$$\sqrt{\frac{2\pi}{3}}\frac{1}{3k}\left(\frac{\cosh(x)}{x}\right)$$

Proof of Theorem 1. We have that

$$\sum''' = \sum_{0<3k<n, 0<h<3k, (h,3k)=1} \omega_{h,k} \exp\left(-\frac{2\pi i h}{3k}\right) k^{-3/2} \frac{2\pi}{9} \left[\frac{\cosh(x)}{x}\right]_{x=\frac{\pi}{3k}\sqrt{2(n-1/24)/3}} =$$

$$\sum_{k^2<\frac{28n}{9}} A_k(n) k^{-3/2} \frac{2\pi}{9} \left[\frac{\cosh(x)}{x}\right]_{x=\frac{\pi}{3k}\sqrt{2(n-1/24)/3}}$$

plus an error of at most $20n^{1/4}$. Therefore, since $g(n) = \sum' + \sum''' + \sum''''$, we have that

$$g(n) = \sum_{k^2<\frac{28n}{9}} A_k(n) k^{-3/2} \frac{2\pi}{9} \left[\frac{\cosh(x)}{x}\right]_{x=\frac{\pi}{3k}\sqrt{2(n-1/24)/3}} + E_n$$

where

$$|E_n| < 200n^{1/4}.$$

(This assumes that $\left[\sqrt{\frac{28n}{9}}\right]^2 > 24n$, though it is easy to verify this for all other n)

□

Proof of Corollary 2. We have that

$$g(n) = a_n \sqrt{\frac{1}{6}} m^{-1/2} e^{\frac{\pi}{3}\sqrt{2m/3}} + F_n$$

where

$$a_n = e^{-\pi i/9} e^{-2n\pi i/3} + e^{\pi i/9} e^{2n\pi i/3} = \begin{cases} 1.87... & \text{if } n \equiv 0(3), \\ -1.53... & \text{if } n \equiv 1(3), \\ -.347... & \text{if } n \equiv 2(3), \end{cases}$$

and

$$m = n - 1/24$$

and

$$\begin{aligned} |F_n| &\leq 200n^{1/4} + \sum_{k^2<\frac{28n}{9}} \sqrt{\frac{2}{3}} k^{-1/2} m^{-1/2} e^{\frac{\pi}{6}\sqrt{2m/3}} \\ &\leq 200n^{1/4} + \sqrt{\frac{1}{6}} \frac{28}{9} m^{1/2} e^{\frac{\pi}{6}\sqrt{2m/3}} \\ &\leq 200n^{1/4} + 2m^{1/2} e^{\frac{\pi}{6}\sqrt{2m/3}}. \end{aligned}$$

Now the absolute value of the main term is at least

$$.1m^{-1/2} e^{\frac{\pi}{3}\sqrt{2m/3}}.$$

Therefore, the absolute value of the main term is bigger than the absolute value of F_n if $n > 500$. Therefore, $g(n)$ has the correct sign for $n > 500$, and it is easy to verify the conjecture for all smaller n.

□

Examples. Here we illustrate Theorem 1 for $n = 300, 301$ and 302. It turns out that

$$g(300) = 119587,$$

and that first few partial sums in the approximation are to the nearest integer

$$\begin{gathered} 119590 \\ 119580 \\ 119586 \end{gathered}$$

and the error of the approximation is

$$-.158...$$

$$g(301) = -99807.$$

The first few partial sums in the approximation are to the nearest .1

$$\begin{gathered} -99785.9 \\ -99810.9 \\ -99807.0 \end{gathered}$$

and the error of the approximation is

$$.312...$$

$$g(302) = -23174.$$

The first few partial sums in the approximation are to the nearest .1

$$\begin{gathered} -23138.8 \\ -23181.7 \\ -23175.2 \end{gathered}$$

and the error of the approximation is

$$-.079...$$

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