ON THE RELATION BETWEEN OPTION AND STOCK PRICES: A CONVEX OPTIMIZATION APPROACH

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The idea of investigating the relation of option and stock prices based just on the no-arbitrage assumption, but without assuming any model for the underlying price dynamics, has a long history in the financial economics literature. We introduce convex and, in particular semidefinite optimization methods, duality, and complexity theory to shed new light on this relation. For the single stock problem, given moments of the prices of the underlying assets, we show that we can find best-possible bounds on option prices with general payoff functions efficiently, either algorithmically (solving a semidefinite optimization problem) or in closed form. Conversely, given observable option prices, we provide best-possible bounds on moments of the prices of the underlying assets, as well as on the prices of other options on the same asset by solving linear optimization problems. For options that are affected by multiple stocks either directly (the payoff of the option depends on multiple stocks) or indirectly (we have information on correlations between stock prices), we find nonoptimal bounds using convex optimization methods. However, we show that it is NP-hard to find best possible bounds in multiple dimensions. We extend our results to incorporate transactions costs.

1. INTRODUCTION

A central question in financial economics is to find the price of a derivative security, given information on the underlying asset. Under the assumption that the price of the underlying asset follows a geometric Brownian motion, and using the no-arbitrage assumption, the Black-Scholes formula provides an explicit and insightful answer to this question. Natural questions arise, however, when making no assumptions on the underlying price dynamics, but using only the no-arbitrage assumption:

(1) What are the best possible bounds for the price of a derivative security with a general payoff function based on the k moments of the price of the underlying asset?

(2) Conversely, given observable option prices, what are the best-possible bounds that we can derive on the moments of the underlying asset?

(3) Given observable option prices, what are the best-possible bounds that we can derive on prices of other derivatives on the same asset?

(4) How can we derive best-possible bounds on derivative securities that are based on multiple underlying assets, either directly (the payoff of the option depends on multiple stocks) or indirectly (we have information on correlations between stock prices), when partial information on the asset prices and their correlations is given?

(5) What is the effect of transaction costs in the above instances?

The idea of investigating the relation of option and stock prices based only on the no-arbitrage assumption, without assuming any model for the underlying price dynamics, has a long history in the financial economics literature. Cox and Ross (1976) and Harrison and Kreps (1979) show that the no-arbitrage assumption is equivalent to the existence of a probability distribution π (the so-called Martingale measure) such that option prices become Martingales under π. The idea that it is possible in principle to infer the Martingale measure from option prices has been introduced by Ross (1976). The idea of using optimization to infer the Martingale measure based on option prices is present in the work of Rubinstein (1994), who, extending earlier work of Longstaff (1990), introduces the idea of deducing the Martingale measure from observed European call prices by solving a quadratic optimization problem that measures the closeness of the Martingale measure to the lognormal distribution. For related work, see Dupire (1994) and Derman and Kani (1994). Closer to the theme of this paper are the papers by Lo (1987), who derives best-possible closed-form bounds on the price of a European call option, given the mean and variance of the underlying stock price; by Grundy (1991), who extended Lo's work for the case when the first and the kth moments of the stock price are known; and by Boyle and Lin (1997), who use semidefinite optimization to find an upper bound on the price of a European call option on the maximum of a number of assets, given the means, variances, and covariances of these assets.

Our overall objective in this paper is to shed new light on the relation of option and stock prices, and to demonstrate that the natural way to address this relation, without

making distributional assumptions for the underlying price dynamics, but using only the no-arbitrage assumption, is the use of convex optimization methods. In particular, we give concrete answers to the previous questions (1)–(5) using convex, and in particular semidefinite, optimization techniques, duality, and complexity theory.

To motivate our overall approach we formulate the problem of deriving optimal bounds on the price of a European call option, given the mean and variance of the underlying stock price. Following Cox and Ross (1976) and Harrison and Kreps (1979), the no-arbitrage assumption is equivalent to the existence of a probability distribution π (the so-called Martingale measure) of the asset price X, such that the price of any European call option with strike price k is given by

$$q(k) = E_\pi[\max(0, X - k)],$$

where the expectation is taken over the unknown distribution π. Note that we have assumed, without loss of generality, that the risk-free interest rate is zero. Moreover, given that the mean and variance of the underlying asset are observable,

$$E_\pi[X] = \mu, \quad \text{and} \quad Var_\pi[X] = \sigma^2,$$

the problem of finding the best-possible upper bound on the call price, written as

$$\max_{X \sim (\mu, \sigma^2)^+} E_\pi[\max(0, X - k)],$$

(where the + operation means that X is defined on $[0, \infty)$) can be formulated as follows:

maximize $E_\pi[\max(0, X - k)],$

subject to $E_\pi[X] = \mu$

$\quad Var_\pi[X] = \sigma^2.$

$\quad \int_0^\infty \pi(x) \, dx = 1$

$\quad \pi(x) \geqslant 0.$

The closed-form solution for this optimization problem is due to Scarf (1958), in the context of an inventory-control problem. Lo (1987) observed the direct application of Scarf's result to option pricing. Grundy (1991) introduced as open problems several of the problems that we solve here: using known option prices, find sharp upper and lower bounds on the moments of the stock price, and on the price of an option with a different strike price. These problems can be formulated as follows:

max / min $E_\pi[X]$, or $E_\pi[X^2]$, or $E_\pi[\max(0, X - k)],$

subject to $E_\pi[\max(0, X - k_i)] = q_i, \quad i = 1, \ldots, n.$

$\quad \int_0^\infty \pi(x) \, dx = 1$

$\quad \pi(x) \geqslant 0.$

For a multidimensional example, suppose we have observed the price q_1 of a European call option with strike k_1 for Stock 1, and the price q_2 of a European call option with strike k_2 for Stock 2. In addition, we have estimated the means μ_1, μ_2, the variances σ_1^2, σ_2^2, and the covariance σ_{12} of the prices of the two underlying stocks. Suppose, in addition, that we are interested in obtaining an upper bound on the price of a European call option with strike k for Stock 1. Intuition suggests that because the prices of the two stocks are correlated, the price of a call option on Stock 1 with strike k might be affected by the available information regarding Stock 2. We can find an upper bound on the price of a call option on Stock 1 with strike k by solving the following problem:

maximize $\quad E_\pi[\max(0, X_1 - k)],$

subject to $\quad E_\pi[\max(0, X_1 - k_1)] = q_1,$

$\quad\quad E_\pi[\max(0, X_2 - k_2)] = q_2,$

$\quad\quad E_\pi[X_1] = \mu_1,$

$\quad\quad E_\pi[X_2] = \mu_2,$

$\quad\quad E_\pi[X_1^2] = \sigma_1^2 + \mu_1^2,$

$\quad\quad E_\pi[X_2^2] = \sigma_2^2 + \mu_2^2,$ \hfill (1)

$\quad\quad E_\pi[X_1 X_2] = \sigma_{12} + \mu_1 \mu_2,$

$\quad\quad \int_0^\infty \int_0^\infty \pi(x_1, x_2) \, dx_1 \, dx_2 = 1,$

$\quad\quad \pi(x_1, x_2) \geqslant 0.$

More generally, Questions (1)–(4) above are special cases of the following general optimization problem:

max / min $\quad E_\pi[\phi(X)],$

subject to $\quad E_\pi[f_i(X)] = q_i, \quad i = 0, 1, \ldots, n.$ \hfill (2)

$\quad\quad \pi(x) \geqslant 0, \quad x \in R_+^m,$

where $X = (X_1, \ldots, X_m)$ is a multivariate random variable, and $\phi : R_+^m \to R$ is a real-valued objective function, $f_i : R_+^m \to R$, $i = 1, \ldots, n$ are also real-valued, so-called *moment functions* whose expectations $q_i \in R$, referred to as *moments*, are known and finite. We assume that $f_0(x) = 1$ and $q_0 = E_\pi[f_0(X)] = 1$, corresponding to the implied probability-mass constraint. Questions (1)–(4) introduced earlier can be formulated as follows:

(1) Question (1) for European call options can be formulated as Problem (2) with

$$\phi(x) = \max(0, x - k), \quad f_i(x) = x^i, \quad i = 1, \ldots, k,$$

where q_i is the ith moment of the price of the underlying asset.

(2) Question (2) for European call options can be formulated as Problem (2) with

$$\phi(x) = x, \quad \text{or} \quad \phi(x) = x^2,$$

and

$$f_i(x) = \max(0, x - k_i), \quad i = 1, \ldots, n.$$

(3) Question (3) for European call options can be formulated as Problem (2) with

$$\phi(x) = \max(0, x - k),$$

and

$$f_i(x) = \max(0, x - k_i), \quad i = 1, \ldots, n.$$

(4) Question (4) for a general option with payoff $\phi(x_1, \ldots, x_m)$ that is based on m underlying assets can be formulated as Problem (2) with

$$f_i(x) = x_i, \quad i = 1, \ldots, m, \quad f_{ij}(x) = x_i x_j, \quad i, j = 1, \ldots m,$$

$$q_i = \mu_i, \quad q_{ij} = \sigma_{ij}^2 + \mu_i \mu_j.$$

When $\phi(x) = \chi_S$ in Problem (2) is the indicator function of a convex set S, and f_i are power functions, then Problem (2) models the problem of finding the best-possible bounds on the probability that a multidimensional random variable X belongs in the convex set S, given some joint moments on X. In this context, Problem (2) received a lot of attention in the 1950s and 1960s. The major duality results from this period are due to Isii (1960) and Karlin (see Karlin and Studden 1966, p. 472) for the univariate case, and by Isii (1963) for the multivariate case. The interested reader is referred to the book of Karlin and Studden (1966) for a comprehensive coverage, to Bertsimas and Popescu (1999) for a modern treatment, and to Smith (1995) for applications in decision analysis.

The contributions and structure of this paper are as follows:

1. We provide in §2 an efficient (polynomial time) algorithm for Question (1) for a general payoff function $\phi(x)$ by solving a single semidefinite optimization problem, thus generalizing earlier work of Lo (1987), Grundy (1991), and Boyle and Lin (1997). This result leads to an unexpected connection between finance and semidefinite optimization.

2. We derive in §3 closed-form optimal bounds on call and put prices, given prices of other calls and puts on the same stock, thus answering Question (2).

3. We derive in §4 best-possible bounds on the mean and variance of the underlying stock price when prices of options on this stock are given, thus answering Question (3).

4. In §5 we extend the previous results by taking into account transaction costs, thus answering Question (3).

5. We present in §6 an efficient (polynomial time) algorithm to provide bounds (although not best-possible ones) for options that are affected by multiple stocks, using convex, and in particular semidefinite, optimization methods, thus answering Question (4). We also show that it is NP-hard to find optimal bounds in multiple dimensions.

2. BOUNDS ON OPTION PRICES GIVEN MOMENT INFORMATION

We are given the n first moments $(q_1, q_2, \ldots, q_n)$, (we let $q_0 = 1$) of the price of an asset, and we are interested in finding the best-possible bounds on the price of an option with payoff $\phi(x)$. An example is a European call option with payoff $\phi(x) = \max(0, x - k)$. In §2.1 we propose an efficient algorithmic solution for general payoff functions, while in §2.2 we provide a new proof based on duality of the closed-form upper bound of the price of a European call option derived by Lo (1987).

2.1. Bounds Based on Semidefinite Optimization

As we discussed in the previous section, the problem of finding the best upper bound on the price of a European call option with strike k can be formulated as follows:

$$\text{maximize} \quad E_\pi[\max(0, X - k)]$$
$$= \int_0^\infty \max(0, x - k)\pi(x)\,dx,$$
$$\text{subject to} \quad E_\pi[X^i] = \int_0^\infty x^i \pi(x)\,dx = q_i \quad (3)$$
$$i = 0, 1, \ldots, n,$$
$$\pi(x) \geqslant 0.$$

In the spirit of linear programming theory (see Smith 1995 and Bertsimas and Popescu 1999), we write the dual of Problem (3) by associating a vector of dual variables $y = (y_0, y_1, \ldots, y_n)$ to each of the constraints in Problem (3). We obtain the following problem:

$$\text{minimize} \quad \sum_{r=0}^n y_i q_i,$$
$$\qquad\qquad\qquad\qquad\qquad\qquad (4)$$
$$\text{subject to} \quad \sum_{r=0}^n y_r x^r \geqslant \max(0, x - k), \quad \forall x \in R_+.$$

Isii (1960) shows that strong duality holds, i.e., the optimal solution values of Problems (3) and (4) are equal. Thus, by solving Problem (4) we obtain the desired sharp bound. In this section, we show that the general problem (3) can be reformulated as a semidefinite optimization problem for which very efficient algorithmic solutions are known, both from a theoretical (see Nesterov and Nemirovski 1994 and Vandenberghe and Boyd 1996) and practical standpoint (see Fujisawa et al. 1998). The results in the following proposition are inspired by Ben-Tal and Nemirovski (1998, p. 140–142). The proofs are in the Appendix.

PROPOSITION 1. (a) The polynomial $g(x) = \sum_{r=0}^{2k} y_r x^r$ satisfies $g(x) \geqslant 0$ if and only if there exists a positive semidefinite matrix $X = [x_{ij}]_{i,j=0,\ldots,k}$, such that

$$y_r = \sum_{i,j:\, i+j=r} x_{ij}, \quad r = 0, \ldots, 2k, \quad X \succeq 0. \quad (5)$$

(b) The polynomial $g(x) = \sum_{r=0}^n y_r x^r$ satisfies $g(x) \geqslant 0$ for all $x \in [0, a]$ if and only if there exists a positive

semidefinite matrix $X = [x_{ij}]_{i,j=0,...,n}$, *such that*

$$0 = \sum_{i,j:\ i+j=2l-1} x_{ij}, \qquad l = 1, \ldots, n,$$

$$\sum_{r=0}^{l} y_r \binom{k-r}{l-r} a^r = \sum_{i,j:\ i+j=2l} x_{ij}, \qquad l = 0, \ldots, n, \qquad (6)$$

$$X \succeq 0.$$

(c) *The polynomial* $g(x) = \sum_{r=0}^{n} y_r x^r$ *satisfies* $g(x) \geqslant 0$ *for all* $x \in [a, \infty)$ *if and only if there exists a positive semidefinite matrix* $X = [x_{ij}]_{i,j=0,...,n}$, *such that*

$$0 = \sum_{i,j:\ i+j=2l-1} x_{ij}, \qquad l = 1, \ldots, n,$$

$$\sum_{r=l}^{k} y_r \binom{r}{l} a^r = \sum_{i,j:\ i+j=2l} x_{ij}, \qquad l = 0, \ldots, n, \qquad (7)$$

$$X \succeq 0.$$

(d) *The polynomial* $g(x) = \sum_{r=0}^{k} y_r x^r$ *satisfies* $g(x) \geqslant 0$ *for all* $x \in [a, b]$ *if and only if there exists a positive semidefinite matrix* $X = [x_{ij}]_{i,j=0,...,n}$, *such that*

$$0 = \sum_{i,j:\ i+j=2l-1} x_{ij}, \qquad l = 1, \ldots, n,$$

$$\sum_{m=0}^{l} \sum_{r=m}^{k+m-l} y_r \binom{r}{m} \binom{k-r}{l-m} a^{r-m} b^m$$

$$= \sum_{i,j:\ i+j=2l} x_{ij}, \qquad l = 0, \ldots, n, \qquad (8)$$

$$X \succeq 0.$$

Theorem 1 shows that Problem (3) can be solved as a semidefinite optimization problem.

THEOREM 1. *The best upper bound on the price of a European call option with strike* k, *given the* n *first moments* $(q_1, \ldots, q_n)(q_0 = 1)$ *of the underlying stock, is given by the solution of the following semidefinite optimization problem:*

minimize $\sum_{r=0}^{n} y_i q_i$

subject to $0 = \sum_{i,j:\ i+j=2l-1} x_{ij}, \quad l = 1, \ldots, n,$

$$\sum_{r=0}^{l} y_r \binom{k-r}{l-r} k^r = \sum_{i,j:\ i+j=2l} x_{ij}, \quad l = 0, \ldots, n,$$

$$0 = \sum_{i,j:\ i+j=2l-1} z_{ij}, \quad l = 1, \ldots, n,$$

$$(y_0 + k) + (y_1 - 1)k + \sum_{r=2}^{k} y_r k^r = z_{00}, \qquad (9)$$

$$(y_1 - 1)k + \sum_{r=2}^{k} y_r r k^r = \sum_{i,j:\ i+j=2} z_{ij},$$

$$\sum_{r=l}^{k} y_r \binom{r}{l} k^r = \sum_{i,j:\ i+j=2l} z_{ij}, \quad l = 2, \ldots, n,$$

$$X, Z \succeq 0.$$

PROOF. We note that the feasible region of Problem (4) can be written as

$$\sum_{r=0}^{n} y_r x^r \geqslant 0 \quad \text{for all } x \in [0, k],$$

$$(y_0 + k) + (y_1 - 1)x + \sum_{r=2}^{n} y_r x^r \geqslant 0 \quad \text{for all } x \in [k, \infty).$$

By applying Proposition 1 (b) and (c) we reformulate Problem (4) as the semidefinite optimization Problem (9). □

We next consider an option with payoff function given as follows:

$$\phi(x) = \begin{cases} \phi_0(x), & x \in [0, k_1], \\ \phi_1(x), & x \in [k_1, k_2], \\ \quad \vdots & \quad \vdots \\ \phi_{d-1}(x), & x \in [k_{d-1}, k_d], \\ \phi_d(x), & x \in [k_d, \infty), \end{cases} \qquad (10)$$

where the functions $\phi_r(x)$, $r = 0, 1, \ldots, d$ are polynomials. Given the generality of the payoff function (10), we can approximate the payoff of any option using the payoff function (10). In this case the dual problem becomes:

minimize $\sum_{r=0}^{n} y_i q_i$

subject to $\sum_{r=0}^{n} y_r x^r \geqslant \begin{cases} \phi_0(x), & x \in [0, k_1], \\ \phi_1(x), & x \in [k_1, k_2], \\ \quad \vdots & \quad \vdots \\ \phi_{d-1}(x), & x \in [k_{d-1}, k_d], \\ \phi_d(x), & x \in [k_d, \infty), \end{cases}$ (11)

The next theorem shows that the problem of finding best-possible bounds on an option with a general piecewise polynomial payoff function is efficiently solvable both practically and theoretically as a semidefinite optimization problem.

THEOREM 2. *The best-possible bounds for the price of an option with a piecewise polynomial payoff function* $\phi(x)$ *shown in* (10), *given moments of the underlying asset, can be solved efficiently as a semidefinite optimization problem.*

PROOF. The constraint set for Problem (11) can be written as follows:

$$\sum_{r=0}^{n} y_r x^r \geqslant \phi_i(x), \quad x \in [k_{i-1}, k_i], \qquad i = 1, \ldots, d+1,$$

with $k_0 = 0$, $k_{d+1} = \infty$. Let $\phi_i(x) = \sum_{r=0,...,m_i} a_{ir} x^r$, and assume without loss of generality that $m_i \leqslant n$. Then, the

constraint set for Problem (11) can be equivalently written as

$$\sum_{r=0}^{m_i}(y_r-a_{ir})x^r + \sum_{r=m_i+1}^{n} y_r x^r \geqslant 0, \quad x \in [k_{i-1},k_i],$$

$$i = 1,\ldots,d+1.$$

For the interval $[k_0,k_1]$ we apply Proposition 1(b); for the intervals $[k_{i-1},k_i]$, $i = 2,\ldots,d$, we apply Proposition 1(d); and for the interval $[k_d,\infty)$, we apply Proposition 1(c), to express Problem (11) as a semidefinite optimization problem. □

2.2. Closed-Form Bounds

In this section, we provide a new proof from first principles of a closed-form optimal bound of the price of a European call option with strike k.

THEOREM 3 (OPTIMAL UPPER BOUND ON OPTION PRICES, LO 1987). *The optimal upper bound on the price of an option with strike k, on a stock whose price at maturity has a known mean μ and variance σ^2, is computed by*

$$\max_{X \sim (\mu,\sigma^2)^+} E[\max(0, X-k)]$$

$$= \begin{cases} \dfrac{1}{2}\left[(\mu-k)+\sqrt{\sigma^2+(\mu-k)^2}\right], & \text{if } k \geqslant \dfrac{\mu^2+\sigma^2}{2\mu}, \\[2mm] \mu-k+k\dfrac{\sigma^2}{\mu^2+\sigma^2}, & \text{if } k < \dfrac{\mu^2+\sigma^2}{2\mu}. \end{cases}$$

PROOF. The optimal upper bound on the price of a European call option with strike k is given as the solution of Problem (4), which in this case is formulated by associating dual variables y_0, y_1, and y_2 with the probability-mass, mean, and variance constraints, respectively. We obtain the following dual formulation:

minimize $(\mu^2+\sigma^2)y_2+\mu y_1+y_0,$

subject to $g(x)=y_2 x^2+y_1 x+y_0 \geqslant \max(0,x-k), \quad \forall x \geqslant 0.$

A dual feasible function $g(\cdot)$ is any quadratic function that, on the positive orthant, is nonnegative and lies above the line $(x-k)$. In an optimal solution, such a quadratic should be tangent to the line $(x-k)$, so we can write $g(x)-(x-k)=a(x-b)^2$, for some $a \geqslant 0$. The nonnegativity constraint on $g(\cdot)$ can be expressed as $a(x-b)^2+x-k \geqslant 0$, $\forall x \geqslant 0$. Let $x_0 = b-\frac{1}{2a}$ be the point of minimum of this quadratic. Depending on whether or not x_0 is nonnegative, either the inequality at $x=x_0$ or at $x=0$ is binding in an optimal solution. We have two cases:

(a) If $b \geqslant \frac{1}{2a}$, then $-\frac{1}{4a}+b-k=0$ (binding constraint at $x=x_0$);

Substituting $a = \frac{1}{4(b-k)}$ in the objective, we obtain

$$\max_{X \sim (\mu,\sigma^2)^+} E[\max(0, X-k)]$$

$$= \min_b \frac{((\mu-k)+(b-k))^2+\sigma^2}{4(b-k)}$$

$$= \frac{1}{2}\left[(\mu-k)+\sqrt{\sigma^2+(\mu-k)^2}\right],$$

achieved at $b_0 = \frac{\mu^2+\sigma^2}{\mu}$. Let $a_0 = \frac{1}{4(b_0-k)}$.

This bound is valid whenever $b_0 \geqslant \frac{1}{2a_0} = 2(b_0-k)$, that is, $\frac{\mu^2+\sigma^2}{2\mu} \leqslant k$.

(b) If $b < \frac{1}{2a}$, then $ab^2-k=0$ (binding constraint at $x=0$).

Substituting $a = \frac{k}{b^2}$ in the objective, we obtain

$$\max_{X \sim (\mu,\sigma^2)^+} E[\max(0, X-k)] = \min_b \frac{k}{b^2}(\mu^2+\sigma^2) - 2\frac{k}{b}\mu+\mu$$

$$= \mu-k\frac{\mu^2}{\mu^2+\sigma^2}.$$

achieved at $b_0 = \frac{\mu^2+\sigma^2}{\mu}$. Let $a_0 = \frac{k}{b_0^2}$. This bound is valid whenever $b_0 < \frac{1}{2a_0} = \frac{b_0^2}{2k}$, that is, $\frac{\mu^2+\sigma^2}{2\mu} > k$. □

3. BOUNDS ON OPTION PRICES GIVEN OTHER OPTION PRICES

In this section, we derive closed-form optimal upper and lower bounds on the price of a European call option on a single stock, when prices of other options with the same exercise date but different strikes on the same stock are known. For simplicity and without loss of generality, we assume that the risk-free interest rate is zero. In this section, we ignore transaction costs, the effect of which will be discussed in §5.

3.1. Bounds on Call Prices

Let X be the random variable that represents the price of the underlying stock. We are given prices $q(k_i)=q_i=E[\max(0, X-k_i)]$ of call options on the same stock with strikes $0 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$ and the same exercise date, and we want to compute optimal upper and lower bounds on $q(k)=E[\max(0, X-k)]$, for a given strike price k.

For notation purposes we set $k_0=0$ and $q_0=q(0)=E[\max(0, X-0)]=E[X]$. In some cases it may also be useful to assume an upper bound K on the price X of the stock at the time of maturity of the calls. This information can be easily integrated in this framework by defining $k_{n+1}=K$ and $q_{n+1}=q(K)=E[\max(0, X-K)]=0$. If no such upper bound is assumed, then we assume $k_{n+1}=\infty$. We say that a given function $q(\cdot)$ is a valid call-pricing function if there exists a distribution of the stock price X such that $q(k)=E[\max(0, X-k)]$, $\forall k \geqslant 0$.

THEOREM 4 (OPTIMAL BOUNDS ON CALL PRICES). *Given valid prices $q_i = q(k_i) = E[\max(0, X - k_i)]$ of call options with strikes $0 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$, on a stock X, the range of possible valid prices for a call option with strike price k, where $k \in (k_j, k_{j+1})$, for some $j = 0, \ldots, n$ is $[q^-(k), q^+(k)]$, where:*

$$q^-(k) = \max\left(q_j \frac{k - k_{j-1}}{k_j - k_{j-1}} + q_{j-1} \frac{k_j - k}{k_j - k_{j-1}}, \right.$$
$$\left. q_{j+1} \frac{k_{j+2} - k}{k_{j+2} - k_{j+1}} + q_{j+2} \frac{k - k_{j+1}}{k_{j+2} - k_{j+1}} \right),$$
$$q^+(k) = q_j \frac{k_{j+1} - k}{k_{j+1} - k_j} + q_{j+1} \frac{k - k_j}{k_{j+1} - k_j}. \tag{12}$$

To obtain some intuition on the nature of these bounds, we note that for a given function $q(\cdot)$ to be a valid call-pricing function, we need the existence of a nonnegative random variable X such that $q(k) = E[\max(0, X - k)]$, $\forall k \geqslant 0$. Clearly, $q(\cdot)$ is decreasing and convex. What Theorem 4 proves is that the necessary and sufficient conditions for $q(\cdot)$ to define a valid call-pricing function is for it to be decreasing and convex. In particular, the values of $q^-(k)$ and $q^+(k)$ given above are precisely determined by the monotonicity and convexity of the call-pricing function $q(\cdot)$. Figure 1 depicts the construction of the bounds $q^-(k)$ and $q^+(k)$ geometrically in a concrete example. Moreover, the range of prices of a call option with strike price $k \in (k_j, k_{j+1})$ is constrained only by the prices q_{j-1}, q_j of the two options with the closest strikes to the left of k and to the right of k, q_{j+1} and q_{j+2}.

The bounds (12) are relevant only when the given options are correctly priced. Interestingly, this is not always the case, as one can see in the actual examples given in Figure 2, where some of the call-pricing functions are

Figure 1. The optimal upper and lower bounds on the price of a call option, given prices of calls on the same stock, with different strikes and the same maturity date.
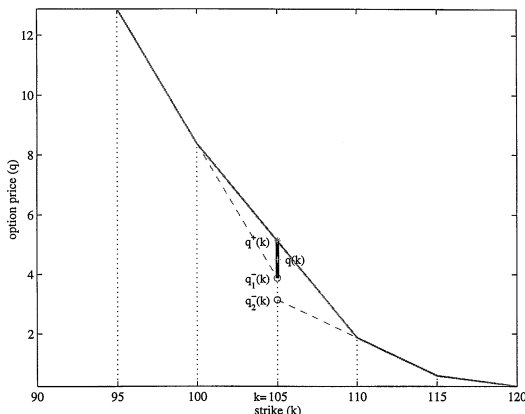
Notes. (Actual data quoted from *The Wall Street Journal*, July 7, 1998: Microsoft July '98 call options with: $k_i = [95, 100, 110, 115, 120]$, $q_i = [12\frac{7}{8}, 8\frac{3}{8}, 1\frac{7}{8}, \frac{5}{8}, \frac{1}{4}]$; $k = 105$, $q(k) = 4\frac{1}{2}$). Note that the bounds are derived by the convexity and monotonicity of the the price function $q(\cdot)$.

clearly nonconvex, so the upper bounds computed by Theorem 4 may turn out smaller than the respective lower bounds (see the explanation in the notes of Figure 2).

PROOF OF THEOREM 4: THE LOWER BOUND PROBLEM. We first consider the lower bound problem and formulate it as a continuous optimization problem over all feasible stock price densities $\pi(x)$, as follows:

$$q^-(k) = \text{minimize} \int_k^\infty (x - k)\pi(x)\, dx,$$

$$\text{subject to} \int_{k_i}^\infty (x - k_i)\pi(x)\, dx = q_i,$$

$$i = 1, \ldots, n, \tag{13}$$

$$\int_0^\infty \pi(x)\, dx = 1,$$

$$\pi(x) \geqslant 0, \quad \forall x \in R_+.$$

If we restrict our horizon to stock price distributions $p_x = P(X = x)$ over a *discrete* range of values $S \subseteq R_+$ that include the strike prices $k_i \in S$, $i = 1, \ldots, n$, we can formulate the restricted problem as

$$q_R^-(k) = \text{minimize} \sum_{x \geqslant k} (x - k)p_x,$$

$$\text{subject to} \sum_{x \geqslant k_i} (x - k_i)p_x = q_i, \quad i = 1, \ldots, n,$$

$$\sum_{x \geqslant 0} p_x = 1, \tag{14}$$

$$p_x \geqslant 0, \quad \forall x \in S.$$

Clearly, $q_R^-(k) \geqslant q^-(k)$, because the minimization in Problem (14) is over a restricted set of distributions. We will show that $q_R^-(k) = q^-(k)$. We construct the corresponding dual problems by associating a dual variable u_i, $i = 1, \ldots, n$ with each of the first n constraints, and a dual variable v for the probability mass constraint. The dual of Problem (13) is

$$q_D^-(k) = \text{maximize} \quad v + \sum_{i=1}^n q_i u_i$$

$$\text{subject to} \quad g(x) = v + \sum_{i \mid k_i \leqslant x} (x - k_i)u_i \tag{15}$$

$$\leqslant \begin{cases} 0, & 0 \leqslant x < k, \\ x - k, & x \geqslant k. \end{cases}$$

The dual problem of the restricted Problem (14) is the same as (15), except the constraints need hold only on the discrete set of points $x \in S$, where X ranges. We denote its optimal solution values as $q_{RD}^-(k)$. Notice that for both problems, the dual feasible function $g(x)$ is piecewise linear, in which the slope changes at the points k_i,

Figure 2. The optimal upper and lower bounds for each call price, determined by the prices of the neighboring calls.
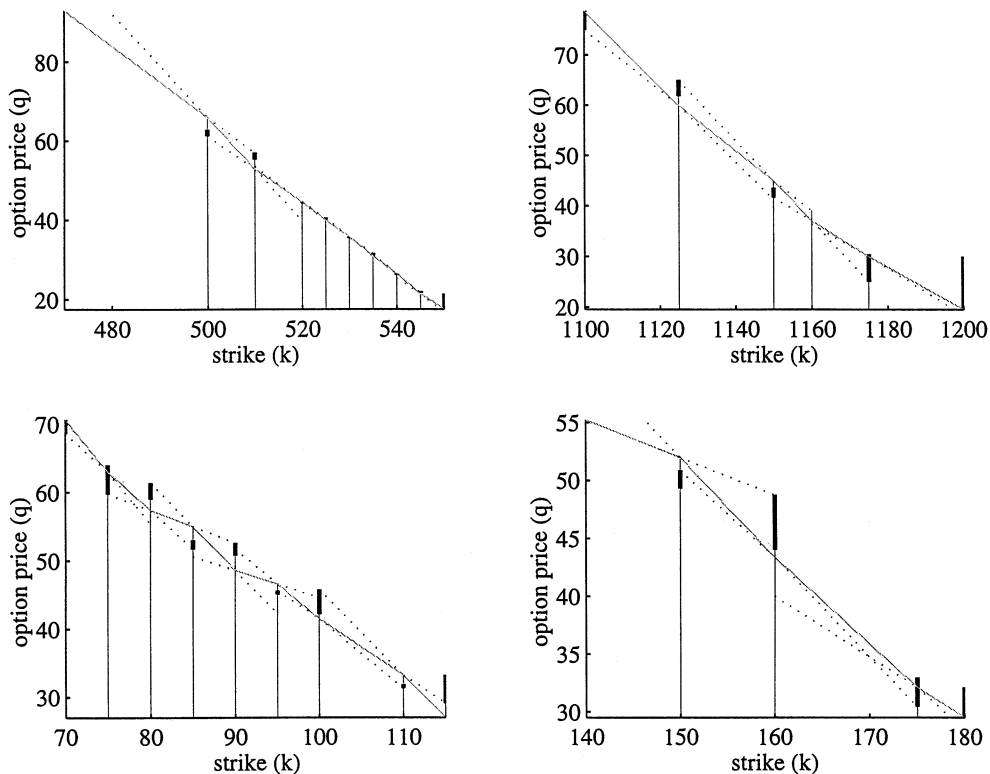
Notes. Clockwise: *S&P*100 July '98, *S&P*500 Sep '98, Yahoo Aug '98, Amazon July '98. Call prices from *The Wall Street Journal*, July 7, 1998. This apparent mispricing can be explained by noting that these are closing prices, so these prices might not all be present simultaneously. Moreover, transaction costs are ignored.

$i = 1, \ldots, n$, and therefore it is sufficient to solve each problem with constraints only at the points k_i. Thus, the two dual problems are equivalent to

$$q_{RD}^-(k) = \text{maximize} \qquad v + \sum_{i=1}^{n} q_i u_i,$$

subject to

$$
\begin{aligned}
g(k_1) &= v &&\leqslant 0 \\
g(k_2) &= v + (k_2 - k_1)u_1 &&\leqslant 0 \\
&\;\;\vdots \;\; \vdots && \vdots \\
g(k_j) &= v + (k_j - k_1)u_1 + \cdots + (k_j - k_{j-1})u_{j-1} &&\leqslant 0 \\
g(k) &= v + (k - k_1)u_1 + \cdots + (k - k_j)u_j &&\leqslant 0 \\
g(k_{j+1}) &= v + (k_{j+1} - k_1)u_1 + \cdots + (k_{j+1} - k_j)u_j &&\leqslant k_{j+1} - k \\
&\;\;\vdots \;\;\;\;\;\; \vdots \;\; \vdots && \vdots \\
g(k_n) &= v + (k_n - k_1)u_1 + \cdots + (k_n - k_{n-1})u_{n-1} &&\leqslant k_n - k \\
& \quad u_1 + u_2 + \cdots + u_n &&\leqslant 1,
\end{aligned}
$$

(16)

where the last constraint is meant to capture the limiting situation as $x \to \infty$. We have $q_D^-(k) = q_{RD}^-(k)$, and weak duality holds for both primal-dual pairs, which means: $q_R^-(k) \geqslant q^-(k) \geqslant q_D^-(k)$. Moreover, strong duality holds for the discretized version (14), because these are linear opti-

mization problems, and therefore, $q_R^-(k) = q_{RD}^-(k) = q_D^-(k)$. This shows that

$$q^-(k) = q_R^-(k) = q_D^-(k). \tag{17}$$

Moreover, there exists a discrete stock-price distribution that achieves the bound $q^-(k)$.

We next proceed to solve Problem (16). This is a linear optimization problem with $n+2$ constraints and $n+1$ variables whose optimum, if it exists, is achieved at a basic feasible solution. In an optimal basic feasible solution, $n+1$ of the constraints must be binding, including the one at k, that is, the constraint $g(k) \leqslant 0$. In this case, the constraints $g(k_j) \leqslant 0$ and $g(k_{j+1}) \leqslant k_{j+1} - k$ cannot be simultaneously binding. We have two cases:

Case 1. Constraint $g(k_j) \leqslant 0$ is not binding. In this case we obtain the following optimal solution:

$$
g(x) = \begin{cases}
0, & x \leqslant k_{j-1}, \\
\dfrac{k_j - k}{k_j - k_{j-1}}(x - k_{j-1}), & x \in (k_{j-1}, k_j), \\
x - k, & x \geqslant k_j;
\end{cases}
$$

that is, the corresponding dual variables are

$$u_{j-1} = \frac{k_j - k}{k_j - k_{j-1}},$$

$$u_j = \frac{k - k_{j-1}}{k_j - k_{j-1}},$$

$$u_i = 0, \quad \forall i \neq j-1, j,$$

$$v = 0.$$

The corresponding dual-optimal objective value in this case is

$$q_1^-(k) = q_j \frac{k - k_{j-1}}{k_j - k_{j-1}} + q_{j-1} \frac{k_j - k}{k_j - k_{j-1}}.$$

Case 2. Constraint $g(k_{j+1}) \leqslant k_{j+1} - k$ is not binding. In this case we obtain the following optimal solution:

$$g(x) = \begin{cases} 0, & x \leqslant k_{j+1}, \\ \dfrac{k_{j+2} - k}{k_{j+2} - k_{j+1}}(x - k_{j+1}), & x \in (k_{j+1}, k_{j+2}), \\ x - k, & x \geqslant k_{j+2}. \end{cases}$$

The corresponding dual variables are

$$u_{j+1} = \frac{k_{j+2} - k}{k_{j+2} - k_{j+1}},$$

$$u_{j+2} = \frac{k - k_{j+1}}{k_{j+2} - k_{j+1}},$$

$$u_i = 0, \quad \forall i \neq j+1, j+2,$$

$$v = 0.$$

The corresponding dual-optimal objective value in this case is

$$q_2^-(k) = q_{j+1} \frac{k_{j+2} - k}{k_{j+2} - k_{j+1}} + q_{j+2} \frac{k - k_{j+1}}{k_{j+2} - k_{j+1}}.$$

The desired optimal lower bound is given by: $q^-(k) = \max(q_1^-(k), q_2^-(k))$, which leads to the lower bound expression in Equation (12). Note that an extremal distribution of the stock price X that achieves this bound is given by the corresponding optimal solution of the discretized primal problem.

The Upper Bound Problem. Using the same procedure, we formulate the optimal upper bound problem as a continuous optimization problem over all feasible stock price densities $\pi(x)$:

$$q^+(k) = \text{maximize} \quad \int_k^\infty (x - k)\pi(x)\,dx,$$

$$\text{subject to} \quad \int_{k_i}^\infty (x - k_i)\pi(x)\,dx = q_i,$$

$$i = 1, \dots, n, \quad (18)$$

$$\int_0^\infty \pi(x)\,dx = 1,$$

$$\pi(x) \geqslant 0, \quad \forall x \in R_+,$$

and solve the corresponding dual problem:

$$q_D^+(k) = \text{minimize} \quad v + \sum_{i=1}^n q_i u_i$$

$$\text{subject to} \quad g(x) = v + \sum_{i \mid k_i \leqslant x} (x - k_i)u_i \quad (19)$$

$$\leqslant \begin{cases} 0, & 0 \leqslant x < k, \\ x - k, & x \geqslant k. \end{cases}$$

Similar to the lower bound problem, we prove that strong duality holds, the primal is equivalent to its discretized version, and it is sufficient to solve the dual problem with constraints only at the points k_i, $i = 1, \dots, n$. We obtain that

$$g(x) = \begin{cases} 0, & x \leqslant k_j, \\ \dfrac{k_{j+1} - k}{k_{j+1} - k_j}(x - k_j), & x \in (k_j, k_{j+1}), \\ x - k, & x \geqslant k_{j+1}, \end{cases}$$

and the corresponding dual variables are

$$u_j = \frac{k_{j+1} - k}{k_{j+1} - k_j},$$

$$u_{j+1} = \frac{k - k_j}{k_{j+1} - k_j},$$

$$u_i = 0, \quad \forall i \neq j, j+1,$$

$$v = 0.$$

The corresponding dual optimal objective value in this case is

$$q^+(k) = q_j \frac{k_{j+1} - k}{k_{j+1} - k_j} + q_{j+1} \frac{k - k_j}{k_{j+1} - k_j}. \quad \square$$

3.2. Bounds on Prices of Mixed Options

We now extend the results of the previous section to *put* options and combinations of *calls* and *puts*. Let $p(k)$ be the price of a put option with strike k and the same exercise date. Then $p(k) = E[\max(0, k - X)]$. Because the price of a call with the same strike satisfies $q(k) = E[\max(0, X - k)]$, then

$$p(k) - q(k) = k - E[X].$$

Clearly, the function $p(\cdot)$ is increasing and concave. Similar to the case of calls, these conditions are in fact necessary and sufficient for put-pricing functions to be valid. In other words, if the prices p_i of only puts are known with strikes k_i, then the best-possible bounds for the price of a put with strike $k \in (k_j, k_{j+1})$ are $p^-(k) < p(k) < p^+(k)$ with

$$p^-(k) = p_j \frac{k_{j+1} - k}{k_{j+1} - k_j} + p_{j+1} \frac{k - k_j}{k_{j+1} - k_j},$$

$$p^+(k) = \min\left(p_j\frac{k-k_{j-1}}{k_j-k_{j-1}} + p_{j-1}\frac{k_j-k}{k_j-k_{j-1}},\right. \tag{20}$$

$$\left. p_{j+1}\frac{k_{j+2}-k}{k_{j+2}-k_{j+1}} + p_{j+2}\frac{k-k_{j+1}}{k_{j+2}-k_{j+1}}\right).$$

Suppose we are given prices of call and put options with various strikes k_i, and we want to find optimal bounds on prices of a call or put option with strike k. Notice that if we know the prices of both a call and of a put option with a certain strike k_i, then we can derive the expected stock price from the put-call parity formula $E[X] = q_i - p_i + k_i$. Now, if we know the expected price of the stock $E[X]$, the problem can be directly reduced to the one we solved in §3.1 by simply writing all option prices in terms of call prices, using the put-call parity result.

Finally, suppose we are given prices q_i of calls with strikes k_i, $i = 1, \dots, n$ and prices p_i of puts with strikes c_i, $i = 1, \dots, m$, such that $c_i \neq k_j$ for all i, j. We are interested in finding the best-possible bounds for a call with strike k. In this case we cannot determine $E[X]$ uniquely from the put-call parity. By the put-call parity we can transform the given put prices to corresponding call prices q_i' given by

$$q_i' = p_i + E[X] - c_i.$$

We sort the strikes k_i and c_i. We want the call prices q_i' to be consistent. We apply Theorem 4 for the calls with strikes c_i as well as the call with strike k. This leads to linear inequalities involving the two unknowns $q(k)$ and $E[X]$. Solving the resulting linear optimization problem with the objective of maximizing or minimizing $q(k)$ gives optimal upper and lower bounds on $q(k)$.

4. BOUNDS ON THE VARIANCE OF THE STOCK PRICE, GIVEN OPTION PRICES

In this section, we determine optimal bounds on the variance of a stock price X, when prices of options with different strikes and same exercise date T on that stock are known.

4.1. Bounds on the Mean

Typically, the risk-neutral mean of the price of a stock is known (it is the present price). For completeness, however, we derive bounds in case the present price is not observable.

Call Options. The bounds on the expected stock price given call prices information are easy to derive, because we can interpret $E[X] = E[\max(0, X - 0)] = q(0)$ as the price of a call option with zero strike. The result of Theorem 4 can be applied in this case to find the following optimal bounds on $E[X] = q(0) \in [q^-(0), q^+(0)]$:

$$q^-(0) = \frac{q_1 k_2 - q_2 k_1}{k_2 - k_1} = M^-,$$

$$q^+(0) = q_1 + k_1 = M^+. \tag{21}$$

Notice that these bounds depend only on the prices of the two calls with smallest strikes.

Mixed Put and Call Options. Now suppose we are given miscellaneous prices of either calls or puts with different strikes. The optimal bounds M^- and M^+ on the expected stock price $E[X]$ can be determined by converting the put prices into call prices by the put-call parity result, and then constraining the call-pricing function to be decreasing and convex, using the bounds from Theorem 4.

4.2. Upper Bounds on the Variance

To compute bounds on the variance we need to assume a finite upper bound K on the stock price X at time T. We incorporate this information by introducing a call option with strike K with price equal to zero, i.e., $k_{n+1} = K$ and $q_{n+1} = q(k_{n+1}) = 0$.

Call Options. Suppose first that our information consists of call prices only. We want to determine optimal upper bounds on the variance $Var[X] = E[X^2] - E[X]^2$ of a stock price X, when prices q_i of call options with strikes k_i, $i = 1, \dots, n$ on that stock are known.

We can formulate this as an optimization problem as follows:

$$V^+ = \text{maximize} \quad \int_0^K x^2 \pi(x)\,dx - \left(\int_0^K x\pi(x)\,dx\right)^2,$$

$$\text{subject to} \quad \int_{k_i}^K (x - k_i)\pi(x)\,dx = q_i,$$

$$i = 1, \dots, n, \tag{22}$$

$$\int_0^K \pi(x)\,dx = 1$$

$$\pi(x) \geqslant 0, \quad x \in [0, K].$$

THEOREM 5 (OPTIMAL UPPER BOUND ON THE VARIANCE OF THE STOCK PRICE). *(a) Given prices q_i of European calls with strikes k_i, and assuming that the mean M of the stock price is known, the optimal upper bound on the variance of the stock price is given by*

$$V^+(M) = \sum_{i=1}^n (k_{i+1} - k_{i-1})q_i + k_1 M - M^2. \tag{23}$$

(b) If the mean price is not known, then the optimal upper bound is given by

$$V^+ = \begin{cases} \displaystyle\sum_{i=1}^n (k_{i+1} - k_{i-1})q_i & \text{if } \dfrac{q_1 k_2 - q_2 k_1}{k_2 - k_1} < \dfrac{k_1}{2}, \\ \quad + \dfrac{k_1^2}{4}, & \\ \displaystyle\sum_{i=1}^n (k_{i+1} - k_{i-1})q_i & \text{if } \dfrac{q_1 k_2 - q_2 k_1}{k_2 - k_1} \geqslant \dfrac{k_1}{2}. \\ \quad + k_1\dfrac{q_1 k_2 - q_2 k_1}{k_2 - k_1} & \\ \quad - \left(\dfrac{q_1 k_2 - q_2 k_1}{k_2 - k_1}\right)^2, & \end{cases} \tag{24}$$

PROOF. (a) We first solve Problem (22) for a fixed value $M = E[X]$, obtaining an optimal value $V^+(M)$, which we then optimize over all feasible values of $M \in [M^-, M^+]$. We solve the following auxiliary problem with arbitrary fixed expected stock price $E[X] = M$.

$$Z^+(M) = V^+(M) + M^2$$

$$= \text{maximize} \quad \int_0^K x^2 \pi(x)\,dx,$$

$$\text{subject to} \quad \int_0^K x\pi(x)\,dx = M = q_0,$$

$$\int_{k_i}^K (x - k_i)\pi(x)\,dx = q_i,$$

$$i = 1, \dots, n,$$

$$\int_0^K \pi(x)\,dx = 1,$$

$$\pi(x) \geqslant 0, \quad x \in [0, K].$$

We consider the corresponding dual problem:

$$Z_D^+(M) = \text{minimize} \quad v + u_0 M + \sum_{i=1}^n u_i q_i,$$

$$\text{subject to} \quad g(x) = v + u_0 x$$

$$+ \sum_{i \mid k_i \leqslant x} (x - k_i) u_i \geqslant x^2,$$

$$0 \leqslant x \leqslant K.$$

Again, the optimum is obtained by forcing the constraints to be binding at the points k_i: $g(k_i) = k_i^2, i = 0, \dots, n+1$. The corresponding dual solution is $v = k_0 = 0$, $u_0 = k_1$, $u_i = k_{i+1} - k_{i-1}, i = 1, \dots, n$, and the optimal objective value is $Z^+(M) = Z_D^+(M) = k_1 M + \sum_{i=1}^n (k_{i+1} - k_{i-1}) q_i$. Thus, given the mean price M, the optimal upper bound on the variance of X is given by Equation (23).

(b) We next optimize over all feasible values of $M \in [M^-, M^+]$ to determine the upper bound on the variance

$$V^+ = \text{maximize} \quad \sum_{i=1}^n (k_{i+1} - k_{i-1}) q_i + k_1 M - M^2,$$

$$\text{subject to} \quad M^- = \frac{q_1 k_2 - q_2 k_1}{k_2 - k_1} \leqslant M \leqslant M^+ = q_1 + k_1.$$

This is a concave quadratic-optimization problem that can be solved in closed-form, leading to the closed-form bound (24). □

Mixed Put and Call Options. Suppose now that we are given either call or put prices for various strikes $q_i = q(k_i)$ for all $i \in Q$, and $p_i = p(k_i)$ for all $i \in P$, where P and Q are two sets of indices so that $P \cup Q = \{1, \dots, n+1\}$ and $n+1 \in Q$, because we assumed $q_{n+1} = q(k_{n+1}) = q(K) = 0$. We transform the puts to corresponding calls with prices:

$$q_i' = p_i - k_i + M, \quad i \in P.$$

We sort the strikes in $P \cup Q$, and apply the bound (23) for the sequence of calls with prices $q_i, i \in Q$ and $q_i', i \in P$. Note that if P and Q are not disjoint, then we can determine the value of $M = E[X]$ from the put-call parity result for a pair (p_j, q_j) with $j \in P \cap Q$. When the sets P and Q are disjoint, we can obtain an interval $[M^-, M^+]$ in which M lies, using the technique of §4.1. Applying the bound (23), we will find that an optimal upper bound for the variance of the price given M is a concave quadratic function $V^+(M)$ of M. Then, the optimal upper bound on the variance given M is given by

$$V^+ = \max_{M \in [M^-, M^+]} V^+(M).$$

4.3. Lower Bounds on the Variance

Call Options. Suppose for now that the available information consists of call prices only. We denote $q_0 = M$ and $k_0 = 0$. We prove the following result.

THEOREM 6 (OPTIMAL LOWER BOUND ON THE VARIANCE OF THE STOCK PRICE). (a) *Given prices q_i of European calls with strikes k_i, and assuming the mean $M = q_0$ of the stock price is known, the optimal lower bound on the variance of the price is given by*

$$V^-(M) = \sum_{i=0}^{n+1} t_i k_i^2 - Z(M) - M^2, \tag{25}$$

where $Z(M)$ is the objective value of the following quadratic network flow problem:

$$Z(M) = \text{minimize} \quad \sum_{i=0}^{n+1} t_i d_i^2,$$

$$\text{subject to} \quad d_i + d_{i-1} \geqslant c_i, \quad i = 1, \dots, n+1, \tag{26}$$

$$d_i \geqslant 0, \quad i = 0, \dots, n+1,$$

with $t_i = T_{i-1} - T_i \geqslant 0$, $T_i = -\frac{q_{i+1} - q_i}{k_{i+1} - k_i}, i = 0, \dots, n, (T_{-1} = 1, T_{n+1} = 0)$, and $c_i = k_i - k_{i-1} \geqslant 0$.
(b) *If the mean price is not known, then the optimal lower bound is given by*

$$V^- = \min_{M \in [M^-, M^+]} V^-(M), \tag{27}$$

with $M^- = \frac{q_1 k_2 - q_2 k_1}{k_2 - k_1}$ and $M^+ = q_1 + k_1$.

PROOF. (a) Formulating a minimization optimization problem analogous to the maximization problem (22), and taking the dual, we obtain that given $M = q_0$, the lower bound is given by

$$V^-(M) + M^2$$

$$= \text{maximize} \quad v + \sum_{i=0}^n q_i u_i,$$

$$\text{subject to} \quad g(x) = v + \sum_{i \mid k_i \leqslant x} (x - k_i) u_i \leqslant x^2,$$

$$0 \leqslant x \leqslant K.$$

For the upper bound problem, we only needed the constraints at the points k_i to be binding, namely: $g(k_i) = k_i^2$, $i = 0, \ldots, n+1$. This is not sufficient for the lower bound. To ensure feasibility, we also need to make sure that the line segment $(g(k_i), g(k_{i+1}))$ lies below the quadratic x^2 on each interval $x \in [k_i, k_{i+1}]$. This can be interpreted geometrically as follows. Consider the line tangent from the point $(k_i, g(k_i))$ to the quadratic x^2. The constraint says that if the tangency point occurs within the interval (k_i, k_{i+1}), then the line segment connecting $g(k_i)$ and $g(k_{i+1})$ has to lie below the tangent.

To express this algebraically, notice that the constraints at the points $k_i : g(k_i) \leqslant k_i^2$, can be formulated by denoting $d_i^2 = k_i^2 - g(k_i) \geqslant 0$, $i = 0, \ldots, n+1$, with $d_i \geqslant 0$. Then the x-coordinate of the tangency point equals $k_i + d_i$, hence the slope of the tangent is $2(k_i + d_i)$. The constraint for the interval (k_i, k_{i+1}) can be expressed as follows:

If $k_i + d_i \leqslant k_{i+1}$, then

$$k_{i+1}^2 - d_{i+1}^2 \leqslant k_i^2 - d_i^2 + 2(k_i + d_i)(k_{i+1} - k_i).$$

The last inequality can be written as $d_{i+1}^2 \geqslant (d_i - (k_{i+1} - k_i))^2$. By definition $d_i \geqslant 0$, $i = 0, \ldots, n+1$, so the constraints can be restated as $d_{i+1} \geqslant \max(0, k_{i+1} - k_i - d_i)$, $\forall i = 0, \ldots, n$.

In terms of d_is, we can write:

$$u_i = \frac{(k_{i+1}^2 - d_{i+1}^2) - (k_i^2 - d_i^2)}{k_{i+1} - k_i} + \frac{(k_i^2 - d_i^2) - (k_{i-1}^2 - d_{i-1}^2)}{k_i - k_{i-1}}$$

for all $i = 1, \ldots, n$,

$$u_0 = \frac{(k_1^2 - d_1^2) - (k_0^2 - d_0^2)}{k_1 - k_0}$$

and

$$v = k_0^2 - d_0^2 = -d_0^2.$$

By regrouping the terms in the objective, we can write the dual problem in terms of the d_is as follows:

$$V^-(M) + M^2 = \text{maximize} \quad \sum_{i=0}^{n+1} (k_i^2 - d_i^2)(T_{i-1} - T_i),$$

$$\text{subject to} \quad d_i \geqslant \max(0, k_i - k_{i-1} - d_{i-1}),$$

$$i = 1, \ldots, n+1,$$

where $T_i = -\frac{q_{i+1} - q_i}{k_{i+1} - k_i}$, $i = 0, \ldots, n$, $T_{-1} = 1$, and $T_{n+1} = 0$. The optimal bound $V^-(M)$ can thus be rewritten as

$$V^-(M) = \sum_{i=0}^{n+1} t_i k_i^2 - Z(M) - M^2,$$

where $Z(M)$ is the objective value of the following quadratic network-flow problem:

$$Z(M) = \text{minimize} \quad \sum_{i=0}^{n+1} t_i d_i^2,$$

$$\text{subject to} \quad d_i + d_{i-1} \geqslant c_i, \quad i = 1, \ldots, n+1,$$

$$d_i \geqslant 0, \quad i = 0, \ldots, n+1,$$

where we denoted $t_i = T_{i-1} - T_i \geqslant 0$, $c_i = k_i - k_{i-1} \geqslant 0$.

(b) To compute the overall optimal lower bound on the variance, independent from the mean, it remains to minimize $V^-(M)$ over all feasible values $M \in [M^-, M^+]$, where M^-, M^+ are given from the bounds in Equation (21). $\square$

Problem (26) is a separable quadratic optimization problem with network-flow constraints. Because of its special structure it can be solved by the following dynamic-programming algorithm:

Choose d_0 and let

$$d_{i+1} = \begin{cases} k_{i+1} - k_i - d_i, & \text{if } k_{i+1} \geqslant k_i + d_i, \\ 0, & \text{if } k_{i+1} < k_i + d_i. \end{cases}$$

To obtain the optimal solution, one has to optimize over all initial choices of $d_0 \geqslant 0$. A heuristic solution that performs very well in practice starts with $d_0 = 0$.

Geometrically, this iterative construction can be visualized (see Figure 3) as follows: At each step $(i+1)$, from the point $(k_i, g(k_i))$ draw the positive slope tangent to the curve $f(x) = x^2$. The x-coordinate of the tangency point equals $k_i + d_i$, and according to whether or not this falls within the next interval $[k_i, k_{i+1}]$, we have two cases:

• If $k_i + d_i \leqslant k_{i+1}$, then draw the next segment of $g(x)$, $x \in [k_i, k_{i+1}]$ to be the tangent.
• If $k_i + d_i > k_{i+1}$, then draw the next segment of $g(x)$, $x \in [k_i, k_{i+1}]$ so that $g(k_{i+1}) = k_{i+1}^2$.

Mixed Put and Call Options. In the case when both call and put prices are given, we first transform all information in terms of calls and find the best possible bounds

Figure 3. The dynamic-programming algorithm.
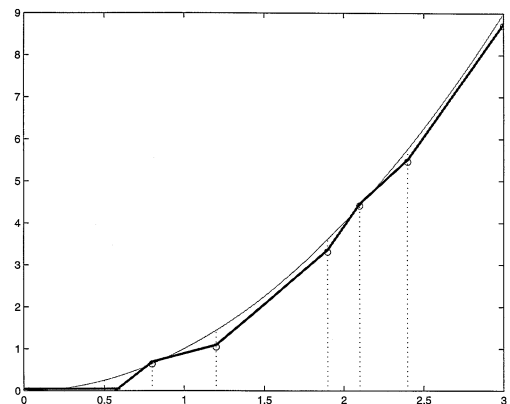
M^-, M^+ on the mean M using Theorem 4. To find an optimal lower bound on the variance, we again solve the problem in Theorem 6(b).

4.4. Computational Results

In this section, we discuss the quality of the upper and lower bounds on the mean and variance of a stock price. In Table 1, we report results on the January '99 Microsoft stock price, computed using information on European call prices from *The Wall Street Journal* of July 7, 1998. The current stock price is $S = 107\frac{13}{16}$, and the listed call options have strikes $k = [80, 95, 100, 110, 120, 140]$ and are sold at closing for $q = [31, 19, 16, 10, 6, 2.25]$. We also incorporate into our calculations the listed risk-free interest rate, listed as $r = 0.0557$. Call prices are given by

$$q(k) = e^{-r(T-t)}E[\max(0, X-k)],$$

where $T - t$ is the time to maturity, measured in years. In this case $T - t = 0.5$.

Using the Black-Scholes option-pricing formula, we estimate from the data an implied volatility of $s_{BS} = 0.3241$. The corresponding estimates for the mean and standard deviation of the forward stock price, under the risk-neutral valuation, are $M_{BS} = S\,e^{r(T-t)} = 110.8573$ and $\sigma_{BS} = S e^{r(T-t)}\sqrt{e^{S^2(T-t)} - 1} = 26.7394$.

Table 1. Optimal bounds on the standard deviation for various values of the mean M given option prices.

M	σ^-	σ^+
97.6829	49.1709	49.2101
98.1723	48.5859	48.6306
98.6617	47.9888	48.0391
99.1511	47.3790	47.4352
99.6405	46.7563	46.8184
100.1299	46.1199	46.1882
100.6193	45.4693	45.5440
101.1087	44.8040	44.8852
101.5981	44.1232	44.2111
102.0875	43.4262	43.5212
102.5769	42.7122	42.8145
103.0663	41.9804	42.0903
103.5557	41.2298	41.3476
104.0451	40.4594	40.5854
104.5345	39.6679	39.8027
105.0239	38.8542	38.9980
105.5133	38.0168	38.1702
106.0027	37.1540	37.3175
106.4921	36.2642	36.4384
106.9815	35.3451	35.5307
107.4709	34.3946	34.5924
107.9603	33.4099	33.6207
108.4497	32.3878	32.6128
108.9391	31.3248	31.5651
109.4285	30.2165	30.4736
109.9179	29.0577	29.3333
110.4073	27.8422	28.1384
110.8967	26.5621	26.8815
111.3861	25.2076	25.5535
111.8755	23.7660	24.1427

Using $M = 110.8573$ and assuming an upper bound $K = 160$ on the stock price, we apply the upper bound given in Equation (23) and the lower bound given in Equation (25) to obtain that the standard deviation of the stock price σ belongs in the interval: $\sigma \in (26.6677, 26.9851)$. If from the standard deviation we were to compute the implied volatility, as implied by the Black-Scholes formula, we would obtain $s \in (0.3223, 0.3259)$, which indeed is very close to the direct Black-Scholes forecasts.

If we do not use any information from the Black and Scholes model, but we apply only the bounds on the mean given by Equations (21), we obtain that the average stock price M is in the interval $M \in [M^-, M^+]$ with $M^- = 97.6829$, and $M^+ = 111.8755$. In Table 1, we vary the mean M in the interval [97.6829, 111.8755], and report the corresponding interval $[\sigma^-, \sigma^+]$ of the standard deviation of the stock price. For a given M, we observe that the bounds we derive on the standard deviation are extremely tight. As M varies in the interval [97.6829, 111.8755], we obtain that the standard deviation is within $23.7660 \leqslant \sigma \leqslant 49.2101$.

5. BOUNDS WITH TRANSACTION COSTS

Up until now we have assumed a frictionless economy and developed our results based on the theory of asset pricing under the no-arbitrage assumption, ignoring transaction costs. In this section, we derive bounds in the presence of transaction costs, using the no-arbitrage assumption. When transaction costs are taken into account, the price of an option is within an interval defined by the bid-ask spread. A call-pricing function is then defined as a pair:

$$q_{tc} : R_+ \to R_+^2, q_{tc}(k) = (q_{bid}(k) = q^-(k), q_{ask}(k) = q^+(k)).$$

In a frictionless market, the asset-pricing theory of Harrison and Kreps (1979) ensures the existence of a risk-neutral Martingale measure that uniquely determines a valid linear-pricing rule for all assets. Suppose we are given n call options with strikes k_i, and bid-ask prices q_i^-, q_i^+, $i = 1, \ldots, n$. In the presence of transaction costs, Jouini and Kallal (1995) show that there is no arbitrage if and only if there exists a probability measure π such that

$$q_i^- \leqslant E_\pi[\max(0, X - k_i)] \leqslant q_i^+, i = 1, \ldots, n.$$

By Theorem 4, this is equivalent to the existence of a convex decreasing function $q^* : R_+ \to R_+$, such that $q^*(k_i) \in [q_i^-, q_i^+]$, for all $i = 1, \ldots, n$. If no convex decreasing function can be fitted between the bid (q^-) and ask (q^+) processes, then the given set of bid-ask spreads is not valid, and an arbitrage opportunity exists. This provides an easy test for arbitrage opportunities in a market with transaction costs.

The next theorem extends the results from the two previous sections by replacing the equality constraints in each respective primal problem by

$$q_i^- \leqslant \int_{k_i}^{\infty} (x - k_i)\pi(x)dx \leqslant q_i^+, i = 1, \ldots, n.$$

We thus introduce corresponding dual variables u_i^-, u_i^+, which are nonnegative for upper bound problems and nonpositive for the lower bounds. The corresponding dual function becomes

$$g(x) = v + \sum_{i|k_i \leqslant x} (x - k_i) u_i,$$

where $u_i = u_i^+ - u_i^-$. With the notation $q_i = q_i^+ - q_i^- (\geqslant 0)$, we can write (for all problems) the dual objective as

$$v + \sum_{i=1}^n (q_i^+ u_i^+ - q_i^- u_i^-) = v + \sum_{i=1}^n q_i^+ u_i + \sum_{i=1}^n q_i u_i^-$$

$$= v + \sum_{i=1}^n q_i^- u_i + \sum_{i=1}^n q_i u_i^+.$$

By optimizing the corresponding dual, and using very similar techniques to those in Theorems 4, 5, and 6, we prove the following result.

THEOREM 7 (BOUNDS UNDER TRANSACTION COSTS). *Given bid and ask prices q_i^- and q_i^+ for European calls with strikes k_i, $i = 1, \ldots, n$, then: (a) The optimal bounds on a call with strike k is given by*

$$q^-(k) = \max\left(q_j^- \frac{k - k_{j-1}}{k_j - k_{j-1}} + q_{j-1}^- \frac{k_j - k}{k_j - k_{j-1}}, \right.$$

$$\left. q_{j+1}^- \frac{k_{j+2} - k}{k_{j+2} - k_{j+1}} + q_{j+2}^- \frac{k - k_{j+1}}{k_{j+2} - k_{j+1}} \right),$$

$$q^+(k) = q_j^+ \frac{k_{j+1} - k}{k_{j+1} - k_j} + q_{j+1}^+ \frac{k - k_j}{k_{j+1} - k_j}.$$

(b) The optimal bounds on the mean stock price are

$$M^- = \frac{q_1^- k_2 - q_2^- k_1}{k_2 - k_1},$$

$$M^+ = q_1^+ + k_1.$$

(c) The optimal lower bound on the variance is

$$V^-(M) = \sum_{i=0}^{n+1} t_i k_i^2 - Z(M) - M^2,$$

where $Z(M)$ is the objective value of the following quadratic network flow problem:

$$Z(M) = minimize \quad \sum_{i=0}^{n+1} t_i d_i^2,$$

$$subject \ to \quad d_i + d_{i-1} \geqslant c_i, \quad i = 1, \ldots, n+1,$$

$$d_i \geqslant 0, \quad i = 0, \ldots, n+1,$$

with $t_i = T_{i-1} - T_i \geqslant 0$, $T_i = -\frac{q_{i+1}^- - q_i^-}{k_{i+1} - k_i}$, $i = 0, \ldots, n$, ($T_{-1} = 1$, $T_{n+1} = 0$), and $c_i = k_i - k_{i-1} \geqslant 0$.
The optimal upper bound on the variance is

$$V^+(M) = \sum_{i=1}^n (k_{i+1} - k_{i-1}) q_i^+ + k_1 M - M^2.$$

6. BOUNDS IN MULTIPLE DIMENSIONS

In this section, we consider generalizations of the bounds we considered in earlier sections when we have information about a set of m different stocks. In particular, we have an option with payoff function $\phi(x)$, $\phi : R_+^m \to R$, and a vector of n moment functions $f = (f_1, \ldots, f_n)$ (we let $f_0(x) = 1$), $f_i : R_+^m \to R$, $i = 0, 1, \ldots, n$, and the corresponding vector of moments $q = (q_1, \ldots, q_n)$ (we let $q_0 = 1$). We address in this section the upper bound problem (2):

$$\begin{aligned} maximize \quad & E_\pi[\phi(X)], \\ subject \ to \quad & E_\pi[f_i(X)] = q_i, \quad i = 1, \ldots, n. \\ & \int_0^\infty \pi(x) \, dx = 1, \\ & \pi(x) \geqslant 0, \quad x \in R_+^m, \end{aligned} \quad (28)$$

where the expectation is taken over all Martingale measures defined on R_+^m. We can solve the lower bound problem by changing the sign of the objective function ϕ in Problem (28).

In Theorem 9 we show that solving Problem (28) is NP-hard. For this reason, we find a weaker bound by optimizing over all Martingale measures defined on R^m as opposed to R_+^m. For this reason we consider the following problem:

$$\begin{aligned} maximize \quad & E_\pi[\phi(X)], \\ subject \ to \quad & E_\pi[f_i(X)] = q_i, \quad i = 1, \ldots, n. \\ & \int_{-\infty}^\infty \pi(x) \, dx = 1 \\ & \pi(x) \geqslant 0, \quad x \in R^m, \end{aligned} \quad (29)$$

and its dual,

$$\begin{aligned} minimize \quad & y_0 + \sum_{i=1}^n y_i q_i, \\ subject \ to \quad & y_0 + \sum_{i=1}^n y_i f_i(x) \geqslant \phi(x), \quad \forall x \in R^m. \end{aligned} \quad (30)$$

Isii (1963) (see also Karlin 1966, p. 472 or Smith 1995) shows that under weak conditions[1] on the moment vector q strong duality holds, i.e., the optimal solution values of Problems (29) and (30) are equal.

The best-possible upper bound corresponds to the optimal solution value of Problem (28). Because Problem (29) is a relaxation of Problem (28), we obtain an upper bound, although not necessarily the optimal one, by solving Problem (29), and by strong duality, Problem (30). In Theorem 8 we identify cases under which we can solve Problem (30) efficiently.

THEOREM 8. *An upper bound on Problem (28) can be solved in polynomial time in the following cases:*
(a) If ϕ and f_i, $i = 1, \ldots, n$ are quadratic or linear functions of the form

$$\phi(x) = x'Ax + b'x + c$$

$$f_i(x) = x'A_i x + b_i'x + c_i, \quad i = 1, \ldots, n, \quad (31)$$

then Problem (30), and thus Problem (29), can be solved in polynomial time by solving the following semidefinite optimization problem:

minimize $\quad \sum_{i=1}^{n} y_i q_i,$

subject to $\quad \begin{bmatrix} \sum_{i=1}^{n} y_i c_i + y_0 - c & \left(\sum_{i=1}^{n} y_i b_i - b\right)' / 2 \\ \left(\sum_{i=1}^{n} y_i b_i - b\right) / 2 & \sum_{i=1}^{n} y_i A_i - A \end{bmatrix} \succeq 0.$ $\quad$ (32)

(*b*) *If* ϕ *and* f_i, $i = 1, \ldots, n$, *are quadratic or piecewise linear functions of the form*

$\phi(x) = x'Ax + b_k'x + c_k \quad x \in D_k, \quad k = 1, \ldots, d,$

$f_i(x) = x'A_i x + b_{ik}'x + c_{ik}, \quad x \in D_k,$ $\quad$ (33)

$\qquad i = 1, \ldots, n, k = 1, \ldots, d,$

over the d disjoint polyhedra $D_1, \ldots, D_d$ *that form a partition of* R^m, *and d is a polynomial in* n, m, *then Problem (30), and thus Problem (29), can be solved in polynomial time.*

PROOF. (a) We first consider the case in which all the functions ϕ and f_i are are quadratic or linear as in Equation (31). In this case, Problem (30) becomes

minimize $\quad y_0 + \sum_{i=1}^{n} y_i q_i$

subject to $\quad g(x) \geqslant 0, \quad \forall x \in R^m,$

where

$g(x) = y_0 + \sum_{i=1}^{n} y_i f_i(x) - \phi(x) = x'\widehat{A}x + \hat{b}'x + \hat{c},$

with

$\widehat{A} = \sum_{i=1}^{n} y_i A_i - A, \quad \hat{b} = \sum_{i=1}^{n} y_i b_i - b, \quad \hat{c} = \sum_{i=1}^{n} y_i c_i + y_0 - c.$

Thus, the constraints $g(x) \geqslant 0$ are equivalent to

$x'\widehat{A}x + \hat{b}'x + \hat{c} \geqslant 0, \quad \forall x \in R^m,$

or equivalently

$\begin{pmatrix} 1 \\ x \end{pmatrix}' \begin{bmatrix} \hat{c} & \hat{b}/2 \\ \hat{b}/2 & \widehat{A} \end{bmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \geqslant 0, \quad \forall x \in R^m.$ $\quad$ (34)

Equation (34) holds if and only if

$\begin{bmatrix} \hat{c} & \hat{b}'/2 \\ \hat{b}/2 & \widehat{A} \end{bmatrix} \succeq 0,$

i.e., the matrix $\begin{bmatrix} \hat{c} & \hat{b}/2 \\ \hat{b}'/2 & \widehat{A} \end{bmatrix}$ is positive semidefinite. Thus, Problem (30) is equivalent to the semidefinite optimization

problem (32), which is solvable in polynomial time (see, for example, Nesterov and Nemirovski 1994 and Vandenberghe and Boyd 1996).

(b) If the functions ϕ or f_i, $i = 1, \ldots, n$ are as in (33), then Problem (30) can be expressed as

minimize $\quad y_0 + \sum_{i=1}^{n} y_i q_i,$

subject to $\quad g_k(x) = x'\widehat{A}x + \hat{b}_k'x + \hat{c}_k \geqslant 0, \quad \forall x \in D_k,$ $\quad$ (35)

$\qquad k = 1, \ldots, d,$

where

$\widehat{A} = \sum_{i=1}^{n} y_i A_i - A, \quad \hat{b}_k = \sum_{i=1}^{n} y_i b_{ik} - b_k,$

$\hat{c}_k = \sum_{i=1}^{n} y_i c_{ik} + y_0 - c_k.$

By the equivalence of separation and optimization (see Grötschel et al. 1981), Problem (35) can be solved in polynomial time if and only if the following separation problem can be solved in polynomial time.

The Separation Problem. Given an arbitrary $y = (y_0, y_1, \ldots, y_n)$, check whether $g_k(x) \geqslant 0$, for all $x \in D_k$, $k = 1, \ldots, n$ and if not, find a violated inequality.

We next show that solving the separation problem reduces to checking whether the matrix $\widehat{A}$ is positive semidefinite, and in this case solving the convex quadratic problems

$\min_{x \in D_k} g_k(x), \qquad k = 1, \ldots, d.$

This can be done in polynomial time using an ellipsoid algorithm (see Grötschel et al. 1981). The following algorithm solves the separation problem in polynomial time.

ALGORITHM A
1. If $\widehat{A}$ is not positive semidefinite, we construct a vector x_0 so that $g_k(x_0) < 0$ for some $k = 1, \ldots, n$. We decompose $\widehat{A} = Q'\Lambda Q$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues of $\widehat{A}$. Let $\lambda_i < 0$ be a negative eigenvalue of $\widehat{A}$. Let u be a vector with $u_j = 0$, for all $j \neq i$, and u_i selected as follows: Let v_k be the largest root of each polynomial if it exists. Let $u_i = \max_k v_k + 1$. If all the polynomials do not have real roots, then u_i can be chosen arbitrarily. Then,

$\lambda_i u_i^2 + (Q\hat{b}_k)_i u_i + \hat{c}_k < 0, \qquad \forall k = 1, \ldots, d.$

Let $x_0 = Q'u$. Because the polyhedra D_k form a partition of R^m, then $x_0 \in D_{k_0}$ for some k_0. Then,

$g_{k_0}(x_0) = x_0'\widehat{A}x_0 + \hat{b}_{k_0}'x_0 + \hat{c}_{k_0}$

$\qquad = u'QQ'\Lambda QQ'u + \hat{b}_{k_0}'Q'u + \hat{c}_{k_0}$

$\qquad = u'\Lambda u + (Q\hat{b}_{k_0})'u + \hat{c}_{k_0}$

$$= \sum_{j=1}^{n} \lambda_j u_j^2 + \sum_{j=1}^{n} (Q\hat{b}_{k_0})_j u_j + \hat{c}_{k_0}$$

$$= \lambda_i u_i^2 + (Q\hat{b}_{k_0})_i u_i + \hat{c}_{k_0} < 0.$$

This produces a violated inequality.

2. Otherwise, if $\widehat{A}$ is positive semidefinite, then we test if $g_k(x) \geqslant 0, \forall x \in D_k$ by solving d convex quadratic optimization problems:

$$\min_{x \in D_k} x'\widehat{A}x + \hat{b}_k' x + \hat{c}_k, \quad \text{for } k = 1, \ldots, d. \quad (36)$$

We denote by x_k^* an optimal solution of Problem (36), and $z_k = g_k(x_k^*)$ the optimal value of Problem (36). If $z_k \geqslant 0$ for all $k = 1, \ldots, d$, then there is no violated inequality. Otherwise, if $z_{k_0} < 0$ for some k_0, then we find x_0^k such that $g(x_k^*) < 0$, which represents a violated inequality.

Thus, Algorithm A solves the separation problem in polynomial time. It follows that Problem (30), and hence Problem (29), can be solved in polynomial time. □

6.1. Examples

Suppose we have observed the price q_1 of a European call option with strike k_1 for Stock 1 and the price q_2 of a European call option with strike k_2 for Stock 2. In addition, we have estimated the means μ_1, μ_2, the variances σ_1^2, σ_2^2, and the covariance σ_{12} of the prices of the two underlying stocks. Suppose in addition that we are interested in obtaining an upper bound on the price of a European call option with strike k for Stock 1. Intuition suggests that because the prices of the two stocks are correlated, the price of a call option on Stock 1 with strike k might be affected by the available information regarding Stock 2. We can find an upper bound on the price of a call option on Stock 1 with strike k by solving the problem we formulated in Equation (1), which is a special case of Problem (28), with $m = 2, n = 7$. From Theorem 8(b), Problem (1) can be solved efficiently. In this case, there are six sets D_k as follows:

$$D_1 = \{(x_1, x_2) \mid x_1 \geqslant k, x_2 \geqslant k_2\},$$

$$D_2 = \{(x_1, x_2) \mid x_1 \geqslant k, x_2 \leqslant k_2\},$$

$$D_3 = \{(x_1, x_2) \mid k_1 \leqslant x_1 \geqslant k, x_2 \geqslant k_2\},$$

$$D_4 = \{(x_1, x_2) \mid k_1 \leqslant x_1 \geqslant k, x_2 \leqslant k_2\},$$

$$D_5 = \{(x_1, x_2) \mid x_1 \leqslant k, x_2 \geqslant k_2\},$$

$$D_6 = \{(x_1, x_2) \mid x_1 \leqslant k, x_2 \leqslant k_2\}.$$

As another example, suppose we are interested in finding an upper bound on the price of an option with payoff

$$\phi(x) = \max(0, a_1' x - k_1, a_2' x - k_2).$$

This option allows its holder to buy two stock indices at maturity: the first one (given by the vector a_1) at price k_1, and the second one (given by the vector a_2) at price k_2.

Suppose we have estimated the mean and covariance matrix of the underlying securities. Again, Theorem 8(b) applies. In this case there are three sets D_k that form a polyhedral partition of R^m:

$$D_1 = \{x \in R^m \mid a_1' x - k_1 \leqslant 0, a_2' x - k_2 \leqslant 0\},$$

$$D_2 = \{x \in R^m \mid 0 \leqslant a_1' x - k_1, a_2' x - k_2 \leqslant a_1' x - k_1\},$$

$$D_3 = \{x \in R^m \mid 0 \leqslant a_2' x - k_1, a_1' x - k_2 \leqslant a_2' x - k_1\}.$$

Note that if $x \in D_1$, $\phi(x) = 0$, while if $x \in D_2$, $\phi(x) = a_1' x - k_1$. Finally, if $x \in D_3$, $\phi(x) = a_2' x - k_2$.

6.2. The Complexity of Optimal Bounds

Theorem 8 provides optimal bounds in polynomial time if we optimize over R^m, but not over R_+^m. The next theorem shows that it is NP-hard to find optimal bounds over R_+^m.

THEOREM 9 (COMPLEXITY OF FINDING OPTIMAL BOUNDS). *The problem of finding the optimal bound*

$$\max_{X \sim (M, \Gamma)^+} E[\phi(X)] \quad (37)$$

is NP-hard even if $\phi(x) = f' x$.

PROOF. The dual of Problem (37) is

$$\text{minimize} \quad y_0 + \sum_{i=1}^{n} y_i \mu_i + \sum_{i=1}^{n} \sum_{j=1}^{i} y_{ij} (\sigma_{ij} + \mu_i \mu_j),$$

$$\text{subject to} \quad x'Yx + y'x + y_0 \geqslant f'x, \quad \forall x \geqslant 0,$$

and the corresponding separation problem becomes:

Separation Problem. Given (Y, y, y_0), check if $\min_{x \geqslant 0} x'Yx + y'x + y_0 - f'x \geqslant 0$, otherwise find a violated inequality.

The separation problem is NP-hard because it reduces to verifying that the matrix Y is co-positive, which is an NP-hard problem (see Murty and Kabadi 1987). Therefore, by the equivalence of separation and optimization (see Grötschel et al. 1981), it is NP-hard to solve Problem (37). □

7. CONCLUDING REMARKS

We have demonstrated that convex optimization is the natural way to address the relation between option and stock prices without making distributional assumptions for the underlying price dynamics, using only the no-arbitrage assumption. For the single stock problem, we have shown that we can efficiently find optimal bounds on option prices, either algorithmically (solving a semidefinite optimization problem) or in closed form. For options that are affected by multiple stocks either directly (the payoff of the option depends on multiple stocks) or indirectly (we have information on correlations between stock prices), we can find bounds (but not optimal ones) using convex optimization methods. However, it is NP-hard to find optimal bounds in multiple dimensions.

APPENDIX

PROOF OF PROPOSITION 1. (a) Suppose (5) holds. Let $e_x = (1, x, x^2, \ldots, x^k)'$. Then

$$
\begin{aligned}
g(x) &= \sum_{r=0}^{2k} \sum_{i+j=r} x_{ij} x^r \\
&= \sum_{i=0}^{k} \sum_{j=0}^{k} x_{ij} x^i x^j \\
&= e_x' X e_x \\
&\geqslant 0,
\end{aligned}
$$

because $X \succeq 0$.

Conversely, suppose that the polynomial $g(x)$ of degree $2k$ is nonnegative for all x. Then, the real roots of $g(x)$ should have even multiplicity, otherwise $g(x)$ would alter its sign in a neighborhood of a root. Let λ_i, $i = 1, \ldots, r$ be its real roots with corresponding multiplicity $2m_i$. Its complex roots can be arranged in conjugate pairs, $a_j + ib_j$, $a_j - ib_j$, $j = 1, \ldots, h$. Then,

$$
g(x) = y_{2k} \prod_{i=1}^{r} (x - \lambda_i)^{2m_i} \prod_{j=1}^{h} \big((x - a_j)^2 + b_j^2\big).
$$

Note that the leading coefficient y_{2k} needs to be positive. Thus, by expanding the terms in the products, we see that $g(x)$ can be written as a sum of squares of polynomials of the form

$$
\begin{aligned}
g(x) &= \sum_{i=0}^{k} \left(\sum_{j=0}^{k} x_{ij} x^j \right)^2 \\
&= e_x' X e_x,
\end{aligned}
$$

with X positive semidefinite, from where Equation (5) follows.

(b) We observe that $g(x) \geqslant 0$ for $x \in [0, a]$ if and only if

$$
(1 + t^2)^k g\left(\frac{at^2}{1 + t^2} \right) \geqslant 0, \quad \text{for all } t.
$$

Because

$$
\begin{aligned}
(1 + t^2)^k g\left(\frac{at^2}{1 + t^2} \right) &= \sum_{r=0}^{k} y_r a^r t^{2r} (1 + t^2)^{k-r} \\
&= \sum_{r=0}^{k} y_r a^r \sum_{l=0}^{k-r} \binom{k-r}{l} t^{2(l+r)} \\
&= \sum_{j=0}^{k} t^{2j} \left(\sum_{r=0}^{j} y_r \binom{k-r}{j-r} a^r \right),
\end{aligned}
$$

by applying Part (a) we obtain Equation (6).

(c) We observe that $g(x) \geqslant 0$ for $x \in [a, \infty)$ if and only if

$$
g(a(1 + t^2)) \geqslant 0, \quad \text{for all } t.
$$

Because

$$
\begin{aligned}
g(a(1 + t^2)) &= \sum_{r=0}^{k} y_r a^r (1 + t^2)^r \\
&= \sum_{r=0}^{k} y_r a^r \sum_{l=0}^{r} \binom{r}{l} t^{2l} \\
&= \sum_{l=0}^{k} t^{2l} \left(\sum_{r=l}^{k} y_r \binom{r}{l} a^r \right),
\end{aligned}
$$

by applying Part (a) we obtain Equation (7).

(d) We observe that $g(x) \geqslant 0$ for $x \in [a, b]$ if and only if

$$
(1 + t^2)^k g\left(a + (b - a) \frac{t^2}{1 + t^2} \right) \geqslant 0, \quad \text{for all } t.
$$

Because

$$
\begin{aligned}
&(1 + t^2)^k g\left(a + (b - a) \frac{t^2}{1 + t^2} \right) \\
&= \sum_{r=0}^{k} y_r (a + bt^2)^r (1 + t^2)^{k-r} \\
&= \sum_{r=0}^{k} y_r \sum_{m=0}^{r} \binom{r}{m} a^{r-m} b^m t^{2m} \sum_{j=0}^{k-r} \binom{k-r}{j} t^{2j} \\
&= \sum_{l=0}^{k} t^{2l} \left(\sum_{m=0}^{l} \sum_{r=m}^{k+m-l} y_r \binom{r}{m} \binom{k-r}{l-m} a^{r-m} b^m \right),
\end{aligned}
$$

by applying Part (a) we obtain Equation (8). $\quad \square$

ENDNOTE

[1] An example of such a condition is as follows: If the vector of moments q is interior to the feasible moment set $\mathcal{M} = \{E[f(X)] \mid X \text{ arbitrary multivariate distribution}\}$, then strong duality holds.

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