Analysis of Stationary Distributions of Infinite Markov Chains Using Lyapunov Functions *

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Abstract

In this paper, we develop a systematic method for deriving bounds on the stationary distribution of stable Markov chains with a countably infinite state space. We assume that a Lyapunov function is available that satisfies a negative drift property (and, in particular, is a witness of stability), and that the jumps of the Lyapunov function corresponding to state transitions have uniformly bounded magnitude. We show how to derive closed form, exponential type, upper bounds on the steady-state distribution. Similarly, we show how to use suitably defined lower Lyapunov functions to derive closed form lower bounds on the steady-state distribution. We apply our results to homogeneous random walks on the positive orthant, and to multiclass single station Markovian queueing systems. In the latter case, we establish closed form exponential type bounds on the tail probabilities of the queue lengths, under an arbitrary work conserving policy.

1 Introduction

In this paper, we consider discrete time stable Markov chains with a countably infinite state space. Our main objective is to provide bounds on its stationary distribution, assuming that stability and the existence of the stationary distribution can be established using a suitable Lyapunov function.

Markov chains are a central object in the theory of discrete time stochastic processes and are used widely for modeling various biological, engineering, and economic processes. The stability of Markov chains has been studied intensively in the last decades, with much of this research being motivated

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within the context of queueing networks. Although, there are various techniques that can establish stability, methods based on Lyapunov functions seem to be most general and have been widely used. The idea is similar to the original use of Lyapunov functions, in establishing stability of deterministic dynamical systems, by showing that a certain "potential energy" (Lyapunov) function must decrease along any trajectory. The analogous property in the stochastic context is to require that, other than a "small" subset of the state space, the expected change of the Lyapunov function is negative; see [23] for a comprehensive survey. Numerous researchers have used the Lyapunov function method for the stability analysis of Markov chains with special structure. For example, the stability of homogeneous random walks in the nonnegative orthant was considered in [9, 20, 21, 22], and exact conditions for stability were derived when the state space is the nonnegative integer lattice Z_+^2 or Z_+^3 [20, 22, 9]. Furthermore, numerous papers have studied the Markov chains that arise in Markovian queueing networks. In this context, special types of Lyapunov functions (quadratic, or piecewise linear) have proven to be very effective tools for stability analysis (see [4, 8, 9, 14, 13, 15, 16, 17, 18]).

Besides the stability problem, which is addressed in the above cited works, a major topic in the study of Markov chains refers to performance analysis. Namely, given a Markov chain, we would like to obtain information on its stationary distribution, if it exists. This is simple, in principle, for finite-state Markov chains because the stationary distribution can be computed exactly. The problem is, however, highly nontrivial when the state space is infinite: before studying the properties of the stationary distribution, one needs to establish its existence, which amounts to the stability problem discussed earlier. Some of the existing literature provides bounds on the moments of the stationary distribution [4, 16, 17]. Closer to our objectives is the work of Hajek [12], who proves exponential type upper bounds on the tail of the distribution of hitting times and of tail of the distribution of the state during a transient period. Exponential type steady-state bounds can also be obtained by taking limits as time converges to infinity. Dai and Harrison [5] have considered the performance problem for a continuous-time Markov process (reflected Brownian motion in a positive orthant). They constructed a numerical (as opposed to closed form) procedure for bounding the stationary distribution of such process.

In this paper, we propose a general method for the performance analysis of infinite state Markov chains, based on Lyapunov functions. By using such functions and by exploiting certain equilibrium equations, we prove exponential type upper and lower bounds on the stationary distribution. The only conditions required are that the Lyapunov function has negative drift (outside a "small" set) and that the jumps of the Lyapunov function are uniformly bounded. Our method does not construct the desired Lyapunov function. Instead, we assume that such a function has already been constructed in the course of stability analysis. This is not a significant drawback, because stability analysis must precede performance analysis, and it typically involves the construction of a Lyapunov function, often by systematic methods. In a sense, our main point is that whenever one succeeds in constructing a Lyapunov function to establish the stability for a Markov chain, the same Lyapunov function can be used to derive performance bounds as well, with little extra effort.

The structure and the contributions of the paper are as follows. In the next section, we introduce our model - Markov chain with a countable state space. We introduce the notions of Lyapunov and lower Lyapunov functions. The lower Lyapunov function is introduced exclusively as a means for establishing lower bounds on stationary distribution, and is not required to be a witness of stability.

In Section 3, we state and prove the main results. We modify the Lyapunov function, using suitable thresholds, we write down the corresponding equilibrium equation, and we derive upper and lower bounds on the steady-state tail probabilities. These bounds are of exponential type and are expressed in terms of primary parameters of the Lyapunov function — drift and maximal jump. Bounds on the expected value of the Lyapunov function (in steady-state) are also obtained. Compared to Hajek's bounds, our bounds are simpler and are expressed in terms of the primary parameters. Also, unlike [12], we construct both upper and lower bounds on stationary tail probabilities. On the other hand, our approach is restricted by the boundedness assumption on the jump sizes of the Lyapunov function, whereas [12] only assumes that the jumps are of exponential type.

In Section 4, we apply our bounds to Markov chains with special structure, namely, homogeneous random walks. In Section 4.1, we derive upper and lower bounds on the tail distribution and expectation of a random walk in steady-state using linear Lyapunov functions. In Section 4.2, we apply our bounds to a Markovian multiclass single station queueing system. Using the results of Section 3, we derive exponential type upper and lower bounds on the steady-state distribution of the queue lengths. The bounds are in closed form and are expressed in terms of the arrival and service rates. We conclude with some open questions and some possible directions for strengthening the results of the paper.

A particularly interesting class of Lyapunov functions that satisfy the bounded jump assumption is the class of piecewise linear Lyapunov functions. Such Lyapunov functions have been used to establish the stability of certain multiclass queueing networks, and there exist systematic methods for constructing them [1], [6], [7], [8]. For multiclass queueing networks with two stations, it has been shown that piecewise linear Lyapunov functions lead to exact conditions for stability (this was proven for fluid networks in [1], and for stochastic networks by Dai and Vande-Vate [6], with the exception of the marginal case where the traffic intensity is equal to 1).

We refer the reader to [2], which concentrates on piecewise linear Lyapunov functions. Using the techniques of the present paper, [2] obtains explicit upper and lower bounds on the tail probabilities and expectations of queue lengths in multiclass queueing networks.

2 Homogeneous Markov Chains and Lyapunov Functions

Let $\{Q(t);\ t = 0, 1, 2, \ldots, \}$ be a discrete-time, discrete-state, time-homogeneous Markov chain that takes values in some countable set $\mathcal{X}$. For any two vectors $x, x' \in \mathcal{X}$, let $p(x, x')$ denote the transition probability

$$p(x, x') = \Pr\Big\{Q(t+1) = x' \,|\, Q(t) = x\Big\}.$$

A *stationary distribution* is defined as a function $\pi : \mathcal{X} \mapsto \Re$ such that $\pi(x) \geq 0$ for all $x \in \mathcal{X}$,

$$\sum_{x \in \mathcal{X}} \pi(x) = 1,$$

and

$$\pi(x) = \sum_{x' \in \mathcal{X}} \pi(x') p(x', x), \qquad \forall x, x' \in \mathcal{X}. \tag{1}$$

The existence (and uniqueness) of a stationary distribution, is often taken as the definition of stability; it is usually established by constructing a suitable Lyapunov function.

We now introduce our definitions of Lyapunov and lower Lyapunov functions. As discussed in the introduction, our goal is to use Lyapunov functions for performance analysis (specifically, upper bounds), under the assumption that the Markov chain is stable. On the other hand, the notion of a lower Lyapunov function is introduced exclusively as a means of getting the lower bounds on the stationary distribution of a Markov chain, and can be unrelated to the Lyapunov function used to establish stability or upper bounds. In the definition below, and in the rest of the paper, we use $\Re_+$ to denote the set $[0, \infty)$.

Definition 1 *A nonnegative function*

$$\Phi : \mathcal{X} \mapsto \Re_+$$

is said to be a Lyapunov function *if there exist some $\gamma > 0$ and $B \geq 0$, such that for any t, and any $x \in \mathcal{X}$ that satisfies $\Phi(x) > B$, there holds*

$$E[\Phi(Q(t+1)) \mid Q(t) = x] \leq \Phi(x) - \gamma.$$

Also a nonnegative function

$$\Phi : \mathcal{X} \mapsto \Re_+$$

is said to be a lower Lyapunov function *if there exists some $\gamma > 0$ such that for any t, and any $x \in \mathcal{X}$, there holds*

$$E[\Phi(Q(t+1)) \mid Q(t) = x] \geq \Phi(x) - \gamma.$$

Remarks:

1. The terms γ and B will be called the *drift* and *exception* parameters, respectively.

2. Usually, for Φ to be a witness of stability, it is required that the set $\{x : \Phi(x) \leq B\}$ be finite. We do not require it here, although in all of the examples to be considered later, this set is just a singleton, namely, the set $\{0\}$. For now, we simply assume that a stationary distribution exists. So, in principle, our Lyapunov function Φ may not be a witness of stability.

3. We could introduce an exception parameter B in the definition of the lower Lyapunov function, as well. This would unnecessarily complicate the derivations of the results, and is not required for the examples here and in [2].

From now on, we assume that the Markov chain $Q(t)$ is positive recurrent. This guarantees that there exists a unique stationary distribution π and $\pi(x)$ can be interpreted as the steady-state probability $\Pr_\pi\Big\{Q(t) = x\Big\}$ of state x. We will use $E_\pi[\cdot]$ we denote the expectation with respect to the probability distribution π . For any function $\Phi : \mathcal{X} \to \Re_+$ let

$$\nu_{max} \equiv \sup_{x,x' \in \mathcal{X}: p(x,x')>0} |\Phi(x') - \Phi(x)|, \tag{2}$$

and

$$\nu_{min} \equiv \inf_{x,x' \in \mathcal{X}: p(x,x')>0, \Phi(x')>\Phi(x)} (\Phi(x') - \Phi(x)). \tag{3}$$

Namely, ν_{max} is the largest possible magnitude of the change in Φ during an arbitrary transition, and ν_{min} is the smallest possible increase of Φ. Also, let

$$p_{max} = \sup_{x \in \mathcal{X}} \sum_{x' \in \mathcal{X}, \Phi(x) < \Phi(x')} p(x, x') \tag{4}$$

and

$$p_{min} = \inf_{x \in \mathcal{X}} \sum_{x' \in \mathcal{X}, \Phi(x) < \Phi(x')} p(x, x') \tag{5}$$

Namely, p_{max} and p_{min} are tight upper and lower bounds on the probability that the value of Φ increases during an arbitrary transition. We will be interested in Lyapunov functions with finite ν_{max}, and lower Lyapunov functions with positive ν_{min} and p_{min}.

As an illustration, consider a multidimensional random walk $Q(t)$ on the nonnegative lattice Z_+^n with unit step lengths. Namely, for any $z = (z_1, \ldots, z_n) \in Z_+^n$ and $z' = (z_1', \ldots, z_n') \in Z_+^n$, we assume that $p(z, z')$ can be positive only if $z' - z = e_k$ for some $k = 1, 2, \ldots, n$, where e_k is the k-th unit vector. For any $z = (z_1, \ldots, z_n) \in Z_+^n$ set $\Phi(z) = \sum_{i=1}^n z_i$. Then Φ is a linear lower Lyapunov function with $\nu_{min} = 1$ and $\gamma = 1$. This is a special case of the random walks that we will consider in Section 4.1.

As another illustration, suppose that the state is an n-dimensional vector, consider a piecewise linear Lyapunov function of the form $\Phi(z) = \max_j L_j' z$, where each L_j is a nonnegative vector in $\Re^n$. Notice that this Lyapunov function has bounded jump size ν_{max}, if the underlying process has bounded jumps, which holds true for random walks that model queueing networks.

3 Upper and Lower Bounds on the Stationary Probability Distribution

In this section, we derive upper and lower bounds on the stationary distribution of Markov chains that admit Lyapunov and/or lower Lyapunov functions. We will show that whenever a Markov chain has a Lyapunov function with a finite maximal step size ν_{max}, certain exponential type upper bounds hold. Similarly, whenever a Markov chain has a lower Lyapunov function with a positive minimal step size ν_{min} and positive p_{min}, certain exponential type lower bounds hold. The key to our analysis is a modified Lyapunov function, defined by

$$\hat{\Phi}(x) = \max\Big\{c, \Phi(x)\Big\} \tag{6}$$

and the corresponding equilibrium equation

$$E_\pi\Big[\hat{\Phi}(Q(t))\Big] = E_\pi\Big[\hat{\Phi}(Q(t+1))\Big]. \tag{7}$$

For all of the results in this paper, we will be assuming that

$$E_\pi[\Phi(Q(t))] < \infty. \tag{8}$$

As we will see later, this is a checkable property for most of the cases of interest. For example, if $Q(t)$ is a homogeneous random walk in a positive orthant (see Subsection 4.1) and Φ is a linear Lyapunov function of the form $\Phi(x) = L'x$, then (8) follows from $E_\pi[Q(t)] < \infty$, which holds whenever the random walk is stable. Property (8) allows us to rewrite (7) in an equivalent form

$$E_\pi\Big[\hat{\Phi}(Q(t+1)) - \hat{\Phi}(Q(t))\Big] = 0, \tag{9}$$

which will be used to derive upper and lower bounds on tail probabilities of the Markov chain.

Lemma 1 *Consider a Markov chain $Q(t)$ with stationary probability distribution π, and suppose that Φ is a Lyapunov function with drift γ and exception parameter B, such that $E_\pi[Q(t)]$ is finite. Then, for any (possibly negative) $c \geq B - \nu_{max}$, there holds*

$$\Pr_\pi\Big\{c + \nu_{max} < \Phi(Q(t))\Big\} \leq \frac{p_{max}\nu_{max}}{p_{max}\nu_{max} + \gamma}\Pr_\pi\Big\{c - \nu_{max} < \Phi(Q(t))\Big\}, \tag{10}$$

where ν_{max} and p_{max} are defined in (2) and (4), respectively.

Proof: Let us fix c as in the statement of the lemma, and consider the function $\hat{\Phi}(x)$ introduced in (6). Since $E_\pi[Q(t)]$ is finite and π is a stationary distribution, we have that (9) holds, which we rewrite as

$$\sum_x \pi(x)\Big(E[\hat{\Phi}(Q(t+1)) \mid Q(t) = x] - \hat{\Phi}(x)\Big) = 0. \tag{11}$$

We decompose the left-hand side of the equation above into three parts and obtain

$$\begin{aligned}
0 \;=\;& \sum_{x:\ \Phi(x) \leq c - \nu_{max}} \pi(x)\Big(E[\hat{\Phi}(Q(t+1)) \mid Q(t) = x] - \hat{\Phi}(x)\Big) && (12)\\
+\;& \sum_{x:\ c-\nu_{max} < \Phi(x) \leq c + \nu_{max}} \pi(x)\Big(E[\hat{\Phi}(Q(t+1)) \mid Q(t) = x] - \hat{\Phi}(x)\Big) && (13)\\
+\;& \sum_{x:\ c+\nu_{max} < \Phi(x)} \pi(x)\Big(E[\hat{\Phi}(Q(t+1)) \mid Q(t) = x] - \hat{\Phi}(x)\Big). && (14)
\end{aligned}$$

As we will see below, the advantage of using the modified function $\hat{\Phi}(x) = \max\{c, \Phi(x)\}$ is that it allows us to exclude the states $x \in \mathcal{X}$ with low values of $\Phi(x)$ from the equilibrium equation (11), without affecting the states x with high value of $\Phi(x)$.

We now analyze each of the summands in (12) separately.

1. To analyze the first summand in Equation (12), observe that for any x, x' with $\Phi(x) \leq c - \nu_{max} < c$ and $p(x, x') > 0$, the definition of ν_{max} implies that

$$\Phi(x') \leq \Phi(x) + \nu_{max} \leq c - \nu_{max} + \nu_{max} = c.$$

Therefore, $\hat{\Phi}(x) = \hat{\Phi}(x') = c$. We conclude that for all x with $\Phi(x) \leq c - \nu_{max}$,

$$E[\hat{\Phi}(Q(t+1)) \mid Q(t) = x] - \hat{\Phi}(x) = c - c = 0.$$

2. We now analyze the third summand in Equation (12). For any x, x' with $\Phi(x) > c + \nu_{max}$ and $p(x, x') > 0$, we again obtain from the definition of ν_{max} that

$$\Phi(x') \geq \Phi(x) - \nu_{max} > c + \nu_{max} - \nu_{max} = c.$$

Therefore, $\hat{\Phi}(x) = \Phi(x)$ and $\hat{\Phi}(x') = \Phi(x')$. Also, by assumption, $c + \nu_{max} \geq B$. We conclude that for all x with $\Phi(x) > c + \nu_{max}$,

$$E[\hat{\Phi}(Q(t+1)) \mid Q(t) = x] - \hat{\Phi}(x) = E[\Phi(Q(t+1) \mid Q(t) = x] - \Phi(x) \leq -\gamma,$$

where the last inequality holds since $\Phi(x) > B$ and Φ is a Lyapunov function with drift parameter γ.

3. Using the definition of $\hat{\Phi}$, it is easily seen that for any two states x, x', there are only the following two possibilities: either

$$0 \leq \hat{\Phi}(x') - \hat{\Phi}(x) \leq \Phi(x') - \Phi(x),$$

or

$$\Phi(x') - \Phi(x) \leq \hat{\Phi}(x') - \hat{\Phi}(x) \leq 0.$$

Therefore, for any x, we have

$$\begin{aligned} E[\hat{\Phi}(Q(t+1)) \mid Q(t) = x] - \hat{\Phi}(x) &= \sum_{x':\hat{\Phi}(x') > \hat{\Phi}(x)} p(x, x')(\hat{\Phi}(x') - \hat{\Phi}(x)) \\ &\quad + \sum_{x':\hat{\Phi}(x') \leq \hat{\Phi}(x)} p(x, x')(\hat{\Phi}(x') - \hat{\Phi}(x)) \\ &\leq \sum_{x':\Phi(x') > \Phi(x)} p(x, x')(\Phi(x') - \Phi(x)) \\ &\leq \sum_{x':\Phi(x') > \Phi(x)} p(x, x')\nu_{max} \\ &\leq p_{max}\nu_{max}. \end{aligned}$$

We conclude that for all $x \in \mathcal{X}$, and, in particular, for all x satisfying $c - \nu_{max} < \Phi(x) \leq c - \nu_{max}$, we have

$$E[\hat{\Phi}(Q(t+1)) | Q(t) = x] - \hat{\Phi}(x) \leq p_{max}\nu_{max}.$$

We now incorporate the analysis of these three case into Equation (12), to obtain

$$0 \;\leq\; 0 + \sum_{x:c-\nu_{max} < \Phi(x) \leq c+\nu_{max}} \pi(x) p_{max} \nu_{max} + (-\gamma) \sum_{x:c+\nu_{max} < \Phi(x)} \pi(x).$$

We then use the equality

$$\sum_{x:c-\nu_{max}<\Phi(x)\le c+\nu_{max}} \pi(x) = \sum_{x:c-\nu_{max}<\Phi(x)} \pi(x) - \sum_{x:c+\nu_{max}<\Phi(x)} \pi(x),$$

to replace the first sum of probabilities in the previous inequality, divide by $p_{max}\nu_{max}+\gamma$, and obtain Eq. (10). ∎

We now state and prove a similar result involving Lyapunov functions and lower bounds on tail probabilities.

Lemma 2 *Consider a Markov chain $Q(t)$ with stationary probability distribution π, and suppose that Φ is a lower Lyapunov function with drift γ, such that $E_\pi\Big[\Phi(Q(t))\Big]$ is finite. Then, for any $c \ge 0$, we have*

$$\Pr_\pi\Big\{c \le \Phi(Q(t))\Big\} \ge \frac{p_{min}\nu_{min}}{p_{min}\nu_{min}+2\gamma}\Pr_\pi\Big\{c - \frac{1}{2}\nu_{min} \le \Phi(Q(t))\Big\}, \tag{15}$$

where ν_{min} and p_{min} are defined in (3) and (5).

Proof: Let us select an arbitrary $c \ge 0$. We again consider the function $\hat\Phi(x) = \max(c, \Phi(x))$, and the equilibrium equation (9), which we rewrite as

$$\begin{aligned}
0 \;=\; & \sum_{x:\ \Phi(x)<c-(1/2)\nu_{min}} \pi(x)\Big(E[\hat\Phi(Q(t+1)) \mid Q(t)=x] - \hat\Phi(x)\Big) && (16)\\
+ & \sum_{x:\ c-(1/2)\nu_{min}\le\Phi(x)<c} \pi(x)\Big(E[\hat\Phi(Q(t+1))|Q(t)=x] - \hat\Phi(x)\Big) && (17)\\
+ & \sum_{x:\ c\le\Phi(x)} \pi(x)\Big(E[\hat\Phi(Q(t+1))|Q(t)=x] - \hat\Phi(x)\Big). && (18)
\end{aligned}$$

We now analyze each of the summands in the above identity.

1. For any x with $\Phi(x) \ge c$, we have $\hat\Phi(x) = \Phi(x)$. So,

$$\begin{aligned}
& \sum_{x:\ \Phi(x)\ge c} \pi(x)\Big(E[\hat\Phi(Q(t+1)) \mid Q(t)=x] - \hat\Phi(x)\Big)\\
= & \sum_{x:\ \Phi(x)\ge c} \pi(x)\Big(E[\hat\Phi(Q(t+1)) \mid Q(t)=x] - \Phi(x)\Big)\\
\ge & \sum_{x:\ \Phi(x)\ge c} \pi(x)\Big(E[\Phi(Q(t+1)) \mid Q(t)=x] - \Phi(x)\Big)\\
\ge & \;(-\gamma) \sum_{x:\ \Phi(x)\ge c} \pi(x).
\end{aligned}$$

2. For any x with $\Phi(x) < c - (1/2)\nu_{min}$, we have $\hat{\Phi}(x) = c$. So,

$$\begin{aligned} & \sum_{x:\ \Phi(x) < c-(1/2)\nu_{min}} \pi(x)\Big(E[\hat{\Phi}(Q(t+1)) \mid Q(t) = x] - \hat{\Phi}(x)\Big) \\ = & \sum_{x:\ \Phi(x) < c-(1/2)\nu_{min}} \pi(x)\Big(E[\hat{\Phi}(Q(t+1)) \mid Q(t) = x] - c\Big) \ \geq \ 0. \end{aligned}$$

3. We next consider the case where x satisfies

$$c - (1/2)\nu_{min} \leq \Phi(x) < c. \tag{19}$$

We have $\hat{\Phi}(x) = c$. If x' is such that $p(x, x') > 0$ and $\Phi(x') > \Phi(x)$, then $\Phi(x') \geq \Phi(x) + \nu_{min} \geq c + (1/2)\nu_{min}$. In particular, $\hat{\Phi}(x') = \Phi(x')$ and

$$\hat{\Phi}(x') - \hat{\Phi}(x) \geq c + (1/2)\nu_{min} - c = (1/2)\nu_{min}.$$

If $p(x, x') > 0$ and $\Phi(x') \leq \Phi(x) < c$, then $\hat{\Phi}(x') = c$, and $\hat{\Phi}(x') - \hat{\Phi}(x) = 0$. We conclude that for any x satisfying (19), we have

$$\begin{aligned} E[\hat{\Phi}(Q(t+1)) \mid Q(t) = x] - \hat{\Phi}(x) &= \sum_{x':\ \Phi(x') > \Phi(x)} p(x, x')\Big(\hat{\Phi}(x') - \hat{\Phi}(x)\Big) \\ &\quad + \sum_{x':\ \Phi(x') \leq \Phi(x)} p(x, x')\Big(\hat{\Phi}(x') - \hat{\Phi}(x)\Big) \\ &= \sum_{x':\ \Phi(x') > \Phi(x)} p(x, x')\Big(\hat{\Phi}(x') - \hat{\Phi}(x)\Big) \\ &\geq p_{min}(1/2)\nu_{min}. \end{aligned}$$

We incorporate our analysis of the three summands into Equation (16), to obtain

$$0 \geq \sum_{x: c-(1/2)\nu_{min} < \Phi(x) \leq c} \pi(x)(1/2)p_{min}\nu_{min} + (-\gamma) \sum_{x: c \leq \Phi(x)} \pi(x).$$

We also have

$$\sum_{x: c-(1/2)\nu_{min} \leq \Phi(x) < c} \pi(x) = \sum_{x: c-(1/2)\nu_{min} \leq \Phi(x)} \pi(x) - \sum_{x: c \leq \Phi(x)} \pi(x).$$

Using this and dividing the previous inequality by $(1/2)p_{min}\nu_{min} + \gamma$, we obtain Eq. (15). ■

We now are ready to state and prove the main result of the paper.

Theorem 3 *Consider a Markov chain $Q(t)$ with stationary probability distribution π.*

1. *If there exists a Lyapunov function Φ with drift term $\gamma > 0$, and exception parameter $B \geq 0$, such that $E_\pi\Big[\Phi(Q(t))\Big]$ is finite, then for any $m = 0, 1, \ldots,$ we have*

$$\Pr_\pi\Big\{\Phi(Q(t)) > B + 2\nu_{max} m\Big\} \leq \Big(\frac{p_{max}\nu_{max}}{p_{max}\nu_{max} + \gamma}\Big)^{m+1}.$$

   As a result,

$$E_\pi[\Phi(Q(t))] \leq B + \frac{2p_{max}(\nu_{max})^2}{\gamma}.$$

2. *If there exists a lower Lyapunov function Φ with drift term $\gamma > 0$, such that $E_\pi\Big[\Phi(Q(t))\Big]$ is finite, then for any $m = 0, 1, \ldots$, we have*

$$\Pr_\pi\Big\{\Phi(Q(t)) \geq (1/2)\nu_{min} m\Big\} \geq \Big(\frac{(1/2)p_{min}\nu_{min}}{(1/2)p_{min}\nu_{min} + \gamma}\Big)^{m}.$$

   As a result,

$$E_\pi[\Phi(Q(t))] \geq \frac{p_{min}(\nu_{min})^2}{4\gamma}.$$

Proof: To prove the first part, let $c = B - \nu_{max}$. By applying Lemma 1, we obtain

$$\begin{aligned}
\Pr_\pi\Big\{B < \Phi(Q(t))\Big\} &\leq \frac{p_{max}\nu_{max}}{p_{max}\nu_{max} + \gamma}\Pr_\pi\Big\{B - 2\nu_{max} < \Phi(Q(t))\Big\} \\
&\leq \frac{p_{max}\nu_{max}}{p_{max}\nu_{max} + \gamma}.
\end{aligned}$$

We continue similarly, using $c = B + \nu_{max}$, $c = B + 3\nu_{max}$, $c = B + 5\nu_{max}, \ldots$. By applying again Lemma 1, we obtain the needed upper bound on the tail distribution.

For the expectation part, note that

$$\begin{aligned}
E_\pi[\Phi(Q(t))] &\leq B \cdot \Pr_\pi\Big\{\Phi(Q(t)) \leq B\Big\} + \\
&\quad \sum_{m=0}^{\infty}(B + 2\nu_{max}(m+1))\Pr_\pi\Big\{B + 2\nu_{max} m < \Phi(Q(t)) \leq B + 2\nu_{max}(m+1)\Big\} \\
&= B \cdot \Pr_\pi\Big\{\Phi(Q(t)) \leq B\Big\} + B\sum_{m=0}^{\infty}\Pr_\pi\Big\{B + 2\nu_{max} m < \Phi(Q(t)) \leq B + 2\nu_{max}(m+1)\Big\} \\
&\quad + 2\nu_{max}\sum_{m=0}^{\infty}(m+1)\Pr_\pi\Big\{B + 2\nu_{max} m < \Phi(Q(t)) \leq B + 2\nu_{max}(m+1)\Big\}.
\end{aligned}$$

But

$$\begin{aligned}
&\sum_{m=0}^{\infty}(m+1)\Pr_\pi\Big\{B + 2\nu_{max} m < \Phi(Q(t)) \leq B + 2\nu_{max}(m+1)\Big\} \\
&\qquad = \sum_{m=0}^{\infty}\Pr_\pi\Big\{B + 2\nu_{max} m < \Phi(Q(t))\Big\}.
\end{aligned}$$

Applying the bounds on the tail distribution, we obtain

$$\begin{aligned} E_\pi[\Phi(Q(t))] &\leq B + 2\nu_{max} \sum_{m=0}^{\infty} \Big(\frac{p_{max}\nu_{max}}{p_{max}\nu_{max} + \gamma} \Big)^{m+1} \\ &= B + \frac{2p_{max}(\nu_{max})^2}{\gamma}. \end{aligned}$$

To prove the second part of the theorem, let $c = \nu_{min}/2$, $c = \nu_{min}$, $c = 3\nu_{min}/2, \ldots$. Then, by applying Lemma 2, we obtain

$$\begin{aligned} \mathrm{Pr}_\pi \Big\{ \Phi(Q(t))(1/2)\nu_{min}m \Big\} &\geq \Big(\frac{(1/2)p_{min}\nu_{min}}{(1/2)p_{min}\nu_{min} + \gamma} \Big)^m \mathrm{Pr}_\pi \Big\{ \Phi(Q(t)) \geq 0 \Big\} \\ &= \Big(\frac{(1/2)p_{min}\nu_{min}}{(1/2)p_{min}\nu_{min} + \gamma} \Big)^m. \end{aligned}$$

For the expectation part, note that

$$E_\pi[\Phi(Q(t))] \geq \frac{1}{2} \sum_{m=0}^{\infty} \nu_{min} m \mathrm{Pr}_\pi \Big\{ (1/2)\nu_{min}m \leq \Phi(Q(t)) < (1/2)\nu_{min}(m+1) \Big\}.$$

But

$$\begin{aligned} & \frac{1}{2} \sum_{m=0}^{\infty} \nu_{min} m \mathrm{Pr}_\pi \Big\{ (1/2)\nu_{min}m \leq \Phi(Q(t)) < (1/2)\nu_{min}(m+1) \Big\} \\ = \; & \frac{1}{2}\nu_{min} \sum_{m=1}^{\infty} \mathrm{Pr}_\pi \Big\{ (1/2)\nu_{min}m \leq \Phi(Q(t)) \Big\}. \end{aligned}$$

Using the lower bounds on the tail probability distribution we obtain

$$E_\pi[\Phi(Q(t))] \geq \frac{1}{2}\nu_{min} \sum_{m=1}^{\infty} \Big(\frac{(1/2)p_{min}\nu_{min}}{(1/2)p_{min}\nu_{min} + \gamma} \Big)^m = \frac{p_{min}(\nu_{min})^2}{4\gamma}.$$

This completes the proof of the theorem. ■

4 Applications

In this section, we apply Theorem 3 to finding bounds for random walks in the positive orthant, and for multiclass single station markovian queueing systems.

4.1 Random Walks in the Nonnegative Orthant

Theorem 3 provides a simple tool for analyzing the stationary probability distribution of homogeneous random walks in the nonnegative integer lattice Z_+^n.

One-dimensional random walk in Z_+

To illustrate the method we start with a simple case of an one-dimensional random walk in Z_+, where exact results can be easily derived. In this way we obtain insight on the quality of the bounds our method produces. Suppose that

$$\Pr\Big\{Q(t+1) = m+1 \mid Q(t) = m\Big\} = p, \qquad m = 0, 1, 2, \dots,$$

$$\Pr\Big\{Q(t+1) = m-1 \mid Q(t) = m\Big\} = 1-p, \qquad m = 1, 2, \dots,$$

and

$$\Pr\Big\{Q(t+1) = 0 \mid Q(t) = 0\Big\} = 1-p.$$

We assume that $p < 1/2$, to ensure stability. For this Markov chain, and with $\Phi(x) = x$, we have $E_\pi[Q(t)] < \infty$, $B = 0$, $\gamma = 1-2p$, $\nu_{max} = \nu_{min} = 1$, and $p_{max} = p_{min} = p$. Applying Theorem 3, we obtain

$$\Pr_\pi\Big\{Q(t) > 2m\Big\} = \Pr_\pi\Big\{Q(t) \geq 2m+1\Big\} \leq \Big(\frac{p}{p+1-2p}\Big)^{m+1} =$$

$$\frac{p}{1-p}\Big(\sqrt{\frac{p}{1-p}}\Big)^{2m}$$

and

$$\Pr_\pi\Big\{Q(t) \geq \frac{1}{2}m\Big\} \geq \Big(\frac{(1/2)p}{(1/2)p+1-2p}\Big)^m = \Big(\frac{p}{2-3p}\Big)^m.$$

Notice that rate of decay $(\frac{p}{1-p})^{\frac{1}{2}}$ in the upper bound is slower than the correct one $\frac{p}{1-p}$ (which can be derived by elementary methods). Nevertheless, this factor converges to zero as p decreases to zero, which is in agreement with the exact result. Similarly, the factor $\frac{p}{2-3p}$ in the lower bound is not exact, but converges to one, when p converges to $1/2$. Regarding expectations, Theorem 3 yields

$$\frac{p}{4(1-2p)} \leq E_\pi[Q(t)] \leq \frac{2p}{1-2p}.$$

Comparing to the correct value of $E_\pi[Q(t)]$, which is $\frac{p}{1-2p}$, we see that our bounds are accurate within a small constant factor.

State-homogeneous random walk in Z_+^n with bounded steps.

We now consider a random walk $Q(t) = (Q_1(t), Q_2(t), \dots, Q_n(t)) \in Z_+^n$, of the form described below. For any $\Lambda \subset \{1, 2, \dots, n\}$, let

$$Z_\Lambda = \Big\{x \in Z_+^n : x_i > 0, \text{ for } i \in \Lambda,\ x_i = 0, \text{ for } i \notin \Lambda\Big\}.$$

As in [10] we call these sets *faces*. We assume that our random walk is *state-homogeneous* in the following sense. For each $\Lambda \subset \{1,2,\ldots,n\}$, there exists a vector $\theta_\Lambda \in \Re^n$ such that for all $x \in Z_\Lambda$

$$E[Q(t+1)|Q(t)=x] = x + \theta_\Lambda.$$

Namely, the *expected step* of the random walk depends only on the face the walk is currently on. This includes the special case where the *distribution of the step* during a transition depends only on the face a two-dimensional random walk above). We refer to that special case as *state-homogeneous in distribution*.

We assume that our random walk has bounded steps, that is,

$$\nu \equiv \sup\Big\{|x'-x| : p(x,x') > 0\Big\} < \infty,$$

where we use the notation $|x| = \sum_{i=1}^n |x_i|$. Our interest in state-hmogeneous random walks with bounded steps is because they can be used to model single-class exponential type queueing networks (classical Jackson networks) and multiclass exponential type queueing networks, operating under a preemptive priority scheduling policy. We now establish that whenever there exists some linear Lyapunov function, certain exponential bounds hold on the stationary probability distribution of such random walks.

Theorem 4 *Consider a state-homogeneous random walk in Z_+^n with expected step θ_Λ for each nonempty $\Lambda \subset \{1,2,\ldots,n\}$. Suppose that there exists a nonnegative vector $L \in \Re^n$ and some $\gamma > 0$, such that for all non-empty $\Lambda \subset \{1,2,\ldots,n\}$, we have*

$$L'\theta_\Lambda \le -\gamma.$$

Then, the random walk is positive recurrent. The corresponding stationary probability distribution π satisfies

$$\Pr_\pi\Big\{L'Q(t) \ge 2\nu m\Big\} \le \Big(\frac{L_{max}\nu}{L_{max}\nu + \gamma}\Big)^m, \qquad m = 0,1,2,\ldots,$$

and

$$E_\pi[L'Q(t)] \le \frac{2(L_{max}\nu)^2}{\gamma},$$

where $L_{max} = \max_{1\le i\le n}\{L_i\}$.

Proof: The assumptions of the theorem imply that $\Phi(x) = L'x$ is a Lyapunov function with drift γ and exception parameter $B = 0$. Positive recurrence then follows from the existence of the Lyapunov function (see [23]). Note that for this Lyapunov function, we have $\nu_{max} \le L_{max}\nu$. Also $p_{max} \le 1$. The result follows directly from the first part of Theorem 3. ∎

We now use the second part of Theorem 3 to get lower bounds that apply to state-homogeneous in distribution random walks (possibly, with $\nu = \infty$), using practically *any* nonnegative linear function with integral components. As mentioned above, for random walks that are state-homogeneous in distribution the distribution of a step depends only on the face that the walk is currently on. Formally, for every $\Lambda \subset \{1, 2, \ldots, n\}$, for every $x_1, x_2 \in Z_\Lambda$ and for every $\Delta x \in Z^n$, the probabilities of transitions from x_1 to $x_1 + \Delta x$ and from x_2 to $x_2 + \Delta x$ are the same, that is,

$$p(x_1, x_1 + \Delta x) = p(x_2, x_2 + \Delta x).$$

Theorem 5 *Consider a positive recurrent random walk which is state-homogeneous in distribution. Let θ_Λ be the expected steps and let π be the stationary distribution. For any nonnegative n-dimensional vector L with integer components, we define*

$$\gamma = -\min_\Lambda \{L'\theta_\Lambda : L'\theta_\Lambda < 0\}.$$

Then

$$\Pr_\pi\Big\{L'Q(t) \geq \frac{m}{2}\Big\} \geq \Big(\frac{(1/2)p^L_{min}}{(1/2)p^L_{min} + \gamma}\Big)^m,$$

for all $m = 0, 1, 2, \ldots,$ and

$$E_\pi[L'Q(t)] \geq \frac{p^L_{min}}{4\gamma},$$

where

$$p^L_{min} = \min_\Lambda \Big\{\Pr\{L'Q(t+1) - L'Q(t) > 0 \mid Q(t) \in Z_\Lambda\}\Big\}.$$

Remark. Note that p^L_{min} depends only on the distributions of 2^n different step vectors Δx, and can be calculated in finite time by considering the 2^n different faces Z_Λ.

Proof: We consider a linear lower Lyapunov function $\Phi(x) = L'x$. By definition, γ is a drift of this function. The minimal increase of the value of this Lyapunov function is not smaller than one, since both L and $Q(t)$ have integer components. Also for this lower Lyapunov function, we have $p_{min} = p^L_{min}$. The result is obtained then by applying the second part of Theorem 3. ∎

4.2 Single Station Markovian Queueing System

In this subsection we apply the results of Theorem 3 to a class of single station Markovian queueing systems, operating under an arbitrary Markovian policy. We construct upper and lower bounds on tail probabilities and expectation of queue lengths and, in particular, we show that the tail probabilities decay exponentially.

The system is described as follows. There is a single arrival stream of customers which are served at a single processing station n times, and then leave the system. Such systems are often called re-entrant lines (see [15]). The arrival process is assumed to be Poisson with rate λ. The service times at subsequent stages (classes) $k = 1, 2, \ldots, n$ are independent and exponentially distributed, with rates $\mu_1, \mu_2, \ldots, \mu_n$ respectively. The processing station serves the customers at different stages in an arbitrary fashion, subject to only two conditions:

(a) The scheduling policy is work conserving: the station must be busy whenever there are customers waiting for service.

(b) The scheduling policy is Markovian. Specifically, the selection of which customer to serve next is a function of the vector of all queue lengths, which constitutes the state of the system. We allow scheduling policies to be preemptive. In fact, most nonpreemptive policies would not have the Markovian property as defined here.

The exponentiality assumption on the interarrival and service times, together with the Markovian assumption, allow us to resort to *uniformization* (see Lippman [19]). This is a method for sampling a continuous time system to obtain a discrete time system with the same steady-state probability distribution. We rescale the parameters, so that $\lambda + \sum_{i=1}^{n} \mu_i = 1$ and consider a superposition of $n+1$ Poisson processes with rates $\lambda, \mu_1, \ldots, \mu_n$, respectively. The arrivals of the first Poisson process correspond to external arrivals into the network. The arrivals of the process with rate μ_k correspond to service completions of class k customers, if the server is actually working on a class k customer, or they correspond to "imaginary" service completions of an "imaginary" customer, if the server is currently idle or working on customers from other classes. Let $\tau_s, s = 1, 2, \ldots$ be the sequence of event times, that is, times at which one of these $n+1$ Poisson processes registers an arrival. Let $Q(t) = (Q_1(t), Q_2(t), \ldots, Q_n(t))$ be the vector of (random) queue lengths at time t. Since the scheduling policy is Markovian, the process $Q(\tau_s), s = 0, 1, \ldots,$ is a Markov chain. It is well known that every work conserving scheduling policy is stable (i.e., the Markov chain $Q(\tau_s)$ is positive recurrent), whenever

$$\rho \equiv \sum_{k=1}^{n} \frac{\lambda}{\mu_k} < 1, \tag{20}$$

and every policy is unstable, if this condition is violated. Let $\rho_k = \lambda/\mu_k$ and

$$\rho_k^+ = \sum_{i=k}^{n} \rho_i.$$

In particular, (20) can be rewritten as $\rho = \rho_1^+ < 1$. From now on, we assume that (20) holds. Let π denote a stationary distribution of the original Markov chain $Q(t)$, corresponding to some work conserving policy w. As a result of uniformization, this is also the stationary distribution of the embedded Markov process, on which we can now concentrate.

Any scheduling policy w satisfying the Markovian assumption can be described as a function $w : Z_+^n \to \{0,1\}^n$, where for any $x \in Z_+^n$, the vector $w(x) = (w_1(x), \ldots, w_n(x))$ is interpreted as follows: we have $w_k(x) = 1$ if the server works on a customer of class k, and $w_k(x) = 0$, otherwise. We require that $w_k(x) = 1$ only if $x_k > 0$. We will use e_k, for $k = 1, \ldots, n$, to denote the k-th unit vector. We also adopt the convention $e_0 = e_{n+1} = 0$. Let us assume that the parameters have been rescaled so that $\lambda + \sum_{i=1}^n \mu_i = 1$. Then, it is easily seen that the transitions of the embedded Markov chain $Q(\tau_s)$ take place as follows:

$$Q(\tau_{s+1}) = \begin{cases} Q(\tau_s) + e_1, & \text{with probability } \lambda, \\ Q(\tau_s) + e_{k+1} - e_k, & \text{with probability } \mu_k w_k(Q(\tau_s)), \\ Q(\tau_s), & \text{with probability } \sum_k \mu_k (1 - w_k(Q(\tau_s)). \end{cases} \tag{21}$$

Consider the vector $L = (\rho_1^+, \rho_2^+, \ldots, \rho_n^+)/\lambda$. We then have

$$E[L'Q(\tau_{s+1})|Q(\tau_s)] = L'Q(\tau_s) + \rho_1^+ + \frac{1}{\lambda}\sum_{k=1}^n \mu_k w_k(Q(\tau_s))(\rho_{k+1}^+ - \rho_k^+).$$

But $\rho_{k+1}^+ - \rho_k^+ = -\rho_k = -\lambda/\mu_k$. Also, whenever $Q(\tau_s) \neq 0$, work conservation implies that $\sum_{k=1}^n w_k(Q(\tau_s)) = 1$. Therefore,

$$E[L'Q(\tau_{s+1})|Q(\tau_s)] = L'Q(\tau_s) + \rho - 1,$$

whenever $Q(\tau_s) \neq 0$.

Assuming that $\rho < 1$, we see that $\Phi(x) = L'x$ is both a Lyapunov and a lower Lyapunov function, with drift $\gamma = 1 - \rho$ and exception parameter $B = 0$, under any work conserving Markovian policy. Note that

$$\Phi(Q(t)) = \frac{1}{\lambda}\sum_{k=1}^n \rho_k^+ Q(t)$$

is the *workload* in the system, i.e., the total time required to process existing customers, assuming that no future arrivals occur. We now determine the parameters ν_{max}, ν_{min}, p_{max} associated with our Lyapunov function $\Phi(x) = L'x$. The arrival of a new customer into the system corresponds to a positive increase, equal to ρ/λ. The departure of a class k customer corresponds to a negative change, equal to $\rho_{k+1}^+ - \rho_k^+ = -\rho_k$. Therefore, $\nu_{min} = \rho/\lambda$. Also, since $\rho \geq \rho_k$ for all k, we also have $\nu_{max} = \rho/\lambda$. Furthermore, $p_{max} = p_{min} = \lambda$. The following result is then obtained.

Theorem 6 *In a single station re-entrant line with arrival rate λ and service rates $\mu_1, \mu_2, \ldots, \mu_n$, for which $\rho < 1$, the following bounds hold on the steady-state number of customers, under any work conserving Markovian policy:*

$$\Pr_\pi\left\{\frac{\sum_{i=1}^n \rho_i^+ Q_i(t)}{2\rho} > m\right\} \leq \rho^{m+1},$$

and

$$\Pr_\pi\left\{\frac{2\sum_{i=1}^n \rho_i^+ Q_i(t)}{\rho} > m\right\} \geq \left(\frac{\rho}{2-\rho}\right)^m,$$

for $m = 0, 1, 2, \ldots$. Also ,

$$\frac{\rho^2}{4(1-\rho)} \leq E_\pi\left[\sum_{i=1}^n \rho_i^+ Q_i(t)\right] \leq \frac{2\rho^2}{1-\rho}.$$

Proof: The result is obtained by applying Theorem 3 and cancelling λ. ■

Remarks.

1. The expected value of $\sum_{k=1}^n \rho_k^+ Q(t)$ can be calculated to be exactly equal to $\frac{\rho^2}{1-\rho}$, using other techniques (see, for example, [3]). Thus, the bounds provided by our methods are are correct up to a small constant factor. However, the bounds on the tail distributions are new results.

2. Theorem 6 can be extended to multiclass single station systems with multiple arrival streams and probabilistic routing. The bounds are again explicit, but are not expressed solely in terms of the load factors ρ, ρ_k (see [11]).

5 Conclusions

We have proposed a general method for the performance analysis of infinite state Markov chains. By considering suitably thresholded Lyapunov functions and by writing down certain equilibrium equations, we have derived exponential type upper and lower bounds on the stationary distribution of a Markov chain, under the assumption that the Lyapunov function has uniformly bounded jump sizes. We have applied our bounds to homogeneous random walks in the nonnegative orthant and to Markovian multiclass single station queueing systems.

It would be interesting to investigate whether Theorem 3 can be extended from Markov chains to continuous-time processes, like reflected Brownian motion or to non-Markovian processes. Also, it would be interesting to determine whether the bounded jump size condition can be relaxed. This would allow us to use broader classes of Lyapunov functions, e.g., we might apply quadratic Lyapunov functions to the study of random walks.

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