**Abstract**—We seek to minimize both the retroactivity to the output and the retroactivity to the input of a phosphorylation-based insulation device by finding an optimal substrate concentration. Characterizing and improving the performance of insulation devices brings us a step closer to their successful implementation in biological circuits, and thus to modularity. Previous works have mainly focused on attenuating retroactivity effects to the output using high substrate concentrations. This, however, worsens the retroactivity to the input, creating an error that propagates back to the output. Employing singular perturbation and contraction theory tools, this work provides a framework to determine an optimal substrate concentration to reach a tradeoff between the retroactivity to the input and the retroactivity to the output.

I. INTRODUCTION

Understanding modularity is one of the most pressing matters in systems biology. Modularity is the property according to which the input/output behavior of a system does not change upon interconnection and has been proposed as one of the possible levels of biological organization [1]. It was suggested, however, that biomolecular systems are not always modular because impedance-like effects at the interconnections, called retroactivity, alter the system’s behavior [2][3] [4]. Fig. 1 shows the system model introduced in [3] to explicitly account for retroactivity. System II, with input \( u \) and output \( y \), is subject to retroactivity to the output \( s \), due to interconnection to a downstream system, and applies retroactivity to the input \( r \) to its upstream system. In a biological circuit, this occurs, for example, when a protein is used as a transcription factor. The downstream process uses the protein in its reactions, directly affecting its dynamics. Retroactivity has also been related to fan-out [5], which is defined as the maximum regulation capacity of a transcription factor.

![Fig. 1. System II with input u, output y, retroactivity to the input r and retroactivity to the output s.](image)

From an engineering point of view, an insulation device can be used to decouple the dynamics of interconnected components. An insulation device is analogous to an insulating amplifier in electrical circuits, where a signal is transmitted to a downstream system while minimizing the impedance effects. It has been suggested that signaling pathways, such as the mitogen-activated protein kinase (MAPK) cascade, can be used as amplifiers and placed in negative feedback loops to obtain insulation from downstream loading [6]. It was shown in [3] that phosphorylation cycles can attenuate retroactivity to the output through a mechanism similar to high gain feedback even without an explicit negative feedback. And indeed, *in vitro* implementations have confirmed this theoretical prediction [7]. Other implementations of insulation devices have also been realized *in vitro*, where the dynamics of a biological oscillator were successfully decoupled from the dynamics of DNA tweezers using a genelet amplifier circuit [8].

An ideal insulation device has the retroactivity to the input \( r \) in Fig. 1 close to zero and the effect of the retroactivity to the output \( s \) on \( y \) is completely attenuated. A fundamental question is whether these two requirements are in conflict with each other. Here, we study this problem when the insulation device is realized with a phosphorylation cycle. It was shown before [3] that as the amounts of cycle substrate and phosphatase are increased, the effect of the retroactivity to the output on the cycle output protein could be attenuated. However, increased amounts of cycle substrate result in an increased retroactivity to the input. In this paper, we propose to parameterize the error of the insulation device output with the cycle substrate and phosphatase concentrations to determine an optimal amount that minimizes both retroactivity effects.

Our approach to characterize this tradeoff is based on singular perturbation and contraction theory. We determine an upper bound on the steady state error between the output of the insulation device under study and an ideal insulation device. This upper bound is a function of the substrate and phosphatase concentrations and can be minimized with respect to these variables. We then show through simulation that the upper bound that we have calculated is tight.

This paper is organized as follows. In Section II, the mathematical tools needed for the problem solution are provided. In Section III, the system model and problem are presented in terms of the chemical reactions and differential equations describing the phosphorylation cycle. Also, the definition of the input error and output error of the insulation device are given. In Section IV, a general solution approach using model reduction techniques is presented. Sections V and VI provide the input and output error in terms of the cycle substrate and phosphatase concentrations. In Section VII, the total error of the insulation device is provided.

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II. MATHEMATICAL TOOLS

Theorem 1. (Contraction Theorem): Adapted from [9]. Consider the $n$-dimensional deterministic system $\dot{x} = f(x, t)$, where $f$ is a smooth nonlinear function. The system is said to be contracting if any two trajectories, starting from different initial conditions, converge exponentially to each other. A sufficient condition is the existence of a matrix measure, $m$, such that there is a $\lambda > 0$ with $m \left( \frac{\partial f(x,t)}{\partial x} \right) \leq -\lambda$ for all $x$ and for all $t \geq 0$. The scalar $\lambda$ defines the contraction rate of the system.

Throughout this paper, the vector norm $|\cdot|$ will refer to the $\ell^2$-norm given by $|x|_2 = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$ and $m_2(A)$ the induced matrix measure given by $m_2(A) = \max_i \left( \lambda_i \left\{ \frac{A\lambda_i A^*}{\lambda_i} \right\} \right)$ where $\lambda_i$ denotes the matrix’s $i^{th}$ eigenvalue.

Lemma 1. (Robustness): Adapted from [9]. Assume that the system $\dot{x} = f(x, t)$ is contracting, with contraction rate $\lambda$, and consider the perturbed system

$$\dot{x}_p = f(x_p, t) + d(x_p, t),$$

where $d(x_p, t)$ is bounded, so there is a $\bar{d} \geq 0$ such that $|d(x_p, t)| \leq \bar{d}$ for all $x_p$ and for all $t \geq 0$. Then, the trajectory of the perturbed system satisfies

$$|x_p(t) - x(t)| \leq e^{-\lambda t} |x_p(0) - x(0)| + \frac{\bar{d}}{\lambda}.$$

Lemma 2. Adapted from [10]. Assume that the system $\dot{x} = f(x, z(t))$ is partially contracting in $x$ with contraction rate $\lambda_x$, so that the solution of the system $f(x_s, z(t)) = 0$ can be written as $x_s = \gamma(z)$, i.e., there is a unique global mapping between $x$ and $z$. Assume further that there exists a $\bar{d} \geq 0$ such that $|\frac{\partial \gamma(z)}{\partial z} \dot{z}| \leq \bar{d}$ for all $x$, for all $z$ and for all $t \geq 0$. Then, any trajectory $x(t)$ satisfies

$$|x(t) - \gamma(z(t))| \leq e^{-\lambda_x t} |x(0) - \gamma(0)| + \frac{\bar{d}}{\lambda_x}.$$

Proof. Let $x$ be the solution of $\dot{x} = f(x, z(t))$ while $x_s = \gamma(z(t))$ is the solution of the “perturbed” system $\dot{x}_s = f(x_s, z(t)) + \frac{\partial \gamma(z)}{\partial z} \dot{z}$ with disturbance $\frac{\partial \gamma(z)}{\partial z} \dot{z}$. Applying the result (3) from Lemma 1 yields bound (4).

Proposition 1. Consider the system (1)-(2) in Lemma 1 and let $|d(x_p, t)| \leq C_0 + \sum_{k=1}^{n} C_k e^{-\lambda_k t}$, $x_0 = |x(0) - x_p(0)|$, the upper bound on $|x(t) - x_p(t)|$ is given by

$$|x(t) - x_p(t)| \leq x_0 e^{-\lambda t} + \frac{C_0}{\lambda} + \sum_{k=1}^{n} C_k e^{-\lambda_k t}.$$

Proof. Let $X(t) = |(x_p(t) - x(t))|$, then $\frac{dx}{dt} + \lambda X \leq |d(x_p(t))| \leq C_0 + \sum_{k=1}^{n} C_k e^{-\lambda_k t}$ and $e^{\lambda t} X(0) \leq e^{\lambda t} (C_0 + \sum_{k=1}^{n} C_k e^{-\lambda_k t})$. The differential equation can now be integrated to obtain (5) where the terms with negative coefficients were neglected for the approximation.

III. SYSTEM MODEL AND PROBLEM

A phosphorylation cycle consists of a set of two reversible enzymatic reactions, where the activation and deactivation of a substrate through the addition/removal of a phosphate group is used to transmit information to a downstream system [11]. Throughout this work, for a given species $X$ its concentration is denoted by $X$ (italics). In a phosphorylation cycle, a kinase labeled $Z$, regulated by $k(t)$, activates substrate $X$ through a phosphate transfer reaction to form $X^*$, while $Y$ deactivates $X^*$ to form $X$. Protein $X^*$ also regulates a downstream system by binding to sites $P$ forming complex $C$. These sites can be DNA promoter sites if $X^*$ is a transcription factor or they can belong to a substrate if $X^*$ is an active kinase. The chemical reactions for the system are:

$$\phi \xrightarrow{k_0} Z, \quad Z \xrightarrow{\phi} X + Z, \quad \beta_1 X + \beta_2 C_1 \xrightarrow{k_1} X^* + Z,$$

where $X^* + Y \xrightarrow{k_{y}1} C_2 \xrightarrow{k_2} X + Y$, and $X^* + p \xrightarrow{k_{p}n} C$, in which $C_1$ is the complex formed by the substrate $X$ kinase $Z$ and $C_2$ is the complex formed by the protein $X^*$ and phosphatase $Y$. The assumed conservation laws are: $X_T = X + X^* + C_1 + C_2 + C$, $Y_T = Y + C_2$, and $p_T = p + C$.

The ODE model of the phosphorylation cycle is given by:

$$\frac{dZ}{dt} = k(t) - \delta Z - \beta_1 X T \left( 1 - \frac{X^*}{X_T} - \frac{C_1}{X_T} - \frac{C_2}{X_T} - \frac{C}{X_T} \right) + (\beta_2 + k_1) C_1,$$

$$\frac{dC_1}{dt} = \beta_2 Z T \left( 1 - \frac{X^*}{X_T} - \frac{C_1}{X_T} - \frac{C_2}{X_T} - \frac{C}{X_T} \right),$$

$$\frac{dC_2}{dt} = -(k_2 + \alpha_2) C_2 + \alpha_1 Y_T X^* \left( 1 - \frac{C_2}{Y_T} \right),$$

$$\frac{dX^*}{dt} = k_1 C_1 + \alpha_2 C_2 - \alpha_1 Y_T X^* \left( 1 - \frac{C_2}{Y_T} \right),$$

$$\frac{dC}{dt} = k_{off} C - k_{on} X^*(p_T - C).$$

Here, $r$ represents the retroactivity to the input and $s$ represents the retroactivity to the output. One can abstract the signal flow in (6) using system $\Sigma$ in Fig. 2. Signal $Z$ drives the $X^*$ dynamics through complex $C_1$ while the binding and unbinding reaction of $Z$ with $X$ creates retroactivity $r$ in the $Z$ dynamics. Similarly, $X^*$ drives the $C$ dynamics downstream, while being affected by the retroactivity $s$. An ideal insulation device should behave as system $\Sigma_i$ in Fig. 2 where the terms under brace $r$ and $s$ in (6) were set to zero. The key tunable parameters in this system are $X_T$ and $Y_T$, which will be kept at a constant ratio $Y_T/X_T = \rho$ throughout the analysis. We seek to adjust the values of these parameters in such a way that the behavior of the system is close to that of an ideal insulation device. This can be better appreciated in Fig. 3, where different substrate concentrations are tested using a sinusoidal input $k(t)$ on system (6). The black line describes the ideal behavior $X_T^*$ given by $r, s = 0$. The red line is the behavior of $X^*$ in the system having $r, s \neq 0$.
As we can see from Fig. 3, having too high or too low values of $X_T$ leads to an error between the actual $X^*$ and the ideal $X^*_T$ device responses. The reason being that a large $X_T$ concentration applies a load to the $Z(t)$ dynamics changing the nominal signal $Z(t)$, while attenuating the effect of $p_T$ on the $X^*(t)$ dynamics. This tradeoff between minimizing the effect of $r$ and $s$ is studied in this paper. Specifically, the total output error $\Delta X_{TOT}^*(t) := X^*(t) - X^*_T(t)$, is quantified by determining positive functions, $A(X_T)$, $B(X_T)$, $\lambda(X_T)$ such that $|\Delta X_{TOT}^*(t)| \leq A(X_T)e^{-\lambda(X_T)t} + B(X_T)$.

IV. SOLUTION APPROACH

This problem will be solved by quantifying the errors in $X^*$ due to $r$ and $s$ in (6) to find $A(X_T)$, $B(X_T)$ and $\lambda(X_T)$. To this end, the virtual system $\Sigma_0$ in Fig. 2 is obtained from $\Sigma$ by setting $s = 0$. The output error is defined as $\Delta X^*_0(t) := X^*(t) - X^*_0(t)$, which accounts for the error in $X^*(t)$ only due to retroactivity $s$. Likewise, the virtual system $\Sigma_1$ in Fig. 2 is obtained from $\Sigma$ by setting $r = 0$ and $s = 0$. The input error, defined as $\Delta X^*_1(t) := X^*_0(t) - X^*_1(t)$, accounts for the error in $X^*_0(t)$ due only to retroactivity to the input $r$. It will be shown that the total output error, given by $\Delta X^*_{TOT}(t) := X^*(t) - X^*_T(t)$, can be upper bounded by $|\Delta X^*_{TOT}(t)| \leq |\Delta X^*_0(t)| + |\Delta X^*_1(t)|$. We proceed to separately determine the output error and input error.

First, a system order reduction will be performed through singular perturbation to obtain a two-state variable model for $Z(t)$ and $X^*(t)$. Processes in system (6) occur in three timescales [13]. The slowest timescale is that of the kinase dynamics due to protein production and decay and the intermediate timescale is that of phosphorylation. The fastest timescale is that of the binding and unbinding reactions to form complexes $C_1$, $C_2$ and $C$. Thus, singular perturbation parameters $\epsilon_1 := \delta/k_1$ and $\epsilon_2 := \delta/k_{off}$ are selected so that $\epsilon_2 \ll \epsilon_1 \ll 1$. We define the parameters: $k_d := k_{off}/k_{on}$, $b_1 := \beta_1 \epsilon_2 / \delta$, $a_1 := \alpha_1 \epsilon_2 / \delta$, $b_2 := \beta_2 \epsilon_2 / \delta$, $a_2 := \alpha_2 \epsilon_2 / \delta$, and $k_x := k_2/k_1$. The transformation $w := Z + C_1$ and $y := X^* + C_2 + C$ is also performed on system (6), converting it to standard singular perturbation form [14]:

$$
\begin{align*}
\frac{dw}{dt} &= k(t) - \delta(w - C_1) \\
\epsilon_1 \frac{dy}{dt} &= \delta C_1 - k_x \delta C_2 \\
\epsilon_2 \frac{dC_1}{dt} &= \delta b_1 (w - C_1)(X_T - y - C_1) - \delta(b_2 + \epsilon_2 / \epsilon_1) C_1 \\
\epsilon_2 \frac{dC_2}{dt} &= \delta a_1 (Y_T - C_2)(y - C_2 - C) - \delta(a_2 + k_x \epsilon_2 / \epsilon_1) C_2 \\
\frac{dC}{dt} &= \delta(b_2 + \delta k_x) (y - C_2 - C)(p_T - C) - \delta C.
\end{align*}
$$

We let $Z(t, \epsilon_1, \epsilon_2)$ and $X^*(t, \epsilon_1, \epsilon_2)$ denote the $Z$ and $X^*$ trajectories of system (7) when transformed back to the original coordinates. This system is the same as described in Example 1 of [13]. Since it satisfies all the required conditions, one can use Lemma 2 (Case 1) of [13], which performs a nested application of Tikhonov’s singular perturbation Theorem, to determine the reduced order dynamics.

V. INPUT ERROR

To determine the input error $\Delta X^*_1 = X^*_0(t) - X^*_1(t)$, the effect of $r$ in the reduced order dynamics of $Z(t)$ will be analyzed. The error produced by $r$ will be written as $\Delta Z(t) := Z(t) - Z_I(t)$, which acts as a disturbance in the dynamics of $X^*_0(t)$ leading to the error $\Delta X^*_1(t)$.

The kinase dynamics evolve in the slowest timescale, thus singular perturbation is performed by setting $\epsilon_1 = 0$ and $\epsilon_2 = 0$. Defining the phosphorylation and dephosphorylation dissociation constants as $k_{d1} = \beta_2 / \beta_1$ and $k_{d2} = \alpha_2 / \alpha_1$, respectively, and assuming $X^* \ll k_{d2}$, $k_d$, and $p_T \ll X^*_T$ (thus working in the linear regime of the Michaelis-Menten functions), the slow manifold is given by $\bar{X}^* = \bar{\psi}_x(\bar{Z}) := \frac{X_T k_{d2}}{Z[k_{d2} r + (k_1 + 1)Y_T] + k_1 k_{d1} Y_T}$, $\bar{C}_2 = \psi_c(\bar{\psi}_x(\bar{Z})) := \frac{\Sigma_2}{k_{d2}} \psi_x(\bar{Z})$, $\bar{C}_1 = \psi_c(\psi_x(\bar{Z})) := \frac{\Sigma_2}{k_{d2}} \psi_x(\bar{Z})$, $C = \psi_c(\bar{\psi}_x(\bar{Z})) := \frac{\Sigma_2}{k_{d2}} \psi_x(\bar{Z})$. Variables $\bar{X}$ and $\bar{Z}$ denote the approximation of $X$ and $Z$ in system (7) once $\epsilon_1 = 0$ and $\epsilon_2 = 0$. The
where

\[ \frac{dZ}{dt} = (1 - R_z(\tilde{Z}))f_z(\tilde{Z}, k(t)), \]

(8)

where

\[ R_z(\tilde{Z}) := \frac{k_{d1}X_T}{\left\{ \frac{\partial \Delta}{\partial X_T} + \frac{(k_1 + 1)}{k_2} \right\}}^2 + k_{d1}X_T, \]

\[ f_z(\tilde{Z}, k(t)) := k(t) - \tilde{\Delta}Z. \]

(9)

By proof of Case(1) in Lemma 2 of [13], one has that

\[ |\tilde{Z}(t) - Z(t, \epsilon_1, \epsilon_2)| = O(\epsilon_1) + O(\epsilon_2), \]

so for \( \epsilon_1, \epsilon_2 \to 0 \), \( \tilde{Z}(t) \) will be taken as a good approximation of \( Z(t, \epsilon_1, \epsilon_2) \), and be denoted by \( Z(t) \) with abuse of notation.

A. Bound for \( \Delta Z(t) \)

From (8), it is notable that the reduced input dynamics have the form of a nominal contracting system with an additive disturbance. The nominal or isolated system is given by setting \( R_z = 0 \) in (8), that is,

\[ \frac{dZ}{dt} = f_z(Z, k(t)). \]

(10)

The connected or perturbed \( Z \) dynamics are given by

\[ \frac{dZ}{dt} = f_z(Z, k(t)) + h_z(Z, k(t)), \]

(11)

where the expression \( h_z(Z, k(t)) \) has been defined as

\[ h_z(Z, k(t)) := -R_z(Z)f_z(Z, k(t)). \]

(12)

In order to apply the robustness result given in Lemma 1 to find a bound on \( \Delta Z \), we first need a bound on the perturbation \( h_z(Z, k(t)) \).

Claim 1. Define \( \tilde{k} := \max_{t \geq 0}|k(t)| \) and assume that

\[ Y_T > \max\left\{ \frac{2k_{d2}/k_1}{\tilde{k}k_{d2}/\delta}, \frac{\tilde{k}k_{d2}/\delta}{k_{d2} + (k_1 + 1)/\delta} \right\} \quad \text{and} \quad k_{d1} > \max\left\{ \frac{4(k_1 + 1)}{\delta \tilde{k}}, \frac{\tilde{k}k_{d2}/\delta}{k_{d2} + (k_1 + 1)/\delta} \right\}. \]

Also let \( z = \gamma_z(\tilde{k}) \) denote the globally unique solution of \( f_z(z, k) = 0 \) and define \( V_z := \max_{t \geq 0}|h_z(t)|. \) Then, the upper bound on \( h_z(Z, k(t)) \) is given by

\[ |h_z(Z, k(t))| \leq C_0 e^{-\gamma_z t} + C_1^z, \]

(13)

where \( C_0^z := \delta \left( \frac{X_T}{X_T + k_{d1}} \right)^\gamma_z \) and \( C_1^z := \left( \frac{X_T}{X_T + k_{d1}} \right)^\gamma_z \),

\[ \gamma_z := |Z(0) - \gamma_z(Z(0))| \]

defining \( \gamma_z := |Z(0) - \gamma_z(Z(0))| \) and \( G_z := \frac{\partial \Delta}{\partial X_T} + \frac{k_{d2}}{k_{d2} + (k_1 + 1)/\delta} \).

Proof. See Appendix A-1.

Claim 2. Let \( \lambda_z = \delta \) be the contraction rate of system (10) and \( \Delta Z_0 = |Z(0) - Z_t(0)| \), then we have

\[ |\Delta Z(t)| \leq \Delta Z_0 e^{-\lambda_z t} + \frac{C_0^z}{\lambda_z} e^{-\lambda_z t} + \frac{C_1^z}{\lambda_z}. \]

(14)

Proof. In order to apply Lemma 1 to system (10) - (11), the contraction rate \( \lambda_z \) of the isolated system (10) is obtained.

\[ \frac{dX^*}{dt} = f_x(X^*_T, Z_t). \]

(18)
The dynamics of $X_0^*$ can now be treated as the perturbed version of (18) with an input $Z(t) = Z_I(t) + \Delta Z(t)$, where $\Delta Z(t)$ satisfies (14):

$$\frac{dX_0^*}{dt} = f_\gamma (X_0^*, Z_I) + h_\gamma (X_0^*, \Delta Z)$$

and $h_\gamma (X_0^*, \Delta Z)$ is defined as $h_\gamma (X_0^*, \Delta Z) := \left( k_1' Y_T X_T + \frac{Y_T}{1 + k_2' Y_T} \right) \frac{\Delta Z}{X_T}$.

Claim 3. Having $\Delta X_0^* = |X_0^*(0) - X_1^*(0)|$, the input error satisfies

$$|\Delta X_1^*(t)| \leq \Delta X_0^* e^{-\lambda_x t} + \frac{C_T^0}{\lambda_x} e^{-\lambda_x t} + \frac{C_T^0}{\lambda_x} e^{-\lambda_x t} + \frac{C_T^0}{\lambda_x} e^{-\lambda_x t},$$

where $C_T^0 := \left( \frac{k_1' Y_T}{1 + k_2' Y_T} \right) \Delta Z_0$, $C_T^1 := \left( \frac{k_1' Y_T}{1 + k_2' Y_T} \right) \Delta Z_0$, and $\lambda_x := \frac{k_1' Y_T}{1 + k_2' Y_T}$.

Proof. Recalling $\Delta Z(t)$ from (14), the disturbance $h_\gamma (X_0^*, \Delta Z)$ satisfies $|h_\gamma (X_0^*, \Delta Z)| \leq \frac{C_T^0}{\lambda_x} e^{-\lambda_x t} + \frac{C_T^0}{\lambda_x} e^{-\lambda_x t} + \frac{C_T^0}{\lambda_x} e^{-\lambda_x t}$. The contraction rate of the isolated system (18) is found as a positive $\lambda_x$ such that $m_\gamma (\frac{\gamma_0 h_\gamma (X_0^*, \Delta Z)}{\lambda_x}) \leq -\lambda_x$, for all $X_1^*$, and $Z_1$, which results in $\lambda_x := \frac{k_1' Y_T}{1 + k_2' Y_T}$. Recalling the constants defined in Claim 3 and using Proposition 1 yields (20). □

The steady state input error can be found as

$$\lim_{t \to +\infty} |\Delta X_1^*(t)| \leq \frac{C_T^0}{\lambda_x} \frac{\Delta Z_0}{k_2' Y_T} := \Delta X_1^{*\infty},$$

which increases as $X_T$ increases. This behavior is captured by Fig. 4, where the steady state error is also shown as obtained from simulation with a periodic input $k(t)$. One can also see that it lies below the calculated upper bound (15) for all values of $X_T$.

![Fig. 4. Simulated input error from model (18) - (19) and estimated error upper bound from (21).](image)

VI. OUTPUT ERROR

The dynamics of $X^*$, subject to both retroactivity effects $r$ and $s$, will be compared to the dynamics of $X_0^*$ which only subject to the retroactivity effect $r$ in order to obtain the output error $\Delta X_0^*$. The dynamics of $X^*$ are given by

$$\frac{dX^*}{dt} = f_\gamma (X^*, Z) + h_\gamma (X^*, Z),$$

where the expression $h_\gamma (X^*, Z)$ has been defined as

$$h_\gamma (X^*, Z) := -R_x f_\gamma (X^*, Z).$$

Equation (22) will be treated as the perturbed version of the $X_0(t)$ dynamics given in (19). In order to apply the robustness result given in Lemma 1 to find a bound on $\Delta X_0^*$, we first need a bound on the perturbation $h_\gamma (X^*, Z)$.

Claim 4. The disturbance $h_\gamma (X^*, Z)$ satisfies

$$|h_\gamma (X^*, Z)| \leq C_T^0 e^{-\gamma_0 t} + C_T^1$$

where $C_T^0 := \left( \frac{C_T^0}{\lambda_x} \right)$ and $C_T^1 := \left( \frac{k_1' Y_T}{1 + k_2' Y_T} \right) \Delta Z_0$.

Proof. See Appendix A-3. □

Claim 5. Having $\lambda_x$ as the contraction rate of (18) and $X_0 := |X^*(0) - X_0^*(0)|$, $|\Delta X_0^*(t)|$ satisfies

$$|\Delta X_0^*(t)| \leq \Delta X_0^{*\infty} e^{-\lambda_x t} + \frac{C_T^0}{\lambda_x} e^{-\lambda_x t} + \frac{C_T^0}{\lambda_x} e^{-\lambda_x t},$$

Proof. Recalling $\lambda_x$ from Claim 3, one can apply Proposition 1 to get (25). □

Assuming $Z(0) = \gamma_0 (k(0))$, after a transient the output error is bounded by

$$\lim_{t \to +\infty} |\Delta X^*(t)| \leq \frac{C_T^0}{\lambda_x} := \Delta X_0^{*\infty},$$

where

$$\Delta X_0^{*\infty} = m_x \left[ \frac{(k_{d1} \rho + Y_T)^2 k_1 V_x}{(2k_{d1} \rho + Y_T)^2} + \frac{\rho^2 k_{d1} \rho (k_{d2} + Y_T)}{2k_{d1} \rho + Y_T} \right]$$

Differentiating with respect to $Y_T$, one has

$$\frac{\partial \Delta X_0^{*\infty}}{\partial Y_T} = -m_x \left[ \frac{(k_{d1} + Y_T) \left[ 8k_{d1} k_{d1} V_x (k_{d1} \rho + Y_T)^2 \right]}{Y_T (2k_{d1} \rho + Y_T)^3} + \frac{\rho^2 k_{d1} \rho (k_{d2} + Y_T)}{2k_{d1} \rho + Y_T} \right]$$

making $\frac{\partial \Delta X_0^{*\infty}}{\partial Y_T} < 0$. Thus, the error decreases as $X_T$ is increased. Fig. 5 captures this behavior. As it is expected, the calculated upper bound lies above the system error simulation for all values of $X_T$. Furthermore, without downstream clients ($p_T = 0$), $\Delta X_0^{*\infty} = 0$ since the error
only accounts for the retroactivity to the output. In (26), since $Y_T = X_T \rho_r$, $\Delta X_{0 \to \infty}$ converges to a value different from zero as $X_T \to \infty$. This is due to the presence of the phosphatase term $Y_T/k_{d2}$ in the $R_x$ expression, which accounts for the phosphatase also placing a load on the output protein $X^*$.

This is different from [3], where the phosphatase term was not accounted for in the retroactivity expression.

Fig. 5. Simulated output error from model (19) -(22), and estimated error upper bound from (26).

VII. TOTAL OUTPUT ERROR

The total output error can be upper bounded using the triangle inequality $|\Delta X_{TOT}^*(t)| \leq |\Delta X^*_0(t)| + |\Delta X^*_I(t)|$. Assuming the initial conditions are the same for connected and isolated systems, and having $A(X_T) = \max \left\{ \frac{A_T}{A_T - \lambda_2}, \frac{A_T}{A_T - \lambda_2}, \frac{A_T}{A_T - \lambda_2} \right\}$, $\lambda(X_T) = \min \{ \lambda_1, \lambda_2, \lambda_3 \}$, and $B(X_T) = \Delta X_{I \to \infty}^* + \Delta X_{0 \to \infty}^*$, the total error takes the form $|\Delta X_{TOT}^*| \leq A(X_T)e^{-\lambda(X_T)T} + B(X_T)$. Evaluating the limit at infinite time we have,

$$\lim_{t \to +\infty} |\Delta X_{TOT}^*(t)| \leq \Delta X_{I \to \infty}^* + \Delta X_{0 \to \infty}^*. \quad (27)$$

The optimal selection of $X_T$, which minimizes both the effects of $r$ and $s$ on signal $X^*(t)$, is given by the minimum of the above expression, which is shown in Fig. 6. The first term $\Delta X_{I \to \infty}^*$ defined in (21), increases with $X_T$ since it comes from the input error, while the second term, defined in (26), decrease as $X_T$ increases since it comes from the output error. This illustrates a tradeoff between the input and output errors. This expression also predicts no error for constant inputs, meaning $V_2 = 0$, making retroactivity for the presented model a purely dynamical effect. A final remark on Fig. 6 is that the bound is tight about the minimum.

Note that if the calculated bounds $\Delta X_{I \to \infty}^*$ and $\Delta X_{0 \to \infty}^*$ fall in the order of $\epsilon_2/\epsilon_1$, then the approximation error due to singular perturbation is not negligible anymore and should be accounted for in the calculations of the error bounds.

VIII. CONCLUSIONS AND FUTURE WORK

This work presents the application of tools from nonlinear systems analysis such as contraction theory and singular perturbation to analyze the tradeoff between input and output retroactivity of a phosphorylation-based insulation device. The analysis shows that while increasing the substrate concentration of the phosphorylation cycle reduces the effect of retroactivity to the output, the cycle becomes a load itself to the upstream system. The error produced by the retroactivity to the input is then propagated to the output, attenuating the output signal. Thus, an optimal substrate concentration was found to be the minimum of a weighted sum of the input and output errors. The results from this work can be used to estimate the minimum upper bound on the error given by the insulation device. This work will be extended to generalized interconnection structures, providing a methodology for the combined minimization of input and output retroactivities.

REFERENCES

The contraction rate from (9) we see that \( h \) is \( \epsilon \)-Lipschitz continuous in \( \epsilon \). Thus it suffices to prove that \( \frac{\partial g_2(k,t)}{\partial k} \) is \( \frac{\partial g_2(x(t),Z)}{\partial x} \). Selecting a \( Y_T \) as in the assumption of Claim 1, the \( X_T \) factor is smaller than \(-\delta k_1/T \), so the contraction rate can be set to

\[
G_z := \delta k_1 X_T/2 + \delta k_1 X_T \tag{30}
\]

Now Lemma 2 can be applied directly using (29) and (30). Letting \( \epsilon_z(t) := Z(t) - \gamma_z(k(t)) \), it follows from Lemma 2 that \( |\epsilon_z(t)| \leq e^{-G_zt} |Z(0) - \gamma_z(k(0))| + V_2e^{G_1t} \). Finally, recalling (12), one has \( |h_z(Z,k(t))| \leq |R_z(Z)||f_z(Z,k(t))| \). From (9), since \( \frac{\partial h_z}{\partial Z} < 0 \), it follows that

\[
|\epsilon_z(Z)| \leq \frac{X_T}{X_T + k_1T}. \tag{31}
\]

One has that \( |f_z(Z,k(t))| = |f_z(\gamma_z(k(t)) + \epsilon_z(t),k(t))| \leq |f_z(\gamma_z,k(t))| + |\Delta_z(t)| \), in which \( \Delta_z(t) := f_z(\gamma_z + \epsilon_z,k(t)) - f_z(\gamma_z,k(t)) \). Since \( f_z(Z,k(t)) \) is Lipschitz continuous in \( Z \) with constant \( \alpha_z = \frac{k_1T}{k_1T} \), it follows that \( |\Delta_z(t)| \leq \alpha_z \epsilon_z(t) \). Also \( f_z(\gamma_z,k(t)),k(t) = 0 \), so that \( |f_z(Z,k(t))| \leq \alpha_z \epsilon_z(t) \). From (31) and (32) we have \( |h_z(Z,k(t))| \leq X_T(e^{-G_zt} + \frac{V_2e^{G_1t}}{X_T}) \). Recalling constants \( C_0^z \) and \( C_1^z \) in Claim 1, we have \( |h_z(Z,k(t))| \leq C_0^z e^{-G_zt} + C_1^z \).

**A-2 Claim:** From system (28), \( m_2 \left( \frac{\partial g_2(Z,k(t))}{\partial Z} \right) \bigg|_{Z=0} \leq m_2 \left( \frac{\partial g_2(x(t),Z)}{\partial x} \right) \).

**Proof:** From (9) we see that \((1 - R_z) \frac{\partial g_2}{\partial Z} \leq 0 \), so \( \frac{\partial g_2}{\partial Z} \leq 0 \). Thus it suffices to prove that \( \frac{\partial g_2}{\partial Z} \leq M \). Defining \( a = k_1X_T (Y_T k_2 + k_2)^2 \), \( b = 1 + Y_T (k_2 + k_2) \), and \( c = Y_T k_2 k_2 / k_2^2 \) we have that

\[
\frac{\partial g_2}{\partial Z} = \frac{\partial g_2}{\partial Z} \left( \frac{1}{Z} \right) = \frac{2ab(c + bZ)}{\left( a + cZ \right)^3} - \frac{a(c + bZ)}{\left( a + cZ \right)^3} \]

which is always negative if \( c > b \frac{k}{\delta} \) as in the assumptions of Claim 1 and noting that \( c^2 < a \).

**A-3 Consider the system**

\[
\frac{dX^*}{dt} = (1 - R_z) f_x(X^*,Z) := g_x(X^*,Z), \tag{33}
\]

The upper bound on \( h_x(X^*,Z) \) from (23) can be obtained by bounding \( X^* \) using Lemma 2 and using Lipschitz continuity, and let \( X_x \) be the globally unique solution of \( g_x(X_x) = 0 \) and denote it \( X_x = \gamma_x(Z) \). In order to apply Lemma 2, we need the bound \( \bar{d} \) in (3) given by \( \frac{\partial g_x(Z)}{\partial Z} \). To obtain bound \( \bar{d} \), it was shown in Appendix A-4 that \( \frac{\partial g_x(Z)}{\partial Z} \leq \frac{1}{K_x \rho} \), also that the bound on \( \frac{\partial Z}{\partial t} \leq V_x \). Thus \( \frac{\partial g_x(Z)}{\partial Z} \leq \frac{V_x}{K_x \rho} \). To complete Lemma 2, the contraction rate \( G_x \) of (33) is determined as a positive number such that \( m_2 \left( \frac{\partial g_x(X,0)}{\partial X} \right) \leq G_x \), for all \( X^* \) and for all \( Z \). One has \( m_2 \left( \frac{\partial g_x(X^*,Z)}{\partial X} \right) \leq \frac{k}{k_1T} Y_T \). Thus, \( \epsilon_x(t) \) can be defined as \( \epsilon_x(t) := X^*(t) - \gamma_x(Z(t)) \) and using Lemma 2, one has

\[
|\epsilon_x(t)| \leq e^{-G_xt} |X(0) - \gamma_x(Z(0))| + \frac{V_x}{G_x K_x \rho}. \tag{34}
\]

Now, recalling (23) one has \( h_x(X^*,Z) := -R_x f_x(X^*,Z) \), so that \( h_x(Z,X^*) \) \( \leq R_x f_x(Z,X^*) \). Given that \( X^* = \gamma_x(Z) + \epsilon_x(t) \), we have that \( |h_x(Z,X^*)| \leq |f_x(\gamma_x(Z) + \epsilon_x(t),Z) + \Delta_x(t)| \), in which \( \Delta_x(t) := f_x(\gamma_x(Z) + \epsilon_x(t),Z) - f_x(\gamma_x(Z),Z) \). Recalling \( W_x \) from Claim 4, and since \( f_x(X^*,Z) \) is Lipschitz continuous with constant \( \alpha_x = k/k_1T + k_1T \), one has \( \Delta_x(t) \leq \alpha_x \epsilon_x(t) \). Also \( f_x(\gamma_x(Z),Z) = 0 \), so that \( |f_x(Z,X^*)| \leq \alpha_x \epsilon_x(t) \). It follows that

\[
|h_x(X^*,Z)| \leq R_x \alpha_x |\epsilon_x(t)|. \tag{35}
\]

Recalling variables \( C_0^x \) and \( C_1^x \) in Claim 4, it follows from (34) and (35) that \( |h_x(X^*,Z)| \leq C_0^x e^{-G_xt} + C_1^x \).

**A-4 Claim:** From system (33), \( \frac{\partial g_x(Z)}{\partial Z} \leq \frac{V_x}{K_x \rho} \).

**Proof.** Using the Implicit Function Theorem [15], one has \( \frac{\partial g_x(Z)}{\partial Z} = \frac{\partial \gamma_x(Z)}{\partial Z} \). It follows that \( \frac{\partial g_x(Z)}{\partial Z} = \frac{1}{k_1T + k_1T} \). Recalling \( K_x \) in Claim 4 and the definition of \( \rho \), we have that \( \frac{\partial g_x(Z)}{\partial Z} \leq \frac{1}{K_x \rho} \).

Now, from (11), one has \( |\epsilon_x(Z)| \leq |1 - R_z||f_x| \). Using (9), \( |1 - R_z| \leq 1 \). From (32), \( |f_x(Z,k(t))| \leq \alpha_z e^{-G_zt} |Z(0) - \gamma_z(k(0))| + V_2e^{G_1t} \). Defining \( V_x := \delta Z(0) - \gamma_z(k(0)) + V_2 \), the upper bound on the time derivative of \( \gamma_x(Z) \) is given by

\[
\frac{d\gamma_x(Z)}{dt} \leq \frac{\partial \gamma_x(Z)|dZ}{dt} \leq \frac{V_x}{K_x \rho}. \]

□