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Induced-gravity inflation and the density perturbation spectrum

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Abstract

Recent experimental determinations of the spectral index describing the scalar mode spectrum of density perturbations encourage comparison with predictions from models of the very early universe. Unlike extended inflation, Induced-gravity Inflation predicts a power spectrum with $0.98 \leq n_s \leq 1.00$, in close agreement with the experimental measurements.

An exciting test for models of the very early universe stems from recent measurements of the power spectrum of density perturbations, as seen in the cosmic microwave background radiation. The scalar spectrum, modeled as $\mathcal{P}(k) \propto k^{n_s}$ [1], functions as a test for models like inflation, independently of the familiar test based on the magnitude of the fluctuations. As pointed out by Andrew Liddle and David Lyth [2], extended inflation predicts a spectral index (n_s) which is tilted too far away from the Harrison-Zel'dovich (scale-invariant) spectrum ($n_s = 1.00$), and hence cannot match the recent Cosmic Background Explorer (COBE) determination. In this Letter, the predictions from a cousin-model of extended inflation, Induced-gravity Inflation, are compared with the experimental values. Unlike extended inflation, Induced-gravity Inflation predicts a spectral index in quite close agreement with recent experimental values.

Like extended inflation [3], Induced-gravity Inflation (IgI) [4–6] incorporates a Generalized Einstein Theory (GET) gravity sector. Yet unlike extended inflation, IgI incorporates only one scalar boson to get all the work of inflation done: the scalar field which couples to the Ricci scalar in Brans-Dicke-like fashion is the *same* field whose potential, $V(\phi)$, drives inflation. This is the crucial difference as far as n_s is concerned: by adopting a potential which leads to a *second* order phase transition (unlike the first order phase transition incorporated in extended inflation), IgI can escape two related problems of extended inflation (discussed below) and lead to an acceptable spectrum of perturbations.

Much of the formalism developed in the literature for calculating n_s assumes an Einsteinian gravitational background [7]; hence it cannot be applied in a straightforward manner to IgI, because of its GET gravity sector. Furthermore, as discussed in [8], the usual strategy of applying a conformal transformation to bring IgI into the canonical Einstein-Hilbert gravitational form [9] may prove problematic when studying the spectrum of perturbations, stemming from ambiguities with semiclassical quantization in the various frames. Therefore, in this Letter, we will restrict attention to the "physical" or "Jordan" frame, in which the nonminimal $\phi^2 R$ coupling is explicit.

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The action for IgI is given by:

$$S = \int d^4x\sqrt{-g}\left(\frac{\phi^2}{8\omega}R - \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi) + L_M\right), \tag{1}$$

where the Brans-Dicke parameter (ω) is related to the nonminimal curvature coupling constant (ξ), often found in the literature, by $\omega = (4\xi)^{-1}$. In this model, L_M only includes contributions from 'ordinary' matter, and does not include a separate Higgs sector; it can henceforth be ignored. The coupled field equations which result are:

$$H^2 + \frac{k}{a^2} = \frac{4\omega}{3\phi^2}V(\phi) + \frac{2\omega}{3}\left(\frac{\dot\phi}{\phi}\right)^2 - 2H\left(\frac{\dot\phi}{\phi}\right) + \frac{2(\omega+1)}{3a^2}\frac{(\partial_i\phi)^2}{\phi^2} + \frac{2}{3a^2}\frac{\partial_i^2\phi}{\phi},$$

$$\ddot\phi + 3H\dot\phi + \frac{\dot\phi^2}{\phi} - \frac{1}{a^2}\frac{(\partial_i\phi)^2}{\phi} - \frac{1}{a^2}\partial_i^2\phi = \frac{2\omega}{3+2\omega}\phi^{-1}\left[4V(\phi) - \phi V'(\phi)\right]. \tag{2}$$

In Eq. (2), $a(t)$ is the cosmic scale factor of the Robertson-Walker metric, and is related to the Hubble parameter by $H \equiv \dot a/a$. Note from the form of Eq. (1) that $4\omega\phi^{-2} = 8\pi G_{\rm eff}$. The k-term is related to the total curvature of the universe, and becomes negligible in the inflationary epoch. If we assume the following approximations for the period of slow-roll,

$$\left|\frac{\dot\phi}{\phi}\right| \ll H,$$

$$\left|\dot\phi^2\right| \ll V(\phi), \tag{3}$$

and further assume that

$$\left|\frac{1}{a^2}\frac{\partial_i^2\phi}{\phi}\right| \ll H^2, \tag{4}$$

then the field equations may be approximated as follows during the slow-roll period:

$$H^2 \simeq \frac{4\omega}{3}\frac{V(\phi)}{\phi^2},$$

$$\ddot\phi + 3H\dot\phi - \frac{1}{a^2}\partial_i^2\phi \simeq \frac{2\omega}{3+2\omega}\frac{(4V - \phi V')}{\phi}. \tag{5}$$

In Eq. (5), terms of order $(\partial_i\phi)^2$ have been neglected. We assume a generic Ginzburg-Landau form for the (zero-temperature) potential,

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2, \tag{6}$$

which describes a second-order phase transition (even when non-zero-temperature corrections are included). Following the ordinary procedure [10], we may write the field $\phi(x,t) = \varphi(t) + \delta\phi(x,t)$, where φ is the homogeneous part of the field, and $\delta\phi$ represents a quantum fluctuation of the field. (This justifies neglecting terms of order $(\partial_i\phi)^2$ above: these terms are quadratic in the small fluctuations.) Then during slow-roll, we may further ignore the $\ddot\varphi$ term, and solve the approximate field equations (5) in terms of $\varphi(t)$:

$$\varphi(t) = \varphi_o + \sqrt{\frac{\lambda\omega}{3\gamma^2}}v^2 t,$$

$$\frac{a(t)}{a_B} = \left(\frac{\varphi(t)}{\varphi_o}\right)^\gamma \exp\left[\frac{\gamma}{2v^2}\left(\varphi_o^2 - \varphi^2(t)\right)\right], \tag{7}$$

where $\gamma \equiv \omega + 3/2$. The quantities φ_o and a_B are values at the beginning of the inflationary epoch. For this Letter, we will be concerned with a 'new inflation'-like scenario, with $\varphi_o \ll v$, instead of the 'chaotic inflation' condition, $\varphi_o \gg v$. Note that with $\varphi_o \ll v$, IgI inflates like a power law for early times, $a(t) \sim t^\gamma$, much like the power-law expansion of extended inflation.

If we define the quantity

$$W(\phi) \equiv \phi^{-1}\left[\phi V' - 4V(\phi)\right], \tag{8}$$

then the equation for the evolution of the fluctuations may be written:

$$\delta\ddot\phi + 3H\delta\dot\phi + \frac{k^2}{a^2}\delta\phi = -\frac{\omega}{\gamma}\left(W'|_\varphi\right)\delta\phi, \tag{9}$$

where $\delta\ddot\phi \equiv \partial^2(\delta\phi)/\partial t^2$. For the particular form of the potential considered here, this gives:

$$W'|_\varphi \to \lambda v^2\left(1 + \frac{v^2}{\varphi^2}\right), \tag{10}$$

and thus

$$\delta\ddot\phi + 3H\delta\dot\phi + \frac{k^2}{a^2}\delta\phi = -\frac{\lambda\omega}{\gamma}v^2\left(1 + \frac{v^2}{\varphi^2}\right)\delta\phi. \tag{11}$$

Note that the $k^2 = \boldsymbol{k} \cdot \boldsymbol{k}$ term in Eqs. (9) and (11) should not be confused with the curvature k term in Eq. (2).

We need to calculate the two-point correlation function for the fluctuations obeying Eq. (11), which in turn will yield the prediction for n_s. The two-point correlation function for a *free*, massless scalar field in a metric expanding as $a(t) \propto t^\gamma$ is by now well known [11,12]. Our task now is to check whether or not the more complicated wave equation for the fluctuations in IgI can be well-approximated by the free wave equation, so that these earlier results for the correlation function may be incorporated. In fact, the free wave equation is a good approximation to Eq. (11), as can be seen by the following. Using Eq. (7), we may write:

$$\frac{\dot{a}}{a} = \gamma \frac{\dot{\varphi}}{\varphi} \left[1 - \frac{\varphi^2}{v^2} \right]. \qquad (12)$$

We will be interested in the two-point correlation function at the time ($t_{\rm HC}$) of last horizon-crossing during inflation of the density perturbations at scales which interest us (from 10^6 to 10^{10} lightyears). The time of last horizon-crossing is very difficult to solve for exactly, but should have happened around 60 e-folds before the end of inflation. As in [5], we may write:

$$e^\alpha \equiv \frac{a(t_{\rm end})}{a(t_{\rm HC})} \sim e^{60}. \qquad (13)$$

Then Eqs. (12) and (13) yield

$$\frac{\alpha}{\gamma} = \ln \left(\frac{v}{\varphi_{\rm HC}} \right) - \frac{1}{2} + \frac{1}{2} \left(\frac{\varphi_{\rm HC}}{v} \right)^2. \qquad (14)$$

Following [4], we may study two limiting cases: (a) $(\alpha/\gamma) \ll 1$ and (b) $(\alpha/\gamma) \gg 1$. These give:

$$\text{(a)} \quad \varphi_{\rm HC} \simeq v \left(1 - \sqrt{\frac{\alpha}{\gamma}} \right),$$

$$\text{(b)} \quad \varphi_{\rm HC} \simeq v \exp \left(-\frac{\alpha}{\gamma} - \frac{1}{2} \right). \qquad (15)$$

The corresponding expressions in [4] are written incorrectly in terms of $\epsilon \equiv 1/(4\omega)$ instead of γ, because those authors made the approximation that $\gamma \equiv \omega + 3/2 \to \omega$ throughout their analysis. As pointed out in [5,6], this is an unnecessary restriction on ω which can lead to qualitatively incorrect results.

We may check that each of these approximate values for $\varphi_{\rm HC}$ falls safely within the domain of the slow-roll approximation by comparing $\varphi_{\rm HC}$ with the value of the field for which the slow-roll approximation breaks down; that is, the point at which $(\ddot{\varphi} + \dot{\varphi}^2/\varphi) = 3H\dot{\varphi}$, instead of being $\ll 3H\dot{\varphi}$. As calculated in [5], this occurs at

$$\varphi_{bd} = v \left[1 + \frac{1 - \sqrt{1 + 6\gamma}}{3\gamma} \right]^{1/2}. \qquad (16)$$

For $\gamma \gg \alpha$ (case (a)), this expression may be written $\varphi_{bd} \simeq v(1 - \sqrt{1/(6\gamma)})$, and it is clear that $\varphi_{\rm HC} < \varphi_{bd}$. There is a lower bound on φ_{bd} which becomes relevant for the case $\gamma \ll \alpha$ (case (b)): even when $\omega \sim 0$, $\gamma \geq 3/2$, and $\varphi_{bd} \geq 3v/(1 + \sqrt{10}) = 0.72v$. Yet for case (b), $\varphi_{\rm HC} \leq v \exp(-3/2) = 0.22v$. Thus $\varphi_{\rm HC} < \varphi_{bd}$ for both cases (a) and (b).

Armed with these expressions for $\varphi_{\rm HC}$, we may return to Eq. (11) for $\delta\phi$:

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} = -\frac{\lambda\omega}{\gamma} v^2 \left(1 + \frac{v^2}{\varphi^2} \right) \delta\phi - \frac{k^2}{a^2}\delta\phi. \qquad (17)$$

We want solutions of this equation for times near $t_{\rm HC}$. The coefficient of the second term on the RHS at $t_{\rm HC}$ is

$$\frac{k^2}{a^2(t_{\rm HC})} = H_{\rm HC}^2 = \frac{\lambda\omega}{3} \frac{\left(v^2 - \varphi_{\rm HC}^2\right)^2}{\varphi_{\rm HC}^2}, \qquad (18)$$

and thus the ratio of the two coefficients on the RHS of Eq. (17) is

$$R \equiv \frac{k^2/a^2(t_{\rm HC})}{\lambda(\omega/\gamma)v^2(1 + v^2/\varphi_{\rm HC}^2)} = \frac{\gamma}{3} \left[\frac{1 - 2(\varphi_{\rm HC}/v)^2 + (\varphi_{\rm HC}/v)^4}{1 + (\varphi_{\rm HC}/v)^2} \right]. \qquad (19)$$

For $\alpha/\gamma \ll 1$ (case (a)), this ratio becomes

$$\text{(a)} \quad R = \frac{\gamma}{3} \left(\frac{\alpha}{\gamma} \right) \left(\frac{4 - 4\sqrt{\alpha/\gamma}}{2 - 2\sqrt{\alpha/\gamma}} \right) = \frac{2\alpha}{3} \gg 1. \qquad (20)$$

Thus, for case (a), the second term on the RHS of Eq. (17) dominates near $t_{\rm HC}$, and the equation for the fluctuations assumes the form for a free, massless scalar field:

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \frac{k^2}{a^2}\delta\phi \simeq 0. \qquad (21)$$

For $\alpha/\gamma \gg 1$ (case (b)), R becomes

$$\text{(b)}\quad R = \frac{\gamma}{3} \times \left(\frac{1 - 2\exp(-2\alpha/\gamma - 1) + \exp(-4\alpha/\gamma - 2)}{1 + \exp(-2\alpha/\gamma - 1)} \right) \simeq \frac{\gamma}{3}, \tag{22}$$

and the fluctuations obey the equation

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \left(1 + \frac{3}{\gamma}\right)\frac{k^2}{a^2}\delta\phi \simeq 0. \tag{23}$$

Note that $(1 + 3/\gamma) \leq 3$ for all γ. (This is an example of how the corrections to [4] can become crucial: even when ω is made arbitrarily small, the ratio R^{-1} remains finite.) The deviation of Eq. (23) from the truly free, massless case is thus small, and, furthermore, does not alter the k-dependence of the two-point correlation function (although it does affect the magnitude of the correlation function). Thus, for both cases (a) and (b), we may import the results from [11,12]: writing the correlation function as

$$|\Delta\phi(\boldsymbol{k})|^2 \equiv k^3 \int \frac{d^3x}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \langle \phi(\boldsymbol{x})\phi(\boldsymbol{0}) \rangle \tag{24}$$

leads to the result

$$|\Delta\phi(\boldsymbol{k})|^2 \propto k^{-2/(\gamma-1)}. \tag{25}$$

Therefore $\delta_H = \delta\tilde{\rho}/\rho \propto k^{-1/(\gamma-1)}$. The spectral index is defined by [1]

$$n_s - 1 \equiv \frac{d\ln\delta_H^2}{d\ln k}, \tag{26}$$

which yields

$$n_s = 1 - \frac{2}{\gamma - 1} \tag{27}$$

for IgI. As calculated in [5], values of ω in the range $10^2 \leq \omega \leq 10^3$ are favored for IgI, based on the upper bound on the quartic self-coupling parameter, λ. (See Fig. 2 in [5].) In particular, the bound on λ is maximized for $\omega_{\mathrm{cr}} = 240$. Eq. (27) yields $n_s(\omega_{\mathrm{cr}}) = 0.99$; for $10^2 \leq \omega \leq 10^3$, IgI predicts $0.98 \leq n_s \leq 1.00$. This is obviously quite close to the $n_s = 1.00$ Harrison-Zel'dovich spectrum.

Recently, the results of the two year data analysis for the COBE Differential Microwave Radiometer (DMR) experiments were announced. [13] The maximum likelihood estimates on n_s were given as 1.22 if the quadrupole contribution were included, and 1.02 if the quadrupole were excluded. The marginal likelihood gave $n_s = 1.10 \pm 0.32$ including the quadrupole, and $n_s = 0.87 \pm 0.36$ excluding the quadrupole. As concluded in [13], these results are completely consistent with an $n_s = 1.00$ spectrum, and hence are in close agreement with the predictions from IgI. Furthermore, it is clear that these data imply a lower limit on ω for IgI, based on Eq. (27). In particular, if we limit $n_s \geq 0.87$, then $\omega \geq 15$.

It should be noted, however, that IgI appears to be incapable of yielding a spectrum with $n_s > 1.00$. This could lead to conflict with the experimental value, if n_s should be shown definitively to hover closer to 1.2, rather than to 1.0. This illustrates how the *spectrum* of perturbations functions as a test for models of the early universe, independently from the test based on the *magnitude* of perturbations: it is always possible (even if aesthetically unappealing) to rescale dimensionless constants of inflationary models in order to match the experimentally-determined magnitude of perturbations. Yet no such rescaling (at least in IgI) can lead to a prediction of $n_s > 1$. Inflationary models which do predict $n_s \geq 1$ are considered in [14,15].

The lower bound on ω, stemming from requirements on n_s, may prove to be very significant when treating the magnitude of density perturbations in IgI. In [6], Redouane Fakir and William Unruh proposed that constraints on the magnitude of perturbations could be met in a 'chaotic inflation' scenario ($\varphi_o \gg v$) of IgI if $\xi \equiv 1/(4\omega)$ were made arbitrarily large. As the above analysis has shown, however, in the context of a 'new inflation' scenario ($\varphi_o \ll v$), ξ *cannot* be made arbitrarily large: as $\omega \to 0$, $\gamma \to 3/2$, and $n_s \to -3$, which is in clear violation of the experimental data. The differing constraints from n_s on ξ (or, equivalently, on ω) for the 'chaotic inflation' versus the 'new inflation' scenarios are further studied in [18].

Finally, we must consider why IgI is able to agree with the experimental determination of n_s, even though the closely-related extended inflation cannot. The result of Eq. (27) is similar in form to that for extended inflation, for which γ in Eq. (27) should be replaced by $\gamma/2$. Yet, as explained in [2], extended inflation is restricted to $\omega \leq 17$, which leads to $n_s \sim 0.76$. This is too steep a tilt away from the nearly scale-invariant, $n_s = 1.00$ spectrum, and is thus a problem for that model. The restriction on ω for extended inflation arises from that model's first order phase transition: values of ω greater than 17 would yield noticeable (and yet unseen) inhomogeneities in the cosmic microwave background radiation, due to the percolation of such a large range in bubble sizes. This has often been referred to as the 'big bubble' or 'ω problem'. [2,16] In IgI, the second order phase transition removes concern with bubble percolation, and there is no analogous upper bound on ω.

The second 'ω problem' for extended inflation, which is also avoided in IgI, concerns the present-day value of the Brans-Dicke parameter ω. Time-delay tests of Brans-Dicke gravitation versus Einsteinian general relativity limit $\omega \geq 500$. Since extended inflation never exits the GET phase, this present-day bound on ω should be obeyed, which is in further violation of the $\omega \leq 17$ bound. In IgI, the phase transition itself ensures the transition from the GET phase to pure Einsteinian general relativity; that is, after the very early phase transition, the universe is no longer described by Brans-Dicke gravitation, and all present-day bounds on ω become irrelevant. IgI delivers the universe into the highly-corroborated Einsteinian gravity, regardless of the value of ω during the early universe. In addition, as discussed in [5], post-inflation reheating for IgI could bring the universe to energies as high as $(10^{-3} - 10^{-2}) M_{\rm Pl}$, where $M_{\rm Pl} = 1.22 \times 10^{19}$ GeV is the present value of the Planck mass, allowing *either* GUT-scale or electroweak baryogenesis [17] to follow the IgI phase.

Thus, Induced-gravity Inflation predicts a spectrum of primordial density perturbations in close agreement with the recent COBE DMR results, with $0.98 \leq n_s \leq 1.00$. Using the experimental data, we may limit $\omega \geq 15$ for Induced-gravity Inflation. Although the functional form of the spectral index predicted by Induced-gravity Inflation is very similar to that for extended inflation, Induced-gravity Inflation is able to avoid *both* of the 'ω problems' which plagued extended inflation. The predicted spectrum, therefore, deviates little from the observed, nearly scale-invariant spectrum.

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References

[1] A.R Liddle and D H Lyth, Phys. Rep. 231 (1993) 1.

[2] A.R. Liddle and D.H Lyth, The End for Extended Inflation? in Texas/PASCOS '92. Relativistic Astrophysics and Particle Cosmology, edited by C W Akerlof and M.A Srednicki, New York Academy of Sciences (1992) p 653.

[3] D. La and PJ Steinhardt, Phys Rev. Lett. 62 (1989) 376, R Holman, E W Kolb and Y. Wang, Phys. Rev. Lett. 65 (1990) 17

[4] FS. Accetta, D J. Zoller and M.S Turner, Phys. Rev. D 31 (1985) 3046.

[5] D I Kaiser, Phys. Rev. D 49 (1994) 6347

[6] R Fakir and W.G Unruh, Phys. Rev D 41 (1990) 1783, 1792,
see also R. Fakir, S Habib and W. Unruh, Astrophys J 394 (1992) 396;
N Makino and M Sasaki, Prog Theo Phys. 86 (1991) 103,
F Lucchin, S. Matarrese and M.D Pollock, Phys Lett B 167 (1986) 163.

[7] See, e.g, E.D. Stewart and D H. Lyth, Phys. Lett B. 302 (1993) 171,
E.W. Kolb and S L Vadas, Relating spectral indices to tensor and scalar amplitudes in inflation, Preprint FERMILAB-Pub-94/046-A, CfPA 94-th-14, astro-ph/9403001

[8] R. Fakir and S Habib, Mod. Phys Lett. A 8 (1993) 2827

[9] See, e g, T. Hirai and K Maeda, Gauge-invariant Cosmological Perturbations in Generalized Einstein Theories, Preprint WU-AP/32/93, astro-ph/9404023;
A. Berkin and K Maeda, Phys Rev D 44 (1991) 1691,
K Maeda, Phys Rev D 39 (1989) 3159.

[10] A H Guth and S.-Y. Pi, Phys. Rev Lett. 49 (1982) 1110

[11] L F Abbott and M.B. Wise, Nucl Phys. B 244 (1984) 541

[12] D.H Lyth and E D Stewart, Phys Lett. B 274 (1992) 168.

[13] K.M Górski, G Hinshaw, A J Banday, C.L. Bennett, E.L Wright, A Kogut, G F Smoot and P. Lubin, On Determining the Spectrum of Primordial Inhomogeneity from the COBE DMR Sky Maps: II Results of Two Year Data Analysis, Preprint astro-ph/9403067;
see also [14] and the references therein.

[14] S. Mollerach, S Matarrese and F Lucchin, Blue Perturbation Spectra from Inflation, Preprint astro-ph/9309054

[15] R. Schaefer and Q Shafi, Constraints on the Power Spectrum and Hot Dark Matter Fraction in Inflationary Models, paper presented at April 1994 American Physical Society meeting, Washington D.C., abstract published in Bulletin of the American Physical Society 39 (1994) 1096.

[16] P.J. Steinhardt and F.S. Accetta, Phys. Rev. Lett. 64 (1990) 2740;
J. García-Bellido and M. Quirós, Phys. Lett. B 243 (1990) 45;
R. Crittenden and P.J. Steinhardt, Phys. Lett. B 293 (1992) 32;
P.J. Steinhardt, Class. Quantum Grav. 10 (1993) S33;
E.W. Kolb, Physica Scripta T36 (1991) 199.

[17] See, e.g., M. Gleiser, Baryogenesis in brief, Preprint DART-HEP-94/03, hep-ph/9407369, and references therein.

[18] D.I. Kaiser, Primordial Spectral Indices from Generalized Einstein Theories, Preprint HUTP-94/A033, DART-HEP-94/05, astro-ph/9408044.