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Metric preheating and limitations of linearized gravity

Bruce A. Bassett ^a, Fabrizio Tamburini ^b, David I. Kaiser ^c, Roy Maartens ^d

^a *Department of Theoretical Physics, Oxford University, Oxford OX1 3NP, UK*

^b *Department of Astronomy, University of Padova, Padova, Italy*

^c *Lyman Laboratory of Physics, Harvard University, Cambridge MA 02138, USA*

^d *School of Computer Science and Mathematics, Portsmouth University, Portsmouth PO1 2EG, UK*

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Abstract

During the preheating era after inflation, resonant amplification of quantum field fluctuations takes place. Recently it has become clear that this must be accompanied by resonant amplification of scalar *metric* fluctuations, since the two are united by Einstein's equations. Furthermore, this "metric preheating" enhances particle production, and leads to gravitational rescattering effects even at *linear order*. In multi-field models with strong preheating ($q \gg 1$), metric perturbations are driven non-linear, with the strongest amplification typically on super-Hubble scales ($k \rightarrow 0$). This amplification is causal, being due to the super-Hubble coherence of the inflaton condensate, and is accompanied by resonant growth of entropy perturbations. The amplification invalidates the use of the linearized Einstein field equations, irrespective of the amount of fine-tuning of the initial conditions. This has serious implications on all scales – from large-angle cosmic microwave background (CMB) anisotropies to primordial black holes. We investigate the (q, k) parameter space in a two-field model, and introduce the time to non-linearity, t_{nl} , as the timescale for the breakdown of the linearized Einstein equations. t_{nl} is a robust indicator of resonance behavior, showing the fine structure in q and k that one expects from a quasi-Floquet system, and we argue that t_{nl} is a suitable generalization of the static Floquet index in an expanding universe. Backreaction effects are expected to shut down the linear resonances, but cannot remove the existing amplification, which threatens the viability of strong preheating when confronted with the CMB. Mode–mode coupling and turbulence tend to re-establish scale invariance, but this process is limited by causality and for small k the primordial scale invariance of the spectrum may be destroyed. We discuss ways to escape the above conclusions, including secondary phases of inflation and preheating solely to fermions. The exclusion principle constrains the amplification of metric perturbations significantly. Finally we rank known classes of inflation from strongest (chaotic and strongly coupled hybrid inflation) to weakest (hidden sector, warm inflation), in terms of the distortion of the primordial spectrum due to these resonances in preheating. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The successes of the inflationary paradigm are based on classical perfection and quantum imperfections. Inflation leads, essentially because of the no-hair theorem, to a nearly maximally symmetric, de Sitter-like state. In contrast, it also leads, via quantum fluctuations, to small metric perturbations with a nearly scale-invariant spectrum [1–4]. The subsequent evolution of these perturbations, in the absence of entropy perturbations, is extremely simple—controlled by conserved quantities [4]. These allow one almost trivially to transfer the spectrum on super-Hubble scales from the end of inflation, typically near the GUT scale $\sim 10^{16}$ GeV, down to photon decoupling where, in an $\Omega = 1$ universe, the large-angle anisotropies in the cosmic microwave background (CMB) are formed, at a scale ~ 1 eV—which spans 25 orders of magnitude in energy.

That such a very simple picture appears consistent with current CMB and large scale structure data is remarkable. In this paper we discuss some limitations of this picture and its fragility in the context of the recent revolution in inflationary reheating known as preheating. We will show that for a wide range of models the post-preheating universe must necessarily have followed a much more complex route than the simple one outlined above. This more complex route typically invokes the full non-linearity of the Einstein field equations and, in its most extreme guises, appears incompatible with current observational cosmology. In more moderate form, however, the preheating route gives new freedom to the inflationary paradigm in the form of its predictions for the CMB, large scale structure and primordial black holes.

Recent study of the reheating era that follows inflation has highlighted a wide array of interesting new physical processes and phenomena. These new features all stem from non-perturbative, non-equilibrium effects in the inflaton's decay. Whereas earlier treatments of post-inflation reheating modeled the inflaton as a collection of independent particles, each decaying into matter fields perturbatively [5–7], these newer analyses focus on collective, coherent effects which may allow the inflaton condensate to transfer much of its energy rapidly into a highly non-thermal distribution of low-momentum particles. The inflaton's decay, in other words, may be *resonant*, coherent and dominated by the peculiarities of quantum statistics [8–28].¹ During preheating, certain Fourier modes of the matter-field fluctuations grow essentially exponentially with time, $\delta\varphi_k \propto \exp(\mu_k t)$, where μ_k is known as the Floquet index or characteristic (Lyapunov) exponent. Such rapid growth of the field fluctuations leads to particle production, with the number density of particles per mode k growing roughly as $n_k \propto \exp(2\mu_k t)$ [8–15, 21, 22, 26–28]. The period of resonant inflaton decay has been dubbed “preheating,” in order to distinguish it from earlier perturbative studies of reheating.

Details of this resonance, such as the rate of particle production and the momenta of particles produced, remain model-dependent. Even so, certain consequences of such exponential growth in the occupation numbers of low-momentum quanta may be investigated. Such effects include the possibility of non-thermal symmetry restoration and defect production [29–35], supersymmetry breaking and GUT-scale baryogenesis

¹ Loosely speaking, preheating is to the old theory of reheating what the high-powered laser is to the household light-bulb.

[36–39]. These kinds of processes follow from the rapid amplification of matter-field fluctuations, driven during the preheating era by the oscillation of the coherent inflaton condensate. The strongly non-perturbative behavior within the matter-fields sector points to important cosmological consequences.

In a recent paper [40] (hereafter Paper I), we highlighted a distinct consequence which *must* in general arise during resonant post-inflation reheating: *the resonant amplification of metric perturbations*. Essentially, this arises because Einstein’s equations directly couple the explosive growth in the field fluctuations to the fluctuations in the gravitational curvature, so that the previous approach of neglecting metric perturbations is inconsistent. Unlike some of the possible effects of preheating for the matter-fields sector, the rapid growth of metric perturbations might leave distinct, observable signals in the CMB radiation, and could dramatically alter primordial black hole formation [40].

Here we develop the results of Paper I, via additional qualitative arguments and extensive numerical simulations of a two-field model. These simulations allow us to extend our discussion of the highly effective *symbiotic* interaction between metric perturbations and particle production during preheating. The source of the energy density for both metric perturbations and particle production is the oscillating inflaton condensate. Because the energy density of gravitational perturbations can carry an opposite sign to the energy density of the matter-field fluctuations, the two kinds of perturbations need not compete for the decaying inflaton energy, but rather can grow resonantly together, coupled by the Einstein equations [40]. (Note that we consider only scalar modes of the metric perturbations here. An interesting extension would be to consider effects on vector or vorticity modes of the perturbed metric during preheating.)

To emphasize this, contrast the resonant evolution of metric perturbations at preheating with that in traditional studies of particle production in curved space-time [41–43]. In the latter case, particle production arises via the polarization of the vacuum due to the shear of the metric and its perturbations, and hence energy conservation leads to a *reduction* in the anisotropy and inhomogeneity of space-time.² Instead, the resonant amplification of metric and field fluctuations described here takes energy from the oscillating *isotropic and homogeneous* part of the gravitational field and so is qualitatively very different: both metric and field fluctuations can grow simultaneously.

In earlier studies of preheating, which neglected the presence and growth of metric perturbations, important distinctions were found between preheating in single-field models as opposed to models with distinct fields coupled to the inflaton [8–15,21,22]. As highlighted in Paper I, and developed further here, it is crucial to attend to these distinctions when studying the amplification of metric perturbations at preheating. When resonant decay is possible at all in the single-field cases, it is typically rather weak, occurring only within a narrow momentum range and governed by a small characteristic exponent μ_k . When distinct scalar fields are coupled to the inflaton, however, qualitatively different behavior results throughout most of parameter space: resonances may be very broad in momentum space, and the rate of particle production may be much greater than in the single-field cases. Whereas the expansion of the universe during preheating is

² The most powerful use of this idea was in the Misner “chaotic cosmology” program [44] whereby it was hoped that arbitrary initial anisotropy and inhomogeneity might be smoothed out by particle production [45–48].

often sufficient to damp dramatically the resonant particle production in the single-field, narrow-resonance cases, broad resonances in the multi-field cases often survive in the expanding universe.

As we demonstrate here, broad resonances in multi-field models can drive the exponential amplification of metric perturbations, and in these models, analyses based on Floquet theory remain useful, even when the expansion of the universe is included. (See Fig. 1.) In general, the effective characteristic exponents $\mu_{k,\text{eff}}$ for the growth of metric fluctuations are greater than or equal to the exponents governing the field fluctuations' growth.

Furthermore, as is true of the non-thermal spectra of matter-field fluctuations amplified during preheating, the strongest resonances (largest μ_k) for the metric perturbations occur for long-wavelength modes. Often these modes have wavelengths larger than the Hubble scale at preheating, $k/a < H$, where $a(t)$ is the cosmic scale factor. As discussed in Paper I, such long-wavelength modes may indeed be amplified causally at preheating; in general, this need not be surprising, since Einstein's field equations necessarily incorporate relativistic causality. *The resonant growth of super-Hubble fluctuations derives from the coherence of the inflaton condensate on these large scales immediately after inflation, itself a natural consequence of inflation.* This growth bypasses the standard conservation law since it is accompanied by *resonant growth of entropy perturbations*, arising from the violent transfer of energy from the inflaton. Earlier studies of metric perturbations induced during (rather than after) inflation in multi-field models of inflation indicated that super-Hubble metric perturbations will not, in general, remain 'frozen' at their horizon-crossing amplitudes, but will rather evolve non-adiabatically [49–52]. The preheating, exponential amplification of super-Hubble

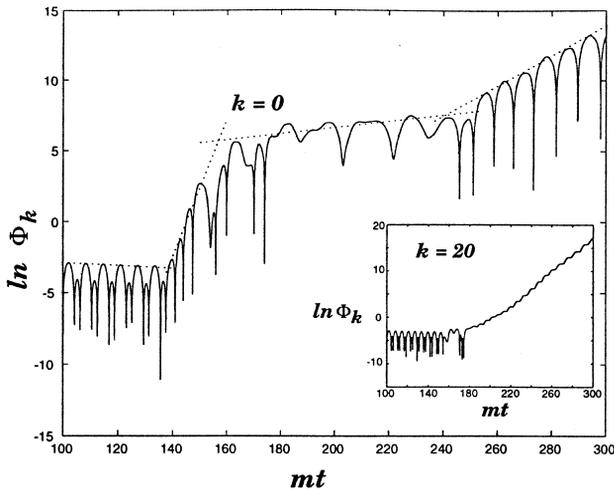


Fig. 1. Growth of $\ln \Phi_k$ versus mt , for $q = 8000$ with $\Phi \sim 10^{-3}$ initially. Φ is the gauge-invariant gravitational potential, m is the inflaton mass, and q is the resonance parameter, as defined in Section 2. The specific model considered here is given by Eq. (8). The main plot is for $k = 0$, and the inset shows growth for $k = 20$; the former is a super-Hubble mode, while the latter is within the Hubble radius. This plot comes from linearized perturbation theory, and so cannot be used to describe the modes' behavior in the non-linear regime. Note, however, the strong breaking of scale invariance in the earliest, linear-regime phase: the two modes go non-linear at very different times.

metric perturbations studied here represents an extreme example of such dynamic, non-adiabatic, long-wavelength growth.

By amplifying a distinct, highly scale-dependent spectrum of metric perturbations after the end of inflation, multi-field preheating could dramatically alter the observational consequences often assumed to hold for generic models of inflation. Rather than being sensitive only to the primordial, ‘stretched’ quantum fluctuations of the inflaton from deep within the Hubble radius during inflation, the power spectrum of density perturbations created during inflation would be deformed by the later preheating dynamics; the spectrum measured by COBE would therefore be sensitive only to these combined effects.

Given the robustness of the amplification of long-wavelength metric perturbations during strong preheating, then, one of the greatest outstanding issues regarding the post-inflationary universe is the non-linear evolution of these perturbations. The numerical simulations presented here stem from integrating the equations of linearized perturbation theory. The time it takes for this linearized approximation to break down may be measured self-consistently, and is discussed below. Typically, modes go non-linear even before the matter-field fluctuations do, and hence well before what had previously been assumed to be the end of preheating. Beyond linear order, mode-mode coupling between metric perturbations of different wavelengths is likely to be crucial for understanding the evolution of the power spectrum. These non-linear effects would in general remove the individual ‘fingerprints’ of specific inflationary models.

Naturally, the challenge remains to study this fully non-linear regime in quantitative detail. In preparation for such future studies, we indicate analytically below some effects which may be expected from such non-linearities. We emphasize that these effects are qualitatively new. The old theory of perturbation evolution based on conserved quantities [4] allowed for large amplification of metric perturbations at reheating, but required no production of entropy perturbations, i.e. purely adiabatic spectra on large scales. Perhaps more important, the conserved quantities were used to find the metric potential Φ during inflation by using the COBE data $\Delta T/T \sim 10^{-5}$ and the Sachs–Wolfe relation $\Delta T/T \approx \frac{1}{3}\Phi$. In the usual account, the large amplification of the $k \rightarrow 0$ modes of Φ between inflation and matter domination then requires fine-tuning of the inflaton mass ($m \sim 10^{-6}M_{\text{pl}}$) and self-coupling ($\lambda \sim 10^{-12}$).

What we discuss departs from this in two important ways. First, entropy perturbations are resonantly amplified during preheating, so that the standard Bardeen parameter ζ is not conserved and Φ is further amplified. Second, and perhaps more important, we cannot use the total amplification naively to fine-tune the initial amplitude of Φ during inflation to match with the $\Phi \sim 10^{-5}$ at decoupling required by the CMB. This second point is discussed further below, in Section 7.

Work on the evolution of scalar metric perturbations through preheating began with study of the $k = 0$ mode in simple single-field models [53,54], for which no super-Hubble resonance was found. At this early stage, a major concern was the periodically singular nature of the evolution equations of the Bardeen potential Φ due to the appearance of $\dot{\varphi}^{-1}$ terms, where φ is the oscillating inflaton background field. In contrast, the evolution of gravitational waves (tensor perturbations) has none of these problems [55–57].

Historically, two-field models were studied soon afterwards [58,59]. In [58], the periodic singularities were removed by employing space-time averaging. This also

removed the source of the parametric resonance and hence no parametric amplification was found. In [59], the oscillations of the inflaton were retained and the singularities avoided by using the so-called Mukhanov variables. A resonance in the scalar metric perturbations was found. However, that study was essentially restricted to classical mechanics – only the two background scalar fields were considered and no reheating to non-condensate modes ($k > 0$) was taken into account. Further, the study was limited to the first (and narrowest) resonance band of the Mathieu equation along the line $A = 2q$. Partly because of this, the authors never considered what happens when metric perturbations go non-linear.

Paper I [40] was a response to the perceived limitations of these earlier works in regard to a realistic treatment of preheating. Work subsequent to Paper I has re-examined the single-field case [60,61] and begun study of the complex numerics of the problem when the metric perturbations become fully non-linear [62]. The need for such a program of study to understand strong preheating is one of the broad theses of this paper.

The remainder of this paper is organized as follows. In Section 2, we review some basic features of preheating when the coupled metric perturbations are neglected altogether, and highlight the qualitative, as well as quantitative, differences between preheating with a single inflaton field, versus preheating when the inflaton is coupled to distinct scalar fields. Section 3 presents a sketch of the important conceptual foundations of the alternative picture of preheating introduced in Paper I and developed further here. Section 4 begins the detailed study of the coupled metric perturbations. We present the gauge-invariant coupled equations for fluctuations of the metric and fields, and discuss the issues of anisotropic stresses and entropy perturbations. In Section 5 we address the question of causality and the amplification of super-Hubble modes.

In Section 6, we focus on a simple two-field model, performing a wide range of numerical simulations to show the growth and evolution of fluctuations, and the scale-dependence of amplification. The time t_{nl} during which the linearized approximation remains adequate for the resonantly amplified metric perturbations is studied in detail. Section 7 turns to the essential role played by non-linear features in understanding the preheating epoch, such as backreaction, mode-mode coupling, and turbulence. We develop some of the consequences such strong non-linearities may have for the perturbation power spectrum, and discuss possible observational consequences of the rapid amplification of long-wavelength metric perturbations. Section 8 discusses possible “escape routes” from the resonant amplification found in our simple model. In Section 9, we describe the hierarchy of inflationary models delineated by the strength of the amplification of super-Hubble metric perturbations. Conclusions follow in Section 10.

2. Preheating basics

2.1. Minkowski space-time

In this section, we review briefly some of the key results for preheating, as derived in the absence of metric perturbations. This section is meant to introduce key ideas and fix

our notation; more complete reviews of preheating may be found in [26–28]. We begin the discussion in Minkowski space-time, and consider after that important deviations from these results for models in an expanding (but unperturbed) space-time.

The equation of motion for any scalar field χ coupled via a $g\phi^2\chi^2$ term to the inflaton, or for example, inflaton fluctuations in a model with a quartic self-coupling, takes the general form in Minkowski space-time:

$$\left[\frac{d^2}{dt^2} + \Omega_k^2(t) \right] \chi_k = 0, \quad \Omega_k^2(t) \equiv \omega_k^2 + g\varphi_0^2 f(\omega t) + \lambda_\chi \Sigma(t). \quad (1)$$

Here $\omega_k^2 = k^2 + m_\chi^2$ is the square of the time-independent natural frequency of the mode χ_k in the absence of interactions; φ_0 and ω are the amplitude and frequency of the inflaton's oscillations, respectively, and $f(\omega t) \equiv \phi^2(t)/\varphi_0^2$. The modes χ_k thus evolve with a time-dependent frequency, $\Omega_k(t)$.

The term $\Sigma(t)$ measures backreaction on the modes χ_k due to non-linear χ self-couplings, such as $\lambda_\chi \chi^4$. A similar term will appear in the equation of motion for the oscillating inflaton field, which encodes the drain of energy from the oscillating inflaton to the amplified fluctuations. It is common in the preheating literature to treat such non-linear terms with either the Hartree–Fock or large- N approximations, in which case $\Sigma(t) \sim |d^3 k| |\chi_k|^2$. As discussed below in Section 7, these approximations capture some elements of the non-linear structure of the quantum theory, but neglect such effects as rescatterings between χ_k modes with different momenta k . These rescattering terms can in many cases become quite important in the later stages of preheating. In any case, $\lambda_\chi \Sigma(t) \ll g\varphi_0^2$ for early times, and this term may be neglected for the first stage of preheating.

Most models of inflation end with the inflaton ϕ oscillating coherently. These include the common examples of chaotic inflation in convex potentials, and new inflation with its second-order phase transition, but excludes models with first-order phase transitions which complete via bubble nucleation, warm inflation, and non-oscillating models. In those cases for which the inflaton does oscillate coherently at the end of inflation, resonance effects may become crucial. For early times after the start of the inflaton's oscillations, when $\lambda_\chi \Sigma(t)$ is negligible, $f(\omega t)$ is *periodic* when studying preheating in Minkowski space-time. It is convenient to write

$$\Omega_k^2(t) = \omega^2 [A_k + 2qf(\omega t)], \quad (2)$$

where $A_k \propto \omega_k^2/\omega^2$, and $q \propto g\varphi_0^2/\omega^2$. Exact relations depend on specifics of the inflaton potential $V(\phi)$; for a massive inflaton without self-couplings, $q = g\varphi_0^2/(4m^2)$, where m is the inflaton mass. (The factor of 4 in the denominator for q is a convention which comes when we write the equation of motion for χ_k in the form of the Mathieu equation.) In this early time regime, the equation of motion for the modes χ_k is a linear second-order differential equation with periodic coefficients. Floquet's theorem then shows that solutions take the form [63]

$$\chi_k(t) = P_k(t) e^{\mu_k t}, \quad (3)$$

where $P_k(t)$ is periodic. For certain values of k , lying within discrete 'resonance bands' Δk , the Floquet index μ_k will have a non-zero real part, indicating an exponential

instability. Modes within these resonance bands thus grow exponentially in time. For the remainder of this paper, we will write μ_k as the real part of the Floquet index only.

Resonantly amplified modes correspond to rapid particle production. The occupation number-density for quanta of momentum k may be written simply in the form

$$n_k = \frac{\Omega_k}{2} \left(|\chi_k|^2 + \frac{1}{\Omega_k^2} |\dot{\chi}_k|^2 \right) - \frac{1}{2}. \tag{4}$$

This follows directly from standard Bogoliubov transformation techniques [41–43]; its form may be understood intuitively, as discussed in [21,22], as $n_k = \rho_k(\chi)/\Omega_k$, where $\rho_k(\chi) = \frac{1}{2}(|\dot{\chi}_k|^2 + \Omega_k^2|\chi_k|^2)$ is the energy density carried by a given χ_k mode. From Eq. (4), it is clear that modes within a resonance band, growing as $\chi_k \sim \exp(\mu_k t)$, correspond to a number density of produced particles for that momentum of $n_k \sim \exp(2\mu_k t)$. This is how preheating produces a highly non-thermal spectrum of particles immediately after inflation: only particles in certain discrete bands Δk get produced resonantly, resulting in a highly non-Planckian distribution. Occupation numbers for quanta lying within the resonance bands grow quickly, and the small k modes may well be described as classical before the end of preheating, in the sense that the density matrix takes a diagonal form with rapidly oscillating off-diagonal terms which time-average (in Minkowski space-time) to zero and are suppressed in the long-wavelength limit [64–67].

The strength (μ_k) and width (Δk) of the resonance depend sensitively upon the ‘resonance parameter’ q . In the narrow resonance regime, $q \leq 1$, analytic results yield $\mu_k \sim \Delta k \sim q\omega$, indicating ‘mild’ exponential growth within narrow resonance bands [8–15,21–25]. In the broad resonance regime, $q \gg 1$, analytic study of the coupled modes is far more involved, though significant progress has been made; in this case, the largest exponents satisfy $\mu_k \geq O(\omega)$, and the widths of the resonances scale as $\Delta k \propto q^{1/4}$ [8–22,68]. In all known cases, the strongest resonances (largest μ_k) occur on average for low-momentum (small- k) modes.

Still stronger resonances occur in models which possess a ‘negative coupling instability’ [68]. This instability affects multi-field models for which the coupling $g\phi^2\chi^2$ between the inflaton and a distinct scalar field χ is negative: $g < 0$. (One can still maintain a positive-definite energy density for such models, and hence a stable vacuum structure, by adding appropriate values of quartic self-couplings for both ϕ and χ .) Now the small- k modes χ_k ‘feel’ an *inverted* harmonic oscillator potential, that is, the parameters of Eq. (2) become approximately

$$A_k \approx \frac{k^2 + m_\chi^2}{m^2} + 2q, \quad q \approx \frac{g\varphi_0^2}{4m^2}, \tag{5}$$

so that $A_k \leq 0$ for $k^2 + m_\chi^2 \leq \frac{1}{2}|g|\varphi_0^2$. Such models lead to highly efficient amplification of these modes χ_k , now within even wider resonance bands than in the ‘usual’ broad-resonance cases. For modes subject to a negative coupling instability, the resonance band widths scale as $\Delta k \sim |q|^{1/2}$ rather than as $q^{1/4}$ [68].

In Minkowski space-time, the resonances which characterize the earliest stages of preheating will end when the backreaction of produced particles, $\Sigma(t)$, significantly damps the inflaton’s oscillations, ending the parametric resonance. This process is typically non-equilibrium and non-Markovian, i.e. it is very badly modeled by a $\Gamma\dot{\phi}$ term in the equation of motion for ϕ [21–25]. One may find this time analytically,

setting $\lambda_\chi \Sigma(t_{\text{end}}) = g\varphi_0^2$ and solving, in the Hartree approximation, for t_{end} which yields $t_{\text{end}} \sim (4\mu_{\text{max}} m)^{-1} \ln(10^{12} mg^{-5} M_{\text{pl}}^{-1})$, where μ_{max} is the maximum value of μ_k [21–25]. After t_{end} the system of coupled oscillators enters the fully non-linear regime; even here, certain modes χ_k may continue to grow due to strong rescattering effects [16–22,69].

Before considering the important changes to these results in an expanding, dynamical background space-time, we first contrast preheating in a single-field model with preheating in multi-field models. Consider a single-field inflationary potential of the form

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4. \quad (6)$$

Considered in Minkowski space-time, this model can produce no inflaton quanta at all when $\lambda = 0$: fluctuations $\delta\phi$ obey the same equation of motion as the background field ϕ , and simply oscillate. When $\lambda \neq 0$ and $m \neq 0$, the resonance parameter for the fluctuations $\delta\phi$ takes the form

$$q \approx \frac{3\lambda\varphi_0^2}{4m^2}. \quad (7)$$

Yet according to the usual treatment of the power spectrum of primordial density perturbations generated during inflation [1–4], both λ and m for this model are constrained by observations of the CMB radiation to take the values $\lambda \sim 10^{-12}$ and $m \sim 10^{-6} M_{\text{pl}}$, where $M_{\text{pl}} = 1.22 \times 10^{19}$ GeV is the Planck mass. Furthermore, for chaotic inflation, the slow-roll phase will end, and the inflaton will begin to oscillate, at $\varphi_0 \sim 0.3 M_{\text{pl}}$. These yield $q < 0.1$. When $m = 0$, the inflaton's frequency of oscillation is well-approximated by $\omega \approx 0.85\sqrt{\lambda}\varphi_0$ [21–25], yielding $q \approx 1$.³ Needless to say, smaller values of the inflaton's amplitude, corresponding for example to GUT-scale inflation, will of course yield far smaller values of q . Thus the *maximum* values of the resonance parameter, q , in single-field models of inflation are restricted to the narrow-resonance regime, $q \leq 1$.

The restriction to the narrow-resonance regime for single-field models of inflation may be contrasted with the situation in multi-field models. Consider, for example, the simple model

$$V(\phi, \chi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}g\phi^2\chi^2. \quad (8)$$

In this case, the resonance parameter for the χ field becomes

$$q = \frac{g\varphi_0^2}{4m^2} \sim g \times 10^{10}, \quad (9)$$

³ When the inflaton potential contains quartic self-couplings, the specific resonance structure for coupled fields is quite different from the case when $\lambda = 0$; in particular, whereas an infinite hierarchy of resonance bands results when $\lambda = 0$, only one single resonance band exists for $\lambda \neq 0, m = 0$. [21–25] Even so, in *both* cases, preheating in a single-field model is restricted to the narrow-resonance regime, for which $q \leq 1$ and $\mu_k \sim q\omega$, and thus these fine-structure points are not relevant to our present discussion.

where the last expression on the right holds for the chaotic inflation case. In this case, CMB observations only restrict the coupling g through radiative loop corrections, which suggest $|g| \leq 10^{-6}$.⁴ Thus even weak couplings between the inflaton and the distinct scalar field χ will produce resonance parameters well into the broad-resonance regime, with $|q| \gg 1$. *Multi-field models of preheating generically lie within the broad-resonance regime, whereas single-field models are necessarily limited to the narrow-resonance regime.* This difference will be crucial when we consider the amplification of metric perturbations at preheating later.

To finish the discussion of preheating in Minkowski space-time, we examine the case of non-periodic inflaton evolution. Perhaps the simplest and most unified way to do this is in terms of the $1 - D$ Schrödinger operator, \mathcal{L} , which is dual to the Klein–Gordon operator in Eq. (1) under the interchange of time and space [71]. Under this exchange, the unstable modes k for which $\mu_k > 0$ are identified with the *complement* of a subset of the spectrum of \mathcal{L} (technically the complement of the absolutely continuous part of the spectrum, σ_{AC} [71]). From this we immediately see why μ_k always had a band structure in the cases considered above, since in the dual picture we simply find the conduction bands of condensed matter models of metals with periodic potentials – the Brillouin zones.

In the case where the inflaton evolution is quasi-periodic, the above duality enables one to show that the band structure of the periodic case is destroyed, leaving only a nowhere-dense, *Cantor* set of stable modes [70,71]. Similarly, in the case that ϕ exhibits partial or complete stochastic behaviour, one finds that all modes grow ($\mu_k > 0$), with measure one [71]. In the dual, condensed matter picture, this simply corresponds to the famous *Anderson localization* of electron eigenfunctions in metals with random disorder [72]. This was found independently by Zanchin et al. [73,74], who also showed that it leads to an increase in μ_k over the purely periodic case at the same q , i.e. noise enhances preheating.

An important application of these results is the case when ϕ evolves chaotically in time, as is expected when there are many fields coupled to the inflaton [75,76]. In the strongly coupled limit one can show that the ϕ motion becomes stochastic and hence one knows that preheating is enhanced in this region of the parameter space [70].

2.2. Unperturbed FRW background

Now consider what happens for preheating in an expanding FRW (unperturbed) universe with scale factor $a(t)$, rather than in Minkowski space-time. In this case, a generic equation of motion for modes coupled to the inflaton takes the form

$$\left[\frac{d^2}{dt^2} + \Omega_k^2(t) \right] \{ a^{3/2}(t) \chi_k \} = 0,$$

$$\Omega_k^2(t) = \frac{k^2}{a^2} + m_\chi^2 + g\phi_0^2(t)f(\omega t) - \frac{3}{4}H^2 - \frac{3}{2}\frac{\ddot{a}}{a} + \lambda_\chi \Sigma(t), \quad (10)$$

⁴This constraint does not exist in (softly broken) supersymmetric theories. Hence the inflaton may be strongly coupled to other fields and the resonance parameters may be arbitrarily large [70].

where $H = \dot{a}/a$ is the Hubble constant. Note that both the physical momentum, $k/a(t)$, and the amplitude of the oscillating inflaton, $\varphi_0(t)$, now redshift with the expansion of the universe. Note also that Floquet’s theorem no longer applies, since the terms within $\Omega_k^2(t)$ are no longer simply periodic. This translates into the physical result that *no resonant amplification occurs in an expanding universe for models restricted to the narrow-resonance regime*. In particular, single-field models of inflation which include a massive inflaton fail to produce any highly efficient, resonant particle production at the end of inflation, when the expansion of the universe is included.⁵

Explosive particle production is still possible in an expanding universe, however, for multi-field models in the broad-resonance regime, with $q \geq 10^3$ [16–22]. (For models in which the χ field has a mass comparable with the inflaton’s mass, resonant growth occurs for $q \geq 10^4$.) In this case, the growth of the χ_k modes does not take a simple exponential form; instead, certain modes χ_k jump *stochastically* each time the oscillating inflaton passes through zero, but evolve quasi-adiabatically in between these sudden jumps [21,22]. To understand these jumps, it is convenient to work in terms of an adiabaticity parameter,

$$r \equiv \frac{|\dot{\Omega}_k|}{\Omega_k^2}. \tag{11}$$

The large amplification of the modes χ_k occurs during those periods when $r > 1$. For definiteness, consider again the model of Eq. (8). In an expanding universe, for early times after the start of the inflaton’s oscillations, the inflaton will evolve as

$$\phi(t) = \varphi_0(t)\sin(mt) \approx \varphi_0(0)\frac{\sin(mt)}{mt}, \tag{12}$$

and an effective resonance parameter for the χ_k modes may be written

$$q_{\text{eff}}(t) = \frac{g\varphi_0^2(t)}{4m^2}. \tag{13}$$

Then, for $q_{\text{eff}} \gg 1$, we have, to leading order in t^{-1} ,

$$r \approx \frac{q_{\text{eff}}^{-1/2} \cos(mt)}{2 \sin^2(mt)}. \tag{14}$$

Given that $q_{\text{eff}} \gg 1$, we see that we will only get strongly non-adiabatic evolution of the modes χ_k near the zeros of $\phi(t)$. Analytic estimates exist for μ_k in this case [21,22]. Two points are of special interest. First, μ_k is largest for $k = 0$, with $\mu_k \propto \ln(1 + 2\exp[-\pi k^2/(2a^2 m^2 \sqrt{q_{\text{eff}}})] + \mathcal{P})$. Hence the amplification of super-Hubble modes is present even in the absence of metric perturbations. Second, μ_k depends on the phase of the wavefunction of χ , through the term \mathcal{P} – a purely quantum effect. The change in

⁵ An exception comes with the model of a *massless*, quartically self-coupled inflaton which is conformally coupled to the curvature, for which weak resonance of the $\delta\varphi$ fluctuations survives in an expanding universe, again with $\mu_k \sim q\omega \leq \omega$, due to the conformal invariance of the theory [21–25].

the phase between the zeros of ϕ is much larger than 2π , and hence the phase (mod 2π) is a pseudo-random variable of time. This stochastic behavior is transmitted to μ_k .

Thus, for the broad-resonance regime, strong amplification is still possible at preheating, though it is no longer of a simple parametric resonance kind. Nevertheless, many concepts from the pure Floquet theory remain helpful, in particular q_{eff} and $\mu_{k,\text{eff}}(t) \equiv \chi_k^{-1} \dot{\chi}_k$ [16–22,68]. As in the Minkowski space-time case, for $q_{\text{eff}} \gg 1$, the effective characteristic exponents may still be large, $\mu_{k,\text{eff}}(t) \geq \omega$, though now they are explicitly time-dependent. One way to understand the evolution of the amplified χ_k modes in this case is that as a given mode, $k/a(t)$, redshifts with the expanding universe, it will pass through a series of broad resonance bands, Δk , and get amplified in each one [21,22,77]. As in Minkowski space-time, the widths of these resonance bands for $q_{\text{eff}} \gg 1$ scale as $\Delta k \sim q_{\text{eff}}^{1/4}$ [21,22]. Similarly, the widths of resonance bands for models with a strong negative-coupling instability in an expanding universe scale as $\Delta k \sim |q_{\text{eff}}|^{1/2}$ [68].

In the old theory of preheating, resonant passing of energy back to the inflaton only occurs at second-order through *rescattering* effects. This was first discovered numerically [16–20] and then explained analytically [21,22] as arising from mode-mode coupling, which acts as a driving term that leads to population of higher-momentum (shorter-wavelength) modes of the inflaton. This provides a way of stemming the build-up of backreaction effects and allows the total variances of the fields to be larger than would be estimated using only the Hartree or $N \rightarrow \infty$ approximations, which miss all rescattering effects. This is of relevance to the discussion of the maximum scale for non-thermal symmetry restoration [29–35]. In Section 4 we show that when the metric perturbations are included, they induce rescattering *even at linear order* and again this helps to raise the maximum variances.

Multi-field models, themselves inherently more realistic than models which include only an inflaton field, lie generically within the broad-resonance regime, with $q_{\text{eff}} \gg 1$. Resonant growth of these coupled field modes, then, marks the immediate post-inflation era, as preheating produces large occupation numbers n_k for quanta in various resonance bands Δk . The highly non-thermal distribution of particles results from a stochastic process, in which modes χ_k slide through a series of resonance bands and receive a large, non-adiabatic ‘kick’ each time the oscillating inflaton passes through a turning point. In the broad-resonance regime, this strongly non-adiabatic, rapidly growing behavior of certain modes χ_k remains robust even when expansion of the universe is included.

As the remaining discussion will demonstrate, it is not only field modes χ_k which enjoy such robust resonant behavior at preheating; metric perturbations do as well. Moreover, the amplified metric perturbations in turn provide one more source to induce increased particle production, by means of gravitational rescattering. By approaching the immediate post-inflation epoch in a fully self-consistent manner, the surprising behavior of preheating becomes all the more striking.

3. Conceptual foundations of the metric perturbation resonances

Before we embark on a detailed study of the amplification of metric perturbations during preheating, it is perhaps appropriate to highlight the important conceptual

differences between metric perturbation evolution in the case of a single field and the case involving many fields interacting *both* via gravity and other forces.⁶

As we will see below (see Eq. (41)), Einstein’s field equations couple metric perturbations Φ_k directly to field fluctuations $\delta\varphi_{Ik}$, for *all* wavenumbers k . This means that metric perturbations can in general be amplified at any k . On small scales, $k/a \geq H$, such resonances could lead to non-linear effects such as pattern and primordial black hole formation [78,79]. On much larger scales, $k/a \ll H$, there is in addition the question of the *causal* amplification of such long-wavelength modes Φ_k .

The schematic in Fig. 2 shows the five main points which are involved in a conceptual understanding of metric perturbation resonances in strong preheating. All except causality are related to the existence of multiple fields in realistic preheating models, and are relevant for large- k and small- k resonances alike.

1. *Non-gravitational forces may dominate*

In the case of fields interacting through non-gravitational forces, the coupling constants may naturally be large, implying that the evolution of metric perturbations is *non-gravitationally* dominated [70]. This can destroy the solution $\Phi = \text{const.}$ for the $k = 0$ growing mode.

2. *Large resonance parameters*

In the single-field case, the natural dimensionless resonance parameter q is order unity or smaller since both the coupling and frequency of oscillations is controlled by the same constant of the theory (e.g. the effective mass or self-coupling). In the multi-field case, the couplings between fields may differ from the inflaton frequency of oscillation m by orders of magnitude, and hence the associated resonance parameters $q_I \equiv g_I \varphi^2 / m^2$ can naturally be very large.

3. *Entropy perturbations*

In the multi-field case, entropy perturbations are generic (see discussion above Eq. (35)). In the presence of strong non-gravitational interactions and the associated explosive transfer of energy between fields with different effective sound speeds, these entropy perturbations are resonantly amplified, thereby invalidating the use of standard conserved quantities to calculate Φ .

4. *Causality is not violated*

Causality is not violated by resonant amplification of the $k \sim 0$ modes. This is discussed further in Section 5 in terms of the unequal-time correlation function for the perturbed energy density. The field equations are relativistic and hence causality is built into their solution. The apparent violation of causality arises essentially from the ‘initial conditions’ at the start of preheating: the homogeneity and coherence of the zero-mode of the inflaton, which is required to solve the horizon problem.

Note that here and throughout this paper, we use the term “super-Hubble” to refer to modes with wavelengths longer than the Hubble radius, that is, with $k/a < H$. It is common, though somewhat misleading, to use the term “superhorizon” for this. As is well-known, the Hubble radius marks the speed-of-light sphere in any given cosmologi-

⁶ The case of many non-interacting fields is well studied, but excludes preheating, and is typically restricted to slow-roll. See e.g. Refs. [49–52].

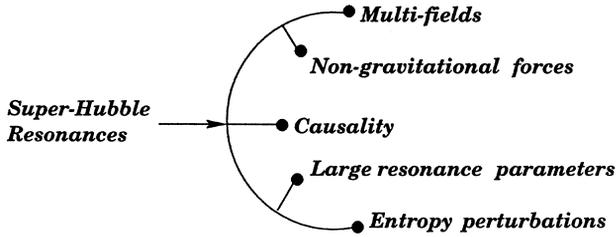


Fig. 2. Conceptual points of importance in understanding the origin of the metric perturbation resonances in strong preheating.

cal epoch, but can be dramatically different from the actual particle horizon. Of course, by solving the horizon problem, inflation generically produces a particle horizon which is many orders of magnitude larger than the Hubble radius during post-inflationary periods in the universe's history. As emphasized in this section and below, the "super-Hubble" resonances described in this paper are thus amplified fully causally; their wavelengths remain smaller than the true particle horizon after inflation, though they stretch beyond the Hubble radius at the end of inflation. Related points have been made in the study of the backreaction of gravitational waves [80]: not only are the super-Hubble modes consistent with causality and locality, but if one imposes a Hubble-scale infrared cut-off and excludes these modes, then energy conservation is violated.

Throughout we simply write $k = 0$ for the longest-wavelength modes which we consider. Physically, the maximum wavelength of perturbations which can ever be amplified during preheating corresponds to the correlation length of the coherent inflaton condensate. Inflation guarantees that the inflaton background field will be coherent (and thus will oscillate at preheating with a space-time-independent phase) over vastly larger length scales than the Hubble radius at preheating. This is what explains the causal amplification of super-Hubble perturbations. Thus, where we write $k = 0$, we really intend something more like $k \sim 10^{-28}$, corresponding to length-scales well within the Hubble radius during inflation, which then were stretched coherently during the inflationary epoch. Of course, in our numerical simulations, these exponentially tiny wavenumbers remain practically indistinguishable from 0. We return to the question of the inflaton condensate's coherence length below, in Section 7.

4. Metric preheating

With the points of the previous section clearly in view, we may proceed to study in detail the behavior of metric perturbations during preheating. We use the gauge-invariant formalism of [4] to investigate the evolution of scalar perturbations of the metric. (The amplification of tensor perturbations, or gravitational waves, at preheating has been considered in [55,56].) In longitudinal gauge, the perturbed metric takes a particularly simple form, expressed in terms of gauge-invariant quantities:

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(t)(1 - 2\Psi)\delta_{ij}dx^i dx^j, \quad (15)$$

where Φ and Ψ are gauge-invariant potentials. On length scales shorter than the curvature scale (which itself is longer than the Hubble length), preheating in an open universe proceeds with little difference from the spatially flat case (see [24,25]), so we restrict attention in this paper to the case of a spatially flat background.

4.1. The single-field case

To begin, consider a single-field model, with the inflaton field separated into a background field and quantum fluctuations, $\phi(x) = \varphi(t) + \delta\varphi(x)$, where $\delta\varphi$ is the gauge-invariant fluctuation in the longitudinal gauge [4]. The Friedmann and Klein–Gordon equations governing the background quantities are

$$H^2 = \frac{1}{3}\kappa^2 \left[\frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right], \tag{16}$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V_\varphi = 0, \tag{17}$$

where $V_\varphi \equiv \partial V / \partial \varphi$ and $\kappa^2 \equiv 8\pi G = 8\pi M_{\text{pl}}^{-2}$. These may be combined to give

$$\dot{H} = -\frac{1}{2}\kappa^2 \dot{\varphi}^2. \tag{18}$$

The energy–momentum tensor

$$T^{\mu\nu} = \nabla^\mu \phi \nabla^\nu \phi - g^{\mu\nu} \left[\frac{1}{2} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi) \right], \tag{19}$$

may be written in the form

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu}, \tag{20}$$

where $u_\mu \propto \nabla_\mu \phi$ is the natural four-velocity defined by the scalar field, which coincides in the background with the preferred 4-velocity. The energy density and effective pressure are

$$\rho = \frac{1}{2} \dot{\phi}^2 + V, \quad p = \frac{1}{2} \dot{\phi}^2 - V,$$

where an overdot denotes the comoving covariant derivative $u^\mu \nabla_\mu$. Thus in the natural frame, $T^{\mu\nu}$ has perfect fluid form, so that the energy flux and anisotropic stress vanish in this frame: $q_\mu = 0 = \pi_{\mu\nu}$. In any other frame which is close to u^μ , i.e. $\tilde{u}^\mu = u^\mu + v^\mu$, where $v_\mu u^\mu = 0$, we have to linear order

$$\tilde{\rho} = \rho, \quad \tilde{p} = p, \quad \tilde{q}_\mu = -(\rho + p)v_\mu, \quad \tilde{\pi}_{\mu\nu} = 0. \tag{21}$$

Thus, to linear order, the scalar field has no anisotropic stress in any frame that is moving non-relativistically relative to the natural frame. Writing $T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$, where $\bar{T}_{\mu\nu}$ is the background energy–momentum tensor and $\delta T_{\mu\nu}$ is the (gauge-invariant) perturbation, it follows that

$$\delta T_j^i \propto \delta_j^i. \tag{22}$$

Then the perturbation of the Einstein equations $\delta G_{\mu\nu} = \kappa^2 \delta T_{\mu\nu}$ gives [4]

$$\left[\nabla_i \nabla^j - \frac{1}{3} \delta_i^j \nabla^2 \right] (\Phi - \Psi) = 0 \quad \text{for } i \neq j, \quad (23)$$

so that $\Psi = \Phi$.

The gauge-invariant perturbations for this single-field case then satisfy the equations

$$3H\dot{\Phi} + (3H^2 - \nabla^2)\Phi = -\frac{1}{2}\kappa^2 \left[\dot{\phi}(\delta\varphi) + V_\varphi \delta\varphi - \dot{\phi}^2 \Phi \right], \quad (24)$$

$$(\delta\varphi)'' + 3H(\delta\varphi)' + [V_{\varphi\varphi} - \nabla^2]\delta\varphi = 4\dot{\phi}\dot{\Phi} - 2V_\varphi\Phi, \quad (25)$$

$$\dot{\Phi} + H\Phi = \frac{1}{2}\kappa^2 \dot{\phi} \delta\varphi. \quad (26)$$

Note that the metric perturbations enter at the same, linear order in Eq. (25) as the usual term, $V_{\varphi\varphi} \delta\varphi$, and must be included in this equation to remain consistent. Eq. (26) can be integrated directly to give

$$\Phi = \frac{\kappa^2}{2a} \int a \dot{\phi} \delta\varphi dt, \quad (27)$$

which allows us to eliminate Φ from Eq. (25) at the expense of introducing terms which depend on the complete time-history of $\dot{\phi}$ and $\delta\varphi$. In the realistic case where the fields exhibit a stochastic component to their evolution [70,71,73,74], this non-Markovian term in the evolution equation is expected to cause new conceptual deviations from the simple Markovian equation in the absence of Φ [23–25].

By Eqs. (19) and (20), the perturbed energy density and pressure in longitudinal gauge are

$$\delta\rho = \dot{\phi}(\delta\varphi)' + V_\varphi \delta\varphi - \dot{\phi}^2 \Phi, \quad (28)$$

$$\delta p = \dot{\phi}(\delta\varphi)' - V_\varphi \delta\varphi - \dot{\phi}^2 \Phi. \quad (29)$$

Eq. (28) shows how the energy density carried by gravitational fluctuations, $\delta\rho[\Phi] = -\dot{\phi}^2 \Phi$, is *negative* when $\Phi > 0$ [40]. The entropy perturbation Γ is defined by [81]

$$p\Gamma \equiv \delta p - \left(\frac{\dot{p}}{\dot{\rho}} \right) \delta\rho.$$

Using Eqs. (24)–(29), we can show that

$$p\Gamma = - \left(\frac{4V_\varphi}{3\kappa^2 a^2 H \dot{\phi}} \right) \nabla^2 \Phi. \quad (30)$$

Thus on all scales $k > 0$, *the single scalar field generically induces non-zero entropy perturbations*. These are easily missed if one neglects the metric perturbations Φ . Slow-roll inflationary conditions will keep these entropy perturbations negligible during inflation, but they will grow in concert with the metric perturbations Φ during the oscillatory, preheating phase.

An important quantity is the Bardeen parameter, i.e. the spatial curvature on uniform density surfaces [4]

$$\zeta = \Phi - \frac{H}{\dot{H}} [\dot{\Phi} + H\Phi], \tag{31}$$

which evolves as

$$\dot{\zeta} = \left(\frac{2H}{\kappa^2 \dot{\phi}^2} \right) \nabla^2 \Phi \tag{32}$$

in the single-field case. It follows from Eqs. (16), (18), (30) and (32) that

$$\dot{\zeta} = \left(\frac{a^2 \rho p}{2\dot{V}} \right) \Gamma.$$

Provided that $\dot{\phi}$ (and hence \dot{H}) is always non-zero, the $k = 0$ mode of ζ is conserved for models with only one scalar field, and the $k = 0$ entropy perturbation is zero. However, during reheating, $\dot{\phi}$ periodically passes through zero, and this can produce entropy perturbations even on super-Hubble scales, as noted in [54,60]. These entropy perturbations will not be strong, since at most weak resonance occurs on super-Hubble scales in single-field models.

4.2. The multi-field case

We now generalize the above discussion and study the behavior of Φ for models with N interacting scalar fields ϕ_I , where $I = 1, \dots, N$, taking the inflaton to be $I = 1$. Each field may be split into a homogeneous part and gauge-invariant fluctuations as $\phi_I(t, \mathbf{x}) = \varphi_I(t) + \delta\varphi_I(t, \mathbf{x})$. The background equations (16), (18) become

$$H^2 = \frac{1}{3} \kappa^2 \left[\frac{1}{2} \sum_I \dot{\phi}_I^2 + V(\varphi_1, \dots, \varphi_N) \right], \tag{33}$$

$$\dot{H} = -\frac{1}{2} \kappa^2 \sum_I \dot{\phi}_I^2, \tag{34}$$

The first question to address in the multi-field models is whether anisotropic stress can arise at linear level. Each field defines its natural frame via $u_I^\mu \propto \nabla^\mu \phi_I$, and these are all close to each other in a perturbed FRW universe, and all reduce to the preferred frame in the background. Given a choice u^μ of global frame that is close to these frames, we have $u_I^\mu = u^\mu + v_I^\mu$, where the relative velocities satisfy $v_I^\mu u_\mu = 0$. The energy-momentum tensor in Eq. (19) becomes

$$T^{\mu\nu} = \sum_I \nabla^\mu \phi_I \nabla^\nu \phi_I - g^{\mu\nu} \left[\frac{1}{2} \sum_I \nabla^\alpha \phi_I \nabla_\alpha \phi_I - V(\phi_1, \dots, \phi_N) \right],$$

and Eq. (20) generalizes to

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu - p g^{\mu\nu} + u^\mu q^\nu + q^\mu u^\nu,$$

where the total energy density, pressure and energy flux are

$$\rho = \frac{1}{2} \sum_I \dot{\phi}_I^2 + V, \quad p = \frac{1}{2} \sum_I \dot{\phi}_I^2 - V, \quad q^\mu = \sum_I \dot{\phi}_I^2 v_I^\mu,$$

and the anisotropic stress vanishes at linear order. Since these total quantities obey the same frame-transformation laws as in Eq. (21), it follows that the anisotropic stress vanishes to first order in any frame close to u^μ . Then, as before, Eqs. (22) and (23) hold, and we have $\Psi = \Phi$. We emphasize that the vanishing of anisotropic stresses in a multi-field universe holds only in linearized gravity (see Refs. [82,83] for the non-linear equations). If fluctuations grow to non-linear levels and relative velocities become relativistic, then anisotropic stresses will emerge and could significantly affect the dynamics (compare Ref. [84]).

The nature of entropy perturbations is another example where the single-field and multi-field cases bifurcate strongly. In the multi-field case, it is known that *entropy perturbations are generic*, arising from differences in the effective sound speeds, from relative velocities, and from interactions (see, e.g., Refs. [81,85]). Since reheating is the conversion of energy from a coherent field to a relativistic, thermalized gas, it is clear that entropy will be generated. Even exactly adiabatic initial conditions at the exit from inflation will not prevent the production of entropy perturbations during reheating, since the fields are interacting non-gravitationally.

The entropy perturbation in a multi-component system is defined by [81]

$$p\Gamma = \sum_I (\delta p_I - c_s^2 \delta \rho_I), \tag{35}$$

where the ‘total’ effective sound speed is given by

$$c_s^2 = \frac{1}{h} \sum_I h_I c_I^2 \quad \text{where } h_I = \rho_I + p_I, \quad h = \sum_I h_I.$$

Then Γ can be related to $\dot{\zeta}$. In the two-field case, $\dot{\zeta}$ is [49–52]

$$\dot{\zeta} = -\frac{H}{\dot{H}} \nabla^2 \Phi + \frac{1}{2} H \left[\frac{\dot{\phi}_1^2 - \dot{\phi}_2^2}{\dot{\phi}_1^2 + \dot{\phi}_2^2} \right] \left(\frac{\delta \varphi_1}{\dot{\phi}_1} - \frac{\delta \varphi_2}{\dot{\phi}_2} \right), \tag{36}$$

which shows explicitly how ζ can grow even for $k = 0$. As we have discussed in Section 2, preheating with more than one scalar field is a period of intensely non-adiabatic evolution, and we may expect that $\dot{\zeta}$ will deviate strongly from 0 even in the $k \rightarrow 0$ limit. This is precisely the kind of non-adiabatic growth which we observe below. This invalidates the use of ζ as a way of finding Φ , and one is forced to explicitly integrate the evolution equations to find Φ , as we do in Section 6. Further discussion of these points is made in Section 7.

Note also that ζ may be written in terms of the Mukhanov perturbative quantities

$$Q_I = \delta \varphi_I + \frac{\dot{\phi}_I}{H} \Phi. \tag{37}$$

In the two-field case [59]

$$\zeta = \frac{H(\dot{\phi}_1 Q_1 + \dot{\phi}_2 Q_2)}{\dot{\phi}_1^2 + \dot{\phi}_2^2}. \quad (38)$$

The linearized equations of motion for the Fourier modes of Φ are (see Ref. [86], where we have corrected the first equation)

$$3H\dot{\Phi}_k + [(k^2/a^2) + 3H^2]\Phi_k = -\frac{1}{2}\kappa^2 \sum_I [\dot{\phi}_I(\delta\varphi_{Ik})' - \Phi_k \dot{\phi}_I^2 + V_I \delta\varphi_{Ik}], \quad (39)$$

$$(\delta\varphi_{Ik})'' + 3H(\delta\varphi_{Ik})' + (k^2/a^2)\delta\varphi_{Ik} = 4\dot{\Phi}_k \dot{\phi}_I - 2V_I \Phi_k - \sum_J V_{IJ} \delta\varphi_{Jk}, \quad (40)$$

$$\dot{\Phi}_k + H\Phi_k = \frac{1}{2}\kappa^2 \sum_I \dot{\phi}_I \delta\varphi_{Ik}. \quad (41)$$

Eq. (41) exhibits a simple generalization of the integral solution of the single field case, Eq. (27). There are, however, important distinctions to appreciate between the single-field and multi-field cases. Note first that when more than one scalar field exists, cross-terms arise in the equations of motion for each of the field fluctuations, $\delta\varphi_I$. That is, the evolution of $\delta\varphi_I$ depends, even at linear order, on all the other $\delta\varphi_J$. These cross-terms have been neglected in previous studies of multi-field models of preheating, which ignored metric perturbations [8–22]. They are present here, in general, because the gravitational potential Φ couples universally with all matter fields, that is, with the entire $\delta T_{\mu\nu}$. Each matter field fluctuation $\delta\varphi_I$ ‘talks’ with the metric perturbations, Φ , which in turn ‘talks’ with all of the other matter field fluctuations. Given the form of Einstein’s equations, then, each of these field fluctuations $\delta\varphi_I$ becomes coupled to each of the others.

Though distinct from non-linear mode-mode coupling of the sort discussed in Section 7, these cross-terms, entering at linear order, can play a large role in making ‘‘gravitational rescattering’’ quite effective at preheating. We adopt the term *rescattering* for these linear-order cross-terms, whereas it is usually reserved for discussing non-linear effects, because the coupled form of the equations in Eqs. (39)–(41) indicates that modes are amplified, and quanta are produced, due to the coupling between fluctuations of different fields. The significance of these linear-order rescattering terms is confirmed in our numerical simulations, presented below.

As discussed in Section 2, further differences remain generic between single-field and multi-field models at preheating, and these distinctions result in dramatic differences for the metric perturbations in each case. In the single-field case, resonances, when they survive the expansion of the universe at all, are quite weak. Furthermore, the amplitude of the inflaton’s oscillations decays with monotonic envelope due both to expansion of the universe and to dissipation from particle production. The right-hand side of the constraint equation, Eq. (26), is therefore either weakly growing or decaying in time. Hence Φ_k in this single-field case is either weakly growing or decaying in time.

In the multi-field cases, we may expect far greater resonance parameters, $q_{\text{eff}} \gg 1$, resulting in much larger $\mu_{k,\text{eff}}$ than in the single-field case. That is, in the multi-field case, the system will in general be dominated by *non-gravitational* forces; the couplings

g_I between the various scalar fields are independent of the gravitational sector. Moreover, the homogeneous parts φ_I , $I > 1$, typically grow during the resonance phase, with the energy coming from the inflaton condensate, φ_1 .

4.3. Homogeneous parts of the fields

There is a subtlety, however, in how to define φ_I for $I > 1$ [21,22]. One approach is to define them implicitly as the homogeneous objects, φ_I , which satisfy the unperturbed Klein–Gordon equations:

$$\ddot{\varphi}_I + 3H\dot{\varphi}_I + V_I = 0, \quad (42)$$

where $V_I \equiv \partial V / \partial \varphi_I$. Our main complaint with this definition is that it is not operational. It does not tell one how to find φ_I from the full fields $\phi_I(t, \mathbf{x})$. More important for the study of preheating, these equations (which simply express the conservation of the homogeneous part of the energy density) miss the gauge-invariant energy contributions from the $k=0$ mode of Φ_k . While in standard analyses this is unimportant, in preheating, a large amount of energy is transferred to this mode, which should be included as a homogeneous backreaction effect. As we stressed in different terms earlier, it is important to appreciate that because gravity has negative specific heat, the energy flowing into $\Phi_{k=0}$ actually *aids* the growth of field-fluctuations $\delta\varphi_{Ik}$.

To include this effect, we define the homogeneous parts of the fields $I > 1$ as

$$\varphi_I \equiv \delta\varphi_{I,k=0}. \quad (43)$$

Thus the background parts of the coupled fields are the homogeneous $k=0$ portions of the fluctuation spectra. Since, as we saw in Section 2, it is the small- k modes which grow the quickest for models with broad resonance, the φ_I might grow at least as quickly as other modes $\delta\varphi_{Ik}$ for $k \neq 0$ but small. Then the combination $\varphi_I \delta\varphi_{Ik}$, which acts as a source for Φ_k , grows even more quickly than either term alone. This, in turn, makes $\mu_{k,\text{eff}}$ for the modes Φ_k greater than that for any of the matter fields. We confirm this behavior below, in Section 6. In the end this is not crucial for the robustness of the resonances however. In Section 6.3 we employ the definition implicit in Eq. (42) and show that the resonances remain, if not quite as strong.

Now with the definition (43), even if all $k > 0$ modes of *each* of the fluctuations $\delta\varphi_{Ik}$ happen to lie within stability bands at a time t , Φ_k will still grow if one of the homogeneous fields φ_I lies within a resonance band. It is a simple combinatorial problem to see that as the total number of fields, N , increases, the probability that *all* fields φ_I and $\delta\varphi_{Ik}$ lie simultaneously within stability bands, and hence that Φ_k receives no growing source terms, decreases quickly.

As an alternative operational definition, one might use the field variances as a basis for defining the homogeneous fields φ_I ($I > 1$)⁷, viz.

$$\varphi_I \equiv \sqrt{\langle (\delta\varphi_I)^2 \rangle} = \left[(2\pi)^{-3} \int d^3k |\delta\varphi_{Ik}|^2 \right]^{1/2}. \quad (44)$$

⁷ See e.g. Ref. [87] for a recent implementation of this definition in inflation.

While the growth rate of φ_I with this definition *may* be less than that when using Eq. (43), it is sure to exhibit resonance since the k -space integral must cross all the instability bands. This is reflected in all numerical studies of preheating which show the rapid growth of the variances [16–20]. Hence the resonant growth will still exist for the metric perturbations Φ_k .

Even using spatial averaging of the fields $\phi_I(x, t)$ to define the background fields φ_I would lead to resonant growth of the φ_I , since the averaging again covers instability bands when viewed in Fourier space.⁸

As we will see in Section 6, these choices for how to define the homogeneous background fields affect quantitative details, but not the qualitative behavior, of the resonant amplification of modes Φ_k during preheating.

In contrast, space-time averaging, as used by Hamazaki and Kodama [58], removes the oscillations (and hence the singularities of their particular evolution equations) of the fields. This is effectively equivalent to taking the limit $q \rightarrow 0$ and no resonant production of entropy perturbations was found. Since the perfect fluid equivalence already exists with oscillating scalar fields, and non-singular systems exist with which to study the problem, we disagree with their procedure and note that this is perhaps the single most important reason why they did not find the super-Hubble resonances. Taruya and Nambu [59], on the other hand, include the inflaton oscillations and find resonant growth of the entropy perturbations (isocurvature mode). They claim however that it has no effect on the adiabatic mode, which they claim is constant. Further, they do not allow the resonance to go non-linear and hence claim that the isocurvature mode decays to zero, essentially leaving no observable traces. Contrary to this, our explicit simulations show that Φ is strongly amplified and once it becomes non-linear, which on general grounds occurs at or before the time at which the matter-field fluctuations do [40], the role of the expansion in damping the amplitude of the modes is not at all clear.

5. Causality

Keeping the important distinction between H^{-1} and the particle horizon in mind [89], we may now consider the role of causality in the immediate post-inflationary universe. *Causality involves correlations across spatial distances.* As emphasized in [90–92], this is best studied via unequal-time correlation functions in real space, rather than in Fourier space. In particular, causality requires

$$\langle \delta T_{00}(\mathbf{r}, \eta) \delta T_{00}(\mathbf{0}, \tilde{\eta}) \rangle = 0 \quad \text{for } r > \eta + \tilde{\eta}, \quad (45)$$

where η is conformal time ($ad\eta = dt$). For simplicity, we assume that one of the fields $\phi_J, J > 1$, dominates preheating ($\mu_J > \mu_I, \forall I$). The correlator (45) is then essentially controlled by the single field ϕ_J . From Eqs. (26) and (28), Eq. (45) shows that it is sufficient to study the unequal-time correlation functions of the field fluctuations, since

⁸ Actually the situation is rather more subtle: no known covariant and gauge-invariant averaging scheme exists, even within *classical* general relativity. However, the above statements are true for volume averaging in synchronous coordinates and will remain true for any reasonable averaging scheme. See [88] for a discussion of backreaction and averaging issues relevant to inflation.

each term within δT_{00} will be proportional to the field fluctuations to a good approximation, for modes within a resonance band. Thus, the function we need to study is $\Delta(\mathbf{r}, \eta, \tilde{\eta}) \equiv \langle \delta\varphi(\mathbf{r}, \eta) \delta\varphi(\mathbf{0}, \tilde{\eta}) \rangle$, i.e.

$$\Delta(\mathbf{r}, \eta, \tilde{\eta}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} \delta\varphi_k^*(\eta) \delta\varphi_k(\tilde{\eta}). \quad (46)$$

Inside a resonance band, these modes will grow as $\delta\varphi_k(\eta) = P_k(\eta) e^{\mu_k \eta}$, where $P_k(\eta)$ is a quasi-periodic, decaying function of time. Evaluating Eq. (46) in the saddle-point approximation, and using Eq. (11.4.29) of Ref. [93], yields

$$\Delta(\mathbf{r}, \eta, \tilde{\eta}) \approx \left[P_k^*(\eta) P_k(\tilde{\eta}) e^{\mu_k(\eta + \tilde{\eta})} \right]_{k_{\max}} \frac{\exp(-r^2/\xi^2)}{2\pi^{3/2}\xi^3}, \quad (47)$$

where k_{\max} is the wavenumber at which μ_k is maximum, and

$$\xi^2 \equiv 4(\eta + \tilde{\eta}) \left| \frac{\partial^2 \mu_k}{\partial k^2} \right|_{k_{\max}}. \quad (48)$$

Causality thus places a constraint on the characteristic exponent:

$$\left(\mu_k \left| \frac{\partial^2 \mu_k}{\partial k^2} \right| \right)_{k_{\max}} < \frac{1}{4}. \quad (49)$$

Note that causality restricts the *shape of the spectrum* of amplified modes, but not directly the wavelength of the perturbations. The approximation sometimes used within studies of preheating, that the distribution of amplified modes falls as a spike, $\delta(k - k_{\text{resonance}})$, violates causality, since this requires that the field fluctuation contain correlations on all length scales, even for $k/a \gg H$. Thus, subject to Eq. (49), the amplification of long-wavelength perturbations may proceed strictly causally. This raises the question of whether or not metric perturbations, amplified at preheating, could affect observable scales even today.

The result in Eq. (49) is of course limited by the single-field and saddle-point approximations, and also by the fact that the true growth in resonance bands will not be so cleanly exponential for multi-field cases in an expanding universe, since, as discussed in Section 2, realistic models of preheating have time-dependent $\mu_{k,\text{eff}}(t)$. Thus the remaining exponential tail in Eq. (49) is an artifact of the various approximations used. The key point is that, regardless of dynamical details and approximation methods, causality is not a question which can easily be answered in Fourier space, and *no* naive estimates of constraints on smallest k modes which could be amplified causally can be made, e.g. those based on comparisons of k with H .

The point we wish to emphasize is that the field equations are relativistic, hence causality is built in.⁹ The governing equations of motion, by themselves, make no distinctions between $k/a < H$ and $k/a > H$ regimes. A solution which satisfies the initial and boundary conditions must, therefore, be causal [see Fig. 3]. No matter or energy is moved superluminally to produce the super-Hubble resonances at preheating.

⁹ We thank John Barrow for making this point to us.

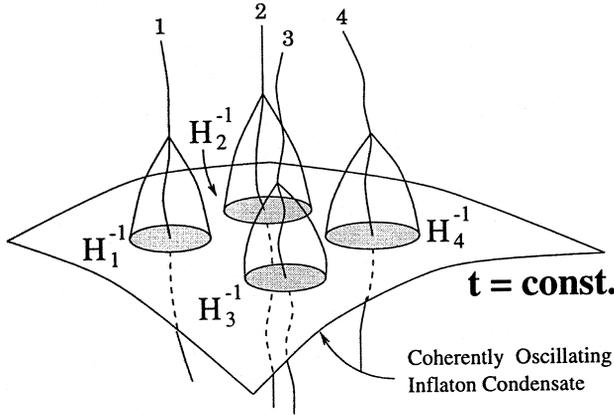


Fig. 3. Schematic plot illustrating how super-Hubble modes can be amplified without violating causality. To solve the horizon problem the inflaton must be correlated on vastly super-Hubble scales at the end of inflation, and hence this zero-mode oscillates with a space-time-independent phase. During preheating, fields at each space-time event react only to physics in their causal past, which since the end of inflation may be very small. However, because the inflaton condensate is exactly the same over vastly larger regions than H^{-1} at the end of inflation, super-Hubble modes can patch together the inflaton behavior in each region to reconstruct (“feel”) the globally oscillating condensate and hence grow resonantly. Due to the isometries of the spacelike surfaces of constant ϕ , all such patches are equivalent and can be mapped onto one another by a translation. Only once second-order backreaction effects destroy the coherence of the inflaton oscillations are these super-Hubble resonances weakened and perhaps ended.

Any apparently acausal behaviour must instead be due to the initial conditions, which, as we have emphasized above, are crucial for stimulating the super-Hubble resonances we have discussed. The resonant amplification of very long-wavelength perturbations at preheating stems from the initial coherence of the inflaton condensate at the end of inflation. We return to the question of the inflaton condensate’s coherence below, in Section 7.

As a final demonstration of this point, let us consider an inflaton distribution which is not a delta-function at $k = 0$ but is rather distributed over k -space. For clarity, let us ignore the metric perturbations completely. In this case, the evolution equation for χ_k with a $g\phi^2\chi^2$ coupling is of the form:

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \frac{k^2}{a^2}\chi_k + \frac{g}{(2\pi)^6} \int \int d^3k' d^3k'' \phi_{k'} \phi_{k-k'} \chi_{k-k'-k''} = 0. \tag{50}$$

We see immediately that if $\phi_k = 0$ for $k < k_{\text{crit}}$, then there *is no* resonant growth of χ_k modes for $k < k_{\text{crit}}$ since the inflaton is not correlated on those scales. From this it is easy to see that χ_k modes are amplified by resonance only up to the maximum scale on which the inflaton is correlated. Of course, inverse cascades (see Section 7) can in principle amplify longer-wavelength modes, but this is another, essentially non-linear, process.

The linear case which we solve numerically is qualitatively very different from the fully non-linear regime where solutions are not known, as occurs for example in the studies of topological defects [94]. In that case, causality can be imposed as a constraint

to determine whether a proposed ansatz is physical (i.e. on-shell) or not. This has been used to great effect in recent years in studies of the evolution of the energy–momentum tensor for causal defects, and in that context, where perturbations *must* come from displacement of energy, causality sets up strong constraints on the possible metric perturbation power spectrum. This is described in detail in Refs. [90–92,94].

6. Numerical results for the 2-field system

We now study a two-field model involving a massive inflaton, ϕ , and a distinct, massless scalar field, χ , with the potential

$$V(\phi, \chi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}g\phi^2\chi^2. \tag{51}$$

Taking χ to be massless is realistic in the sense that there are a large number of effectively massless degrees of freedom ($g_* > 100$) at these energies in all post-Standard Model theories. If $g > 0$, the minimum of the effective potential in Eq. (51) is $\chi = 0$, which we assume holds during inflation. For times after the end of slow-roll in this model, a good approximation for the evolution of the inflaton background field is:

$$\varphi(t) \approx \varphi_0(t_0) \frac{\sin(mt)}{mt}. \tag{52}$$

This approximation improves with time and is quite accurate for $\varphi < M_{\text{pl}}$. We show it in the inset to Fig. 4. This ansatz for the oscillating background field appears to provide two free parameters: $\varphi_0(t_0)$ and t_0 . Actually, there exists only one real degree of

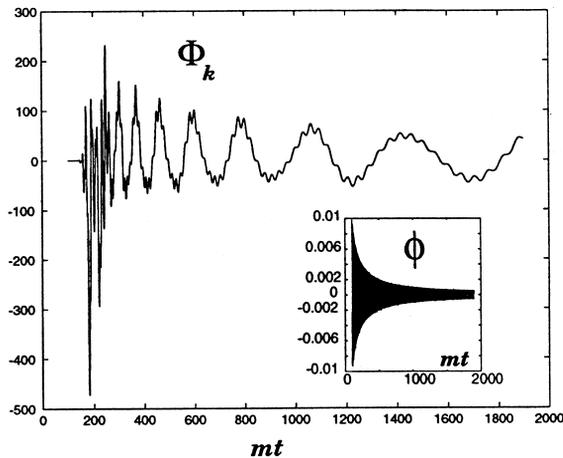


Fig. 4. Φ vs mt for $q = 5 \times 10^3$, for the $k = 0$ mode. Note the initial resonance which invalidates the linearized equations of motion within a few oscillations. The linearized solution then damps away (after $mt \sim 300$) due to the expansion, but this cannot be taken as indicative of the evolution of the non-linear solution. Interestingly there are two characteristic frequencies of oscillation of Φ_k after $mt \sim 250$ – one high frequency and one low frequency. The inset shows the evolution of the inflaton condensate and the damping due to the expansion of the universe.

freedom here, determined by the physical amplitude of the inflaton at the start of preheating. We have fixed $\varphi_0(t_0) = 0.3M_{\text{Pl}}$, so that taking $t_0 \rightarrow 0$ (which gives $\sin(mt_0)/mt_0 \rightarrow 1$) corresponds to a standard chaotic inflation scenario [16–20]; we may think of this as a large-amplitude limit. As becomes clear in the following simulations, the resonances are more effective and modes Φ_k go non-linear more quickly for smaller t_0 ; in particular, see Fig. 15 for a simulation with $mt_0 = 1$. In this case, relevant for simple chaotic inflation, modes Φ_k go non-linear after only $m\Delta t \sim 50$ even for quite weak values of the coupling g (and hence of the resonance parameter $q \sim O(100)$), much earlier than field modes $\delta\varphi_I$ would go non-linear in this simple model when Φ is neglected (compare with Refs. [16–20]).

Increasing the arbitrary parameter t_0 lowers the initial amplitude of inflaton oscillations, and hence allows us to study other, low energy-scale inflationary models, some of which we discuss further in Section 9. Parametrizing our equations this way means that the quoted values for the resonance parameter

$$q = \frac{g\varphi_0^2}{m^2} \quad (53)$$

correspond to the values for $t_0 = 0$; larger start-times, $t_0 > 0$, will correspond to smaller $q_{\text{eff}} < q$. Note that here q differs by a factor 4 from the q that usually appears in the Mathieu equation.

The Friedmann equation gives an approximate scale for the Hubble radius at the start of preheating, $H(t_0) \sim 2\alpha m$, where m is the inflaton mass, and α gives the physical amplitude of the inflaton at the start of preheating: $\varphi(t_0) \equiv \alpha M_{\text{Pl}}$. For chaotic inflation, this yields $H(t_0)/m \sim 0.6$ at the start of preheating; modes with $k/a(t_0) < H(t_0)$ are thus, roughly speaking, super-Hubble modes. For our simulations, we then use the Hubble expansion rate as averaged over a period of the inflaton's oscillations, $H = 2/(3t)$, which is appropriate for a massive inflaton.¹⁰ We also take the homogeneous portion of the χ field to be $\chi = (\delta\chi)_{k=0}$; as discussed above, using $\sqrt{\langle \delta\chi^2 \rangle}$ or the spatial average of χ would make quantitative changes, but would not alter the general features or qualitative behavior presented here.

In each of the simulations presented below, the initial conditions for the metric perturbations are taken to be the generic, scale-invariant spectrum expected to be produced during inflation. That is, we set $\Phi_k(t_0) = 10^{-5}$ for all k . As discussed below in Section 7, changing these initial conditions does not, in general, alter the qualitative behavior presented here. In particular, attempts to fine-tune these initial conditions for Φ_k so as to avoid non-linear evolution during preheating are destined to fail. We have also used the same initial conditions for the matter-field fluctuations, $\delta\varphi_{Ik}(t_0) = 10^{-5}$ and $(\delta\varphi_{Ik}(t_0))' = 0$, where these are measured in units for which $G = 1$, or $\kappa^2 = 8\pi$. The initial conditions were chosen to be scale-invariant (i.e. independent of k) to

¹⁰ The use of such time-averaging for the expansion rate and similar background quantities is acceptable in models dominated by non-gravitational terms, as is the case for the multi-field model we study here. This is in general *not* a good approximation when studying gravitational waves or fields with strong non-minimal couplings to the Ricci curvature scalar, since in these cases oscillations in the Ricci curvature can lead to strong resonances [57].

highlight the strongly k -dependent features of the resonance structure. We return to the discussion of the physical impact of different choices of initial conditions on the resonances below, in Section 8.4. Note that the figures in the following subsection plot natural logarithms (base e), rather than base-10.

6.1. Early growth of perturbations

In our numerical simulations, we integrate the coupled, linearized equations of motion for Φ_k , $\delta\phi_k$, χ , and $\delta\chi_k$, based on Eqs. (40) and (41), with the specific potential of Eq. (51). The resulting eight-dimensional system of first-order ordinary differential equations were integrated using a fourth-order Runge–Kutta routine with variable step-length automatically implemented to ensure accuracy. Two equations are required each for the evolution of $\delta\phi_k$, $\delta\chi_k$, and χ , and two for Φ_k , one with $k > 0$ and one with $k = 0$. We use Planck units for which $G = 1$, $\kappa^2 = 8\pi$, and a dimensionless time-variable $z \equiv mt$. Given the background expansion $H = 2/(3t)$, the scale factor grows on average as $a \propto z^{2/3}$. In other numerical studies of preheating, in the absence of metric perturbations, rescaled conformal time $\tau \equiv m\eta$ is often used; in this case, $z \propto \tau^3$, which facilitates comparisons between our results and earlier results.

Our numerical routine was tested against several limiting cases: (1) The standard results for preheating when metric perturbations are neglected, in which the Klein–Gordon equations for the fluctuations decouple at linear order and the system is four-dimensional with Φ set to zero. In this limit, the standard results of Refs. [16–22,68] are reproduced, as discussed further below. (2) Limiting regions of the parameter space with $q = 0$ were also checked. We neglect backreaction on the inflaton and non-linear mode-mode rescattering in these simulations, but include the linear-order coupling which results between Φ , $\delta\phi_k$ and $\delta\chi_k$. The truly non-linear effects are of course crucial for ending the first, explosive preheating phase, and we discuss some of these effects below, in Section 7.

First consider the growth of metric perturbations Φ_k in our model. Fig. 1 shows the evolution of two different modes, $k = 0$ and $k = 20$, as measured in units of m . The first is clearly super-Hubble scaled, whereas the second lies well within the Hubble radius. As is clear in this figure, *both* modes evolve quasi-exponentially for certain periods of time; that is, the modes slide in and out of various broad resonance bands. The $k = 0$ mode in fact begins to grow considerably after a time $m\Delta t \approx 40$, after fewer than seven inflaton oscillations.

Figs. 1 and 12 reveal the strongly scale-dependent spectrum for Φ_k which results from the early, linear-regime evolution of the coupled system. This is the analog for Φ_k of the strongly non-thermal distribution of particles produced in the first stages of preheating. We will return to the question of the power spectrum for Φ_k during the preheating epoch in Section 7. Of course, the evolution for both modes plotted here beyond $\Phi_k(t) \sim 1$ cannot be trusted based on our linearized equations of motion, and non-linearities will end the resonant growth. We plot the strong growth here as an indication that Floquet theory remains an effective tool for describing the linear regime. Rather than the simple Minkowski space-time results, the effective Floquet indices $\mu_{k,\text{eff}}(t)$ become time-dependent: the modes Φ_k slide into and out of resonance bands, each characterized by a certain exponential-amplification rate.

The coupled scalar field fluctuations $\delta\chi_k$ also experience dramatic quasi-exponential amplification when the coupling to Φ is taken into account, as shown in Figs. 5 and 7.

We may now compare the behavior of the matter-field fluctuations, $\delta\phi_k$ and $\delta\chi_k$, when we include the coupled metric perturbations, and when we neglect these perturbations as in the standard preheating literature. Fig. 6 contains a plot of the inflaton fluctuations $\delta\phi_k$ with and without including the coupling to Φ_k , and Fig. 7 contains a similar plot for $\delta\chi_k$.

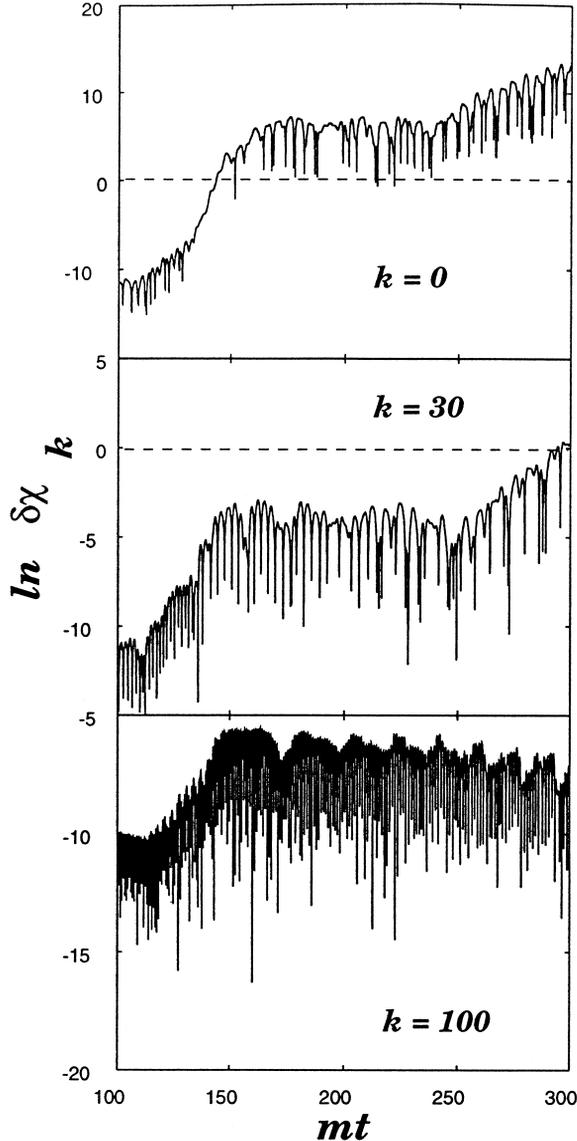


Fig. 5. The evolution of $\ln \delta\chi_k$ for $q = 8000$ and $k = 0$ (top), $k = 30$ (middle), and $k = 100$ (bottom). Note the significant resonant growth of the $k = 0$ mode.

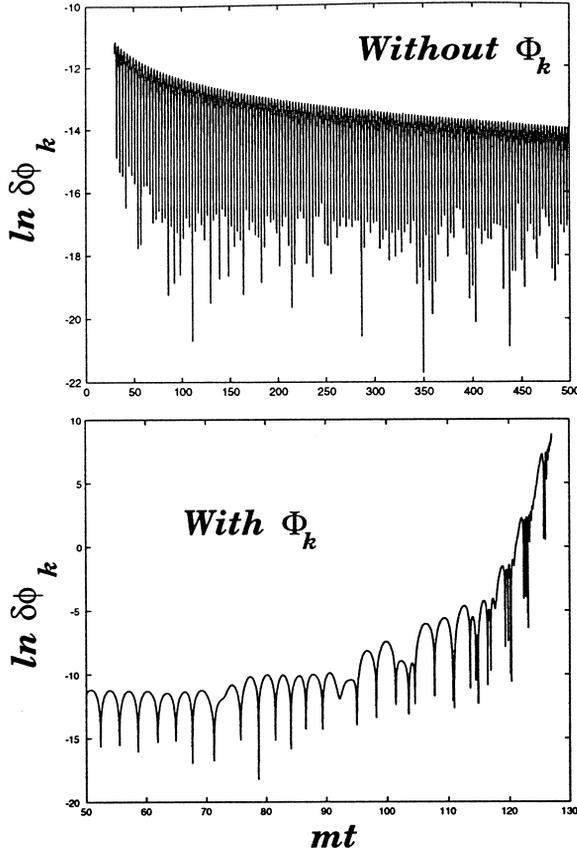


Fig. 6. $\delta\phi_k$ vs mt evolution, $q = 8000$. Top: note the complete lack of particle production in the absence of metric perturbations. Lower: the particle production due to linear rescattering via Φ_k terms is clear. (Note also that for the inflaton field, we maintain a distinction between φ , the oscillating, coherent condensate of zero momentum, and $\delta\phi_{k=0}$. It is the latter which we plot here.)

Note that for the early times plotted here, the behavior of each of these fluctuations when we neglect Φ_k matches well the behavior found in [16–20]. To facilitate comparison, recall that $z \propto \tau^3$, and that we plot here $\chi = (\delta\chi)_{k=0}$ rather than $\langle \delta\chi^2 \rangle$. From Fig. 5, it is clear that the $k=0$ mode of $\delta\chi_k$ dominates the growth, and hence contributes most to $\langle \delta\chi^2 \rangle$, so this makes for a fair comparison. The gradual decay of both $\delta\phi_k$ and $\delta\chi_k$ shown here for early times, when Φ is neglected, matches the behavior found in [16–22]. In these earlier studies, significant amplification of each of these fluctuation fields occurred only for $\tau \geq 10$, or, roughly, $z \geq 10^3$. Instead, when the coupling to the metric perturbations is included, we find significant growth in each of the field fluctuations well before these late times.

The strong, resonant growth of both $\delta\phi_k$ and $\delta\chi_k$ confirms that *the metric perturbations serve as both a source and a pump for the matter-field fluctuations*. We conjectured this above and in Paper I, based both on the form of the perturbed energy density and on the linear-order gravitational rescattering. This is a significant new effect which simply cannot be ignored when studying realistic models of preheating, which

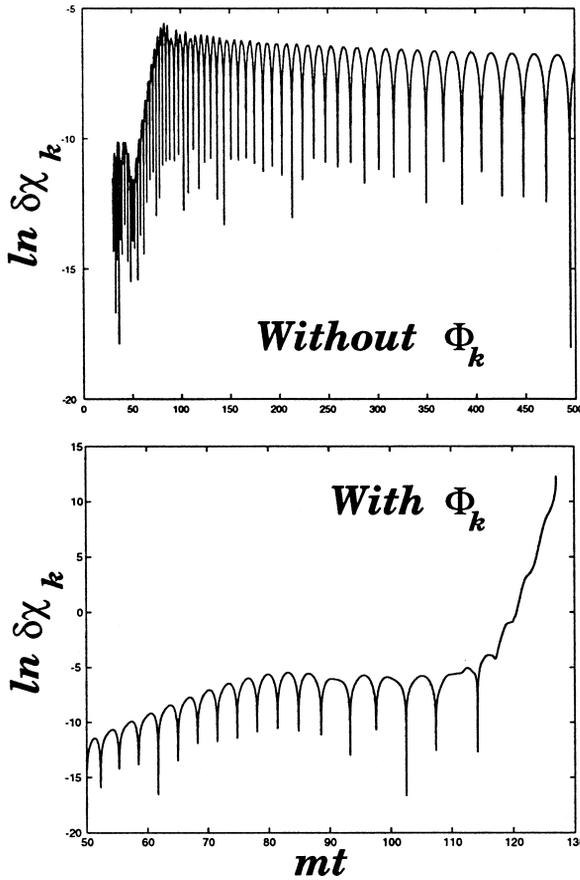


Fig. 7. Here we compare the evolution of $\delta \chi_k$ vs mt ($k = 0$, $q = 8000$, $mt_0 = 30$) with and without the full metric coupling.

involve more than one scalar field. Broad-resonance amplification persists in an expanding universe when $q_{\text{eff}} \gg 1$, and this amplification now affects Φ_k ; the fast-growing Φ_k modes then further stimulate growth in $\delta \phi_k$ and $\delta \chi_k$.

Moreover, note that the rate of growth $\mu_{k,\text{eff}}$ for $\Phi_{k=0}$ is *greater* than that for both $\delta \phi_k$ and for $\delta \chi_k$. This, too, is consistent with our analysis above, based on the form of the constraint equation, Eq. (41), for Φ_k .

Again, we emphasize that the point of these figures is not to study quantitative details of the field and metric perturbations at late times, which is impossible using our linearized evolution equations. Rather it is that quasi-exponential growth of *both* of these types of perturbations will take place early after the start of preheating, and long before it takes place when the metric perturbations are neglected.

6.2. Time to non-linearity

Given the rapid, quasi-exponential growth of these perturbations, the most appropriate physical quantity to study is the time t_{nl} it takes a mode k to saturate its linear

theory bound, that is, $|\Phi_k(t_{nl})| = 1$. Clearly in a system dominated by gravitation alone, $t_{nl} \sim H_0^{-1}$, the current Hubble time, which is much longer than reheating can last. At preheating, however, in multi-field models in the broad-resonance regime, we expect instead that

$$t_{nl} \sim \mu_{k,\text{eff}}^{-1}. \tag{54}$$

Modes lying within a stable band have $\mu_k = 0$ and $t_{nl} = \infty$, whereas modes within a resonance band will saturate their linear-theory limit at some finite time.

In Fig. 8 we show t_{nl} versus the resonance parameter q for moderate values of q and with a cut-off in the integration time of $m\Delta t = 250$. There is a clear phase transition at $q \sim 2800$, where modes start going non-linear for the first time. This is followed by a succession of narrow stability bands and a second phase transition around $q_* \approx 6000$ where no more stable modes are found. For $q > q_*$, essentially all modes go non-linear by $m\Delta t \sim 100$, i.e. after only a few inflaton oscillations.

The inset figure shows t_{nl} for the case of a negative-coupling instability amongst the field modes, which produces a more efficient resonant amplification of all of the perturbations. Note in particular that t_{nl} matches very well the prediction from earlier studies of negative-coupling instabilities that [68]

$$t_{nl} \sim \mu_{k,\text{eff}}^{-1} \propto |q|^{-1/2}. \tag{55}$$

In Figs. 9 and 10 we show the effect of decreasing the starting value of t_0 . Since decreasing t_0 increases the effective inflaton oscillation amplitude, we expect that it should increase the power of the resonance and hence decrease t_{nl} and the number of modes lying in stable bands. This is clearly seen in the figures. Note in particular that for most values of q in the range $1500 \leq q \leq 5000$, and with $5 \leq m t_0 \leq 30$, modes Φ_k go non-linear after $m\Delta t \sim 40 - 100$, well before any such non-linearities appear in studies of preheating which neglect the coupling to Φ .

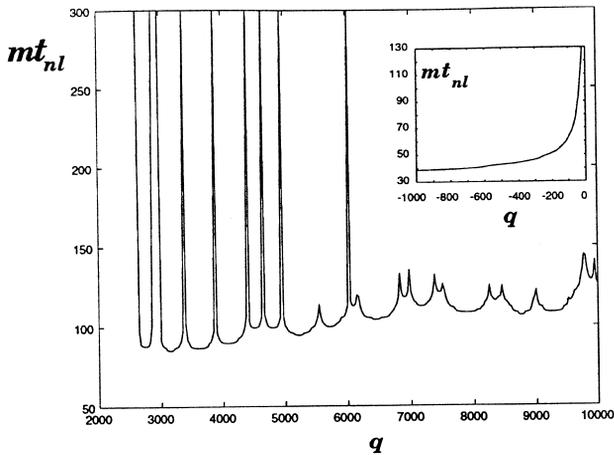


Fig. 8. Time to non-linearity t_{nl} vs q for the $k = 0$ mode of Φ_k . Note the transitions at $q \sim 2800$ and $q_* \sim 6000$. The inset shows t_{nl} vs q for negative q and $k = 0$. There are no stable bands and the time to non-linearity is short. The numerical solution for the negative- q case is closely approximated by the simple scaling law $t_{nl} \sim 1/\sqrt{|q|}$ predicted by Eq. (55).

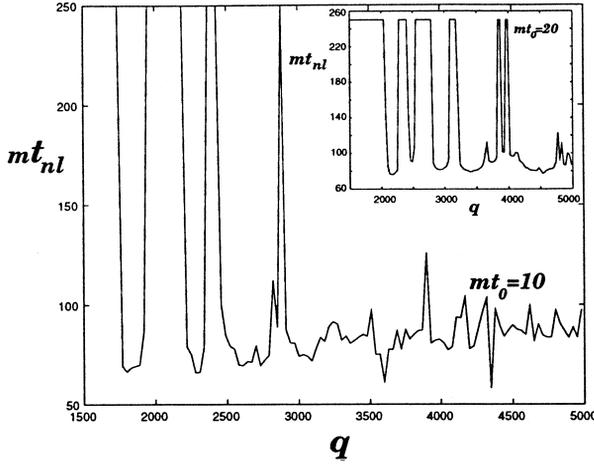


Fig. 9. t_{nl} vs q for the $k = 0$ mode and for $t_0 = 10$ (main figure) and $t_0 = 20$ (inset top right). From this it is clear that varying t_0 is not exactly the same as simply varying q essentially due to the $\dot{\phi}$ terms which are large when t_0 is small (the region where the amplitude of oscillations drops very rapidly). The large q , large t_0 regime is different from the smaller q , smaller t_0 regime which is closer in spirit to the stochastic resonance regime. This is evident between this figure and Fig. 10 which shows ordered resonance and stable bands with $mt_0 = 30$.

Note further that as t_0 is increased, decreasing the strength of the resonance, the resonance band structure becomes more and more clear. This behavior of t_{nl} reveals important qualitative agreement with earlier numerical studies of preheating. For preheating into massless fields χ , significant amplification in an expanding universe had only been found for $q \geq 10^3$ [16–20]. The rapid growth of Φ_k modes found here for our model of the coupled, massless ‘fields’ Φ and χ , falls in this range for q . Moreover, as q_{eff} increases, the dependence of t_{nl} on q becomes more fully stochastic, as seen especially in Fig. 11. In this figure, no stable bands persist for $q \geq 3000$, and t_{nl} varies

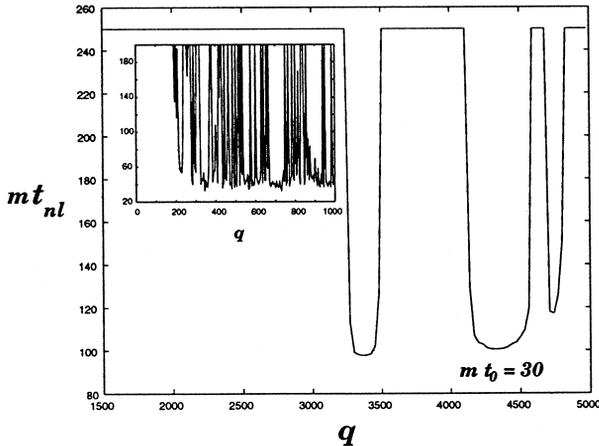


Fig. 10. t_{nl} vs q for Φ_k , $k = 0$ and $mt_0 = 30$. Note the ordered resonance bands. Inset: t_{nl} vs q for $mt_0 = 5$. At this small value of mt_0 modes go non-linear rapidly and no resonance band structure is apparent.

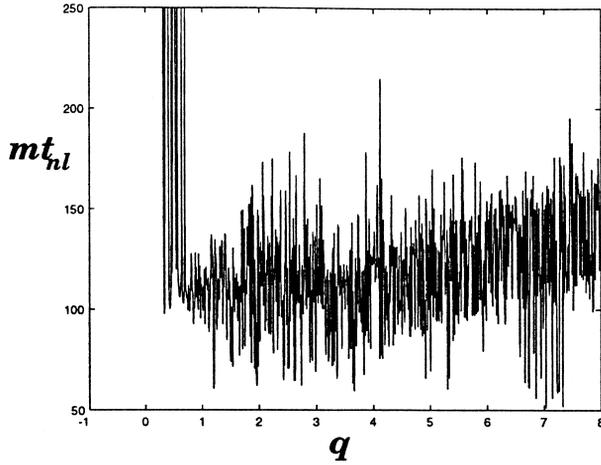


Fig. 11. t_{nl} over a wide range of $q > 0$ with $mt_0 = 30$. The essentially stochastic variation of t_{nl} is clear, as is the transition to non-linear behaviour for $q \sim 3000$. t_{nl} for $q < 0$ is not included here.

non-monotonically with q . These features again match the qualitative behavior found in [21,22] for the stochastic, broad-resonance regime in multi-field models of preheating that neglect metric perturbations. We find this same kind of behavior here for the evolution of Φ_k with $q \gg 1$.

A similar relationship governs the dependence of t_{nl} on k , as shown in Fig. 12. Again, note the absence of stable bands in k -space for this $q \gg 1$ regime. Note, too, that the most rapid amplification, and hence the smallest t_{nl} , occur in the $k \rightarrow 0$ limit. Moreover, this rapid amplification persists smoothly for modes greater than and less

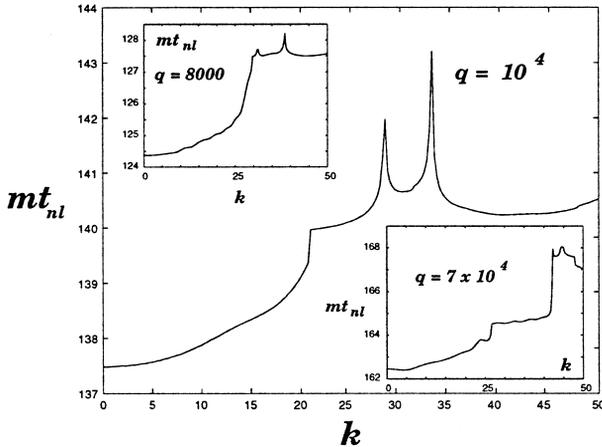


Fig. 12. t_{nl} vs k for $q = 8 \times 10^3$ (top left inset), $q = 10^4$ (main figure) and $q = 7 \times 10^4$ (bottom right inset). It is clear that the small k modes are, on average, the most strongly amplified (with no violation of causality). This is due to their low momentum and hence ease of violating the adiabatic condition, Eq. (11). Note that, as expected on general grounds from Floquet theory, there exist (q,k) -parameter regions where increasing q leads to an increase in t_{nl} showing that the resonance is not monotonically increasing in strength with q .

than the Hubble scale, with t_{nl} varying little in the range $0 \leq k \leq 15$. This provides a further indication that H^{-1} is not a fundamental quantity for preheating with more than one matter field.

6.3. Numerical results with an alternative definition of the background χ

A reasonable concern the reader might have is whether the strong super-Hubble resonances we found in the earlier section are an artifact of our definition of the homogeneous component of χ as the $k=0$ mode of $\delta\chi_k$. As already explained, we made that choice so as to be sensitive to the considerable energy in $\Phi_{k=0}$, and as such may be regarded informally as an attempt to include some backreaction effects.

The standard approach is to require that the background fields satisfy the unperturbed Klein–Gordon equation, which simply expresses energy conservation at zero order. In that case, and for the potential in Eq. (51), we have:

$$\ddot{\chi} + 3H\dot{\chi} + g\phi^2\chi = 0, \quad (56)$$

which misses the driving terms involving the homogeneous $\Phi_{k=0}$ mode. The $k=0$ mode is not in itself gauge-invariant [95], but is physical, since, in the longitudinal gauge which we use, all gauge freedoms are fixed [4]. From a physical point of view, the background quantities would correspond to the coherent condensate, with $k \sim 10^{-28}$ (see discussion at the end of Section 3), and hence are strictly gauge-invariant. It is clearly important to demonstrate that the super-Hubble resonances persist for the above, standard, definition of χ . In Fig. 14 we show the evolution of Φ_k and $\delta\phi_k$ for $k = 10^{-6}$ for $q = 8000$, $t_0 = 10$. In Fig. 13 we show the evolution of $\delta\chi_k$ and the homogeneous χ found from Eq. (56) for the same parameter values. Note that although χ grows

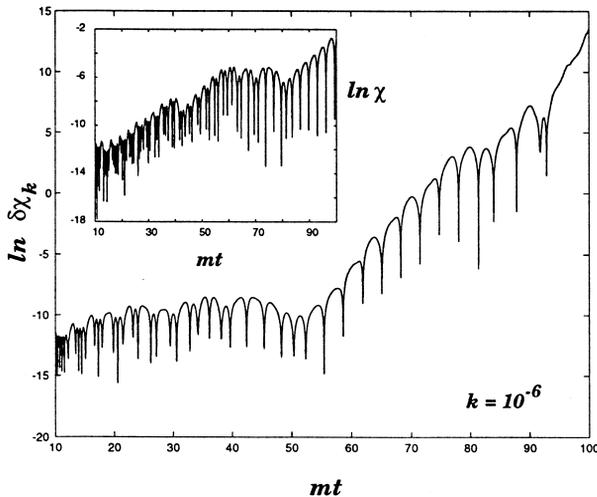


Fig. 13. $\delta\chi_k$ vs mt for $q = 8000$ and $k = 10^{-6}$ and $mt_0 = 10$. Inset: The evolution of the homogeneous mode χ vs mt for the same parameter values and using the evolution equation (56). Notice that although χ remains small throughout the simulation, all the perturbations $\delta\chi_k$, $\delta\phi_k$ and Φ_k have still gone non-linear. This demonstrates that the super-Hubble resonances are robust predictions and not sensitive to the definition of the homogeneous fields, though the precise times to non-linearity are.

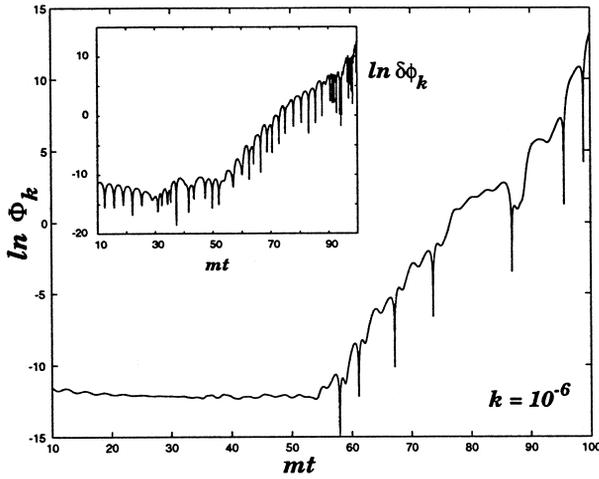


Fig. 14. Φ_k vs mt for $q = 8000$ and $k = 10^{-6}$ and $mt_0 = 10$. **Inset:** The evolution of $\delta\phi_k$ vs mt for the same parameter values.

relatively slowly, the resonances in the $k = 10^{-6}$ modes are still strong, with each of the perturbed quantities going non-linear after $m\Delta t \sim 65$.

It should be clear that none of the essential predictions have changed, as we anticipated earlier in our discussion of the constraint equation. The $g\phi^2\chi$ term in Eq. (56) at large q is sufficient to ensure resonant growth of χ , which leads to resonant growth of Φ_k .

In Fig. 15 we show t_{nl} vs q for $k = 0$ for $mt_0 = 1$, where χ satisfies Eq. (56). Note the fine-structure and the sudden transition to non-linearity, characteristic of the earlier t_{nl} vs q figures. In the usual chaotic inflation scenario, with $mt_0 = 1$, modes Φ_k go non-linear after $m\Delta t \leq 50$ even for low- q values of $q \geq 200$, that is, even for very weak couplings $g \geq 10^{-8}$.

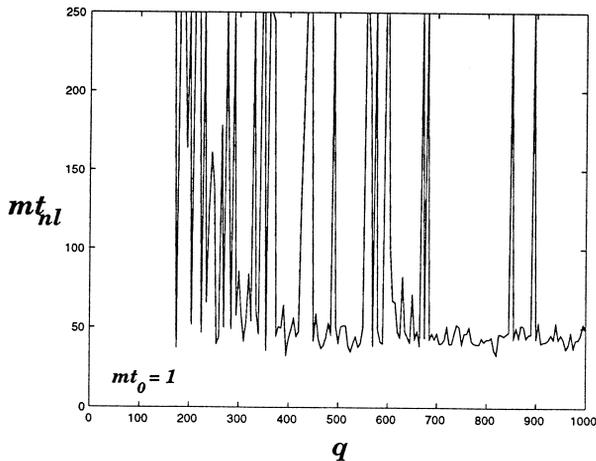


Fig. 15. t_{nl} vs q for $mt_0 = 1$ and $k = 0$ using the conservative definition of χ which satisfies Eq. (56). The complex band structure is still visible.

In conclusion, the existence of resonances for Φ_k is not sensitive to the choice of homogeneous mode for χ . Saturation of linear theory and the need for non-linear study appears to be a robust prediction of strong preheating, independent of the definition of the background fields, as long as they are sensitive to the non-gravitational forces implicit in preheating, as indeed they must be.

7. Backreaction, non-linear aspects and implications

7.1. Non-linearity vs fine-tuning

In the standard theory based on conserved quantities such as ζ , the values of Φ during and after inflation are constrained by a simple relation such as [96]

$$\frac{\Phi(t_i)}{1 + w_i} = \frac{\Phi(t_f)}{1 + w_f}, \quad (57)$$

where $w \equiv p/\rho$. Clearly, $\Phi(t_f) \gg \Phi(t_i)$ if $w_i \sim -1$, as occurs in slow-roll inflation. By fine-tuning $\dot{\phi}$ towards zero during inflation, the amplification of Φ can be made arbitrarily large. In the standard theory, this does not imply that perturbations have gone non-linear. Rather, the problem is run backwards: the $\Phi \sim 10^{-5}$ required by the CMB is used to determine the value of $\Phi(t_i)$ during inflation, which therefore puts constraints on the value of ϕ during inflation. No matter what the amount of amplification, the perturbations can be kept linear by sufficient fine-tuning of the initial value, $\Phi(t_i)$.

Have the previous sections simply been a discovery of a more elegant, realistic, and refined version of this amplification? And why are we claiming that the same fine-tuning cannot be applied in the realistic case of preheating? That is, despite the extra amplification due to the entropy perturbations (which invalidates the use of conserved quantities such as ζ), why can't all the resonant amplification be swept under the carpet, and Φ kept linear, by extra fine-tuning?

Clearly this is a crucial point. The extra amplification due to the quasi-exponential growth of entropy perturbations means that additional fine-tuning would be needed. This would exacerbate the already unpleasant fine-tuning of the couplings in many inflationary models, particularly simple chaotic inflation models.

Yet fine-tuning, while not desirable, is however an issue of taste. Much more serious in any attempt to stop the perturbations from going non-linear are the following points.

(i) First, fine-tuning the value of $\Phi(t_i)$ downwards during inflation will typically require making the couplings (m^2, λ) of the inflaton field smaller. The dependence of $\Phi(t_i)$ on these parameters is typically power-law, e.g. $\Delta T/T \propto m/M_{\text{pl}}$ in the quadratic potential model, and $\Delta T/T \propto \lambda^{1/2}$ in the quartic potential model. However, *assuming* (as is true in simple chaotic inflationary models) that these parameters also control the oscillation frequency of the inflaton during reheating, we see that the resonance parameter $q \propto m_{\text{eff}}^{-2}$ is significantly *increased* when the effective mass is decreased. This makes the resonance stronger and *increases* the effective Floquet index $\mu_{k,\text{eff}}$. Since Φ_k depends exponentially on $\mu_{k,\text{eff}}$, it is unlikely that any fine-tuning gains made

by reducing the coupling constants of ϕ will survive through the end of the resonance in reheating.

(ii) Second, and perhaps of more fundamental importance, concerns the timing of the end (t_{end}) of the initial resonance phase. Floquet theory guarantees that the fluctuations will grow resonantly until the end of this linear phase. Typically in preheating studies [8–15,21–25], t_{end} has been taken to be the time when backreaction became ‘important,’ which was usually taken on dimensional grounds to be when $\phi^2 \sim \langle \chi^2 \rangle, \langle \delta\phi^2 \rangle$, whichever came first. In Paper I we showed that when the metric perturbations are included in the analysis, one expects backreaction to become important in the field fluctuations and metric perturbations around the same time. Our simulations here suggest that at large q (in the multi-field case), it is often the metric fluctuations Φ which go non-linear first.

In the absence of special circumstances (such as a strong χ self-interaction, λ_χ), the end of preheating will occur when second- and higher-order non-linear processes dominate the linear resonance, *no matter what degree of initial fine-tuning is employed*. Thus, regardless of special fine-tuning during inflation, preheating in multi-field models in general will proceed at least until Φ enters the non-linear regime. For this reason, we consider some implications of this non-linearity in the remainder of this section.

7.2. Mode-mode coupling and turbulence

Several backreaction effects have been highlighted in the preheating literature. Two of them may be approximated with Hartree–Fock or large- N schemes: (i) the change to the inflaton amplitude of oscillation, φ_0 , and (ii) the change to the inflaton effective mass, $m_{\text{eff}} = m^2 + g^2 \langle \chi^2 \rangle$, and hence to the frequency of inflaton oscillations [8–15,21,22,66,67]. The first is a strongly non-equilibrium effect, being governed by non-Markovian dynamics rather than by the simple Markovian approximation of adding a $\Gamma\dot{\phi}$ term to the inflaton equation of motion [8–15]. In both of these cases, the effects derive from the growth of the backreaction term Σ introduced in Section 2, $\Sigma_\chi \sim \int d^3k |\chi_k|^2$ and $\Sigma_\phi \sim \int d^3k |\delta\phi_k|^2$. The implications of both (i) and (ii) for the resonance parameter q appear straightforward: given $q \propto \varphi_0^2/m^2$, the resonance parameter will fall as the inflaton amplitude falls; it will likewise fall as the inflaton’s effective mass grows. If the effective mass of the inflaton should decrease, then these two effects would compete.

As treated within the Hartree–Fock or large- N approximations, these backreaction effects are essentially independent of k ; given the form of Σ_l , they simply sum up the growth of modes within certain resonance bands, as a quick way to track the transfer of energy from the coherent inflaton into various k modes. As discovered in lattice simulations [16–20], and explained analytically [21,22], these approximation schemes thereby miss crucial elements of the non-linear evolution. In particular, k -dependent mode-mode coupling can play a large role before the end of preheating. It is clear from experience that any non-linear field theory may induce mode-mode coupling, and this is the origin of rescattering in ordinary preheating.

In a Hilbert space, the Fourier transform of a product is simply the convolution of the individual Fourier transforms, i.e. $\mathcal{F}(f \cdot g) = \mathcal{F}(f) * \mathcal{F}(g)$. The conversion of a non-linear term in the field equations into Fourier space therefore yields one or more convolutions of the spectra of the fields involved. The non-linear term $g\phi^2\chi$ in the

equation of motion for χ , for example, becomes the double convolution in Fourier space (which is symmetric in the arguments of ϕ and χ):

$$g \int \int d^3 \mathbf{k}' d^3 \mathbf{k}'' \phi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''} \phi_{\mathbf{k}-\mathbf{k}'} \chi_{\mathbf{k}}.$$

Assuming statistical isotropy and homogeneity, this reduces to a double scalar integral with measure $k'^2 dk' \cdot k''^2 dk''$. It is precisely this coupling of modes of different momenta which is missed by the Hartree–Fock and large- N approximation schemes.

In the case of perturbations around FRW, we must make a splitting of the scalar matter fields into homogeneous background fields and fluctuations. When this is done, two kinds of terms arise in performing these convolutions: terms for which both $k - k' - k''$ and $k - k'$ are non-zero (i.e. scatterings involving two non-condensate particles), and terms in which one of the ϕ_k particles belongs to the condensate, with $k = 0$. The latter type of term is dominant at the early stages of mode-mode coupling, because the inflaton condensate occupation number is still high [21,22,66,67].

When metric perturbations are included, similar convolution terms (and hence rescatterings) appear beyond linear order, now involving the modes of Φ_k in addition to $\delta\varphi_{1k}$. At first glance, it might appear simply too difficult to study these terms analytically, since at second-order, the equations describing the coupled system become tremendously complicated. The first complication arises because the variables we use, such as Φ and $\delta\varphi$ (and indeed any other known variables), are no longer gauge invariant. Thus a gauge-dependent formalism must be used in practice, although in principle some gauge-invariant generalizations of these variables might be found [97,98]. Further complications arise from the non-zero pressure and the increased variety of cross-terms among the many scalar matter fields. These issues in gravitational perturbation theory have been explored in a cosmological setting in studies of dust Einstein-de Sitter space-time [99].

These complications, of course, are bound to stymie any analytic progress on the fully non-linear evolution of multi-field systems at preheating. Yet despite these complications, we may make a survey of the *kinds* of terms which will be relevant to the specific question of mode-mode coupling and rescattering at second-order. At second-order, the system will contain convolutions of two first-order terms, in addition to new terms formed from background quantities multiplied by explicitly second-order objects. Because the background quantities yield delta-functions in Fourier space, however, these latter terms make no contributions to mode-mode coupling. Thus, modulo subtleties regarding gauge fixing, the non-linear effects from mode-mode coupling arise from combinations of terms which are now familiar from the linearized equations studied above.

Consider, as an example, possible effects of mode-mode coupling for a single-field model with small-amplitude oscillations ($\varphi_0 \ll M_{\text{Pl}}$). In this case, the linear theory predicts [54] no amplification of $k \rightarrow 0$ modes of Φ at preheating. The constraint equation at second-order for this simple case would take the form:

$$\dot{\Phi}_k^{(2)} + H^{(1)} * \Phi^{(1)} + H \Phi_k^{(2)} = \frac{1}{2} \kappa^2 \int \frac{dk' k'^2}{(2\pi)^3} \delta\dot{\varphi}_k^{(1)} \delta\varphi_{k-k'}^{(1)} + \frac{1}{2} \kappa^2 \dot{\varphi} \delta\varphi_k^{(2)}. \quad (58)$$

The integral on the right-hand side can transfer power from high- k to small- k modes (an *inverse cascade*), as well as from small- k to larger k (a forward cascade). Such inverse

cascades could in principle amplify super-Hubble modes of Φ even in such a small-amplitude, single-field model, based on these mode-mode couplings. Inverse cascades of precisely this type are common in non-linear systems [100–102], and indeed evidence for them was found recently in a numerical study of preheating [62].

While still a topic of significant debate in turbulence circles, the classical results of Kolmogorov on scaling in turbulence [103] suggest a $k^{-5/3}$ scaling law for the energy cascade in 3D turbulence. Similar direct cascade behavior has been found in lattice preheating simulations [16–20]. An even more radical possibility is that of inverse cascades, in which power is transferred from small to large scales. Such behavior is believed to occur in energy transport in 2D hydrodynamics, as well as in particle number in the non-linear Schrödinger equation, which signals Bose condensation. In addition, there is the claim [100–102] that energy cascades are inverse in nature in turbulence and magneto-hydrodynamics with power-law spectra $\propto k^\alpha$, if $\alpha > -3$, as is the case in inflation, although see also Ref. [104]. Of course, in the cosmological context causality must play a crucial rôle. Here, the coherence of the inflaton condensate is irrelevant and cascades must obey causality, yielding the usual result that the part of the spectrum added by mode-mode coupling can go on large scales at best like k^4 [90–92,105].

Similar terms involving convolutions over $\Phi_k^{(1)}$ and $\delta\varphi_{lk}^{(1)}$ would appear at second order in the equations of motion for both the gravitational and matter-field fluctuations in multi-field models. For the matter-field fluctuations, these non-linear $\Phi * \delta\varphi_l$ convolutions would therefore add a new source for rescattering to those considered in earlier studies: scattering of ϕ and χ particles directly off the gravitational potential. This non-linear mode-mode gravitational rescattering would complement the gravitational rescattering we found already at linear order, in the form of the $V_{IJ} \delta\varphi_{jk}^{(1)}$ cross-terms examined in Section 4, ultimately leading to larger variances in the matter-field fluctuations than the linear theory alone will produce.

7.3. The post-preheating power spectrum and the CMB

Perhaps more important, we may expect the results found in earlier numerical studies of preheating, regarding turbulence and the final shape of the power-spectrum for fluctuations [16–20], to carry over to the metric perturbations Φ as well. As found there for the case of coupled matter-field fluctuations, turbulence and cascades will lead, in general, to a re-establishment of scale invariance for the power spectra ($P_l(k)$) for each $\delta\varphi_l$. In the case of Φ , then, these mode-mode coupling effects ultimately would lessen the sharply scale-dependent spectrum which results at the end of the linear era of preheating, though as discussed in the previous subsection, causality will limit its effects for very small k . This non-linear effect on $P_\Phi(k)$ could thus produce a nearly scale-invariant spectrum by the end of preheating above a threshold value of k . It is crucial to recognize, however, that this ‘‘late-time’’ (end of preheating) power spectrum $P_\Phi(k)$ would be virtually independent of the scale-invariant spectrum produced *during* inflation [106]. The nearly flat spectrum measured by COBE, which probes $P_\Phi(k)$ over very small k , would therefore be a complicated combination of these two spectra.

However, based on earlier work on second-order perturbations and the CMB [107], it is clear that amplification of super-Hubble modes to the stage where second-order perturbations cannot be neglected is not compatible in naive models with the COBE DMR results that $\Delta T/T \sim 10^{-5}$. Hence some mechanism is required to damp the

spectra or limit the resonances before they can be observationally compatible. Some implementations of this requirement are discussed in Section 8.

An interesting possibility also remains for observational consequences of $P_\phi(k)$ as it evolves through its highly non-linear mode-mode coupling regime in the later stages of preheating. This second effect concerns larger- k (shorter-scale) perturbations. The turbulent, non-linear evolution of Φ_k modes at preheating could effectively mimic generic features of the power spectra predicted by topological defect models. In particular, the broad-resonance, stochastic amplification to non-linearity of certain modes Φ_k could potentially smear out the secondary acoustic Doppler peaks in the $\ell > 100$ portion of the CMB spectrum, if the modes remained non-linear all the way down to decoupling. These sharp secondary Doppler peaks are expected in nearly all inflationary models (when one ignores completely the behavior of Φ during preheating), yet are usually absent from defect models (which feature “active,” incoherent spectra, especially in these higher multipoles) [108–110].

Combined with the spectrum at small k , this could produce a hybrid spectrum for $P_\phi(k)$: (nearly) scale-invariant at scales probed by COBE (and hence naively similar to standard inflationary predictions, though physically of radically different origin), yet lacking any clear secondary Doppler peaks at shorter scales (and hence naively similar to standard predictions from defects models, though again for very different underlying physical reasons). The specific effects of the resonant amplification of Φ_k modes at preheating on the CMB requires careful study, and is currently under investigation by the authors.

7.4. Coherence of the inflaton zero-mode

As we discussed in Section 5, the super-Hubble resonances which form the basic element of this paper owe their existence to the coherent oscillations of the inflaton zero-mode. These oscillations lead to stimulated effects that are missing if one treats the inflaton as an incoherent fluid [21,22]. Hence, in addition to the backreaction mechanisms outlined above, we expect that another important backreaction event will be the reduction in the correlation length of the inflaton condensate. This is expected on very general grounds based on the fluctuation–dissipation theorem [111]: backreaction will cause the long-range coherence of the inflaton condensate to be broken. Eventually the phase of the inflaton’s oscillations will become space-time dependent, with its own correlation length that is expected to decrease rapidly [112].

As a result we expect the treatment of the inflaton as an effectively infinite, coherent condensate to break down on a timescale t_{coh} . However, the reader should note that at first-order in perturbation theory this effect does not appear, and only at second and higher order can the loss of coherence appear consistently. Hence the super-Hubble resonances cannot be eliminated. These higher-order effects, rather, would decrease the time during which the very-long wavelength metric perturbations would be amplified, and hence could limit the final amplitude of these modes.

For times $t > t_{\text{coh}}$, as the inflaton correlation length decreases rapidly and the inflaton begins to lose coherence as higher-momentum modes get amplified, we expect the resonance for the $k/a \ll H$ modes to weaken significantly. This would likely end further resonant growth of the super-Hubble modes. From that time on, the only mechanism which could amplify the $k = 0$ mode would be inverse cascades.

The coherence issue has been discussed briefly previously. In [113] it is argued that the inflaton’s coherence is only lost if the decay rate Γ_ϕ of the condensate, considered as a collection of zero-momentum bosons, is greater than H . In this sense, the expansion of the universe has a ‘healing’ effect on the coherence. For a coupling of the form in Eq. (8), they argue (using only the high-frequency limit to calculate the cross section for Γ_ϕ) that this ratio is

$$\frac{\Gamma_\phi}{H} \approx 1.2 \times 10^5 g^2, \quad (59)$$

so that only for $g > 3 \times 10^{-3}$ is coherence lost. Note that this does not depend immediately on φ_0 and m (m was cancelled in the cross section and the average H was used, so φ_0 did not appear). If this is correct, then in the simplest models of chaotic inflation, Eq. (9) predicts that all values of $q \leq O(10^7)$ lead to coherent inflaton oscillations, regardless of backreaction. In this case, the super-Hubble resonances of the $k \approx 0$ modes would continue until q was significantly reduced via the other q -reducing backreaction mechanisms. This point was considered separately in [114], in which it was argued based on semi-classical quantum gravity that coherence would be lost rapidly after only a few oscillations, though no detailed arguments were given.

Note that while the break-up of the coherence of the inflaton condensate may stunt the super-Hubble resonances, it will not stop resonances on smaller scales. Further, recent work (see Ref. [74]) has shown that parametric resonances persist even in the case where the effective mass is modulated by spatially inhomogeneous white noise. They further conjecture that the noise increases the Floquet exponent, μ_k , for all modes. If this is correct, the break-up of the condensate may not aid in ending the super-Hubble resonances at all.

Quite apart from these gravitational considerations, yet another fundamental difference appears relevant between single-field and multi-field models: the break-up of the super-Hubble coherence of the inflaton condensate may proceed more quickly in multi-field models than in the single-field case. This stems not only from the larger couplings between fluctuations in the multi-field case, but also from the greater number of directions in field space along which the coupled system could exit inflation. In other words, the inflaton condensate coherence length even at the start of preheating is likely to be shorter in multi-field models than in the single-field case.¹¹ Similar effects have been highlighted in recent studies of tunneling in multi-field models of open inflation, in which the multidimensional field-space yields large ‘‘quasi-open’’ domains inside the nucleated bubble, rather than a perfectly homogeneous background [115]. The coherence of the inflaton condensate thus remains a crucial quantity in need of further study; the amplification of super-Hubble modes depends directly upon it [112].

8. Possible escape routes from non-linearity

Given the severity of the consequences of the super-Hubble resonances with regard to observational compatibility, primarily with the CMB, a natural question, and one to

¹¹ DK would like to thank Robert Brandenberger and Richard Easther for helpful discussions on this point.

which we have already alluded previously in the paper, is “can one have large q but still escape the super-Hubble resonances?”

8.1. Escape route 1: secondary phases of inflation

The no-hair theorem [116] assures us that should a second phase of inflation follow preheating, the large super-Hubble resonances will be smoothed with time, the universe re-approaching the de Sitter geometry.

Such secondary phases of inflation are desirable and even expected in certain models. Consider thermal inflation for example [117], proposed as a solution to the moduli problem with a relatively small number (~ 10) of e -folds. Such a phase of inflation following preheating would stretch and smooth, but not eliminate, the signature of the non-linearity achieved during preheating. The challenge is to identify the fingerprint that one would expect on the large-angle CMB in this case.

Secondary phases of inflation are expected in multiple inflationary models based on supersymmetry and supergravity [118]. Studies have already begun examining the effect of the sudden change in the effective mass of the inflaton in the transition from one phase to another on the CMB, though until now the change has been treated neglecting any resonance effects [118,119].

Finally, it appears that the COBE results prefer a relatively low energy scale for the production of the anisotropies, $V^{1/4}/\epsilon^{1/4} \sim 6.7 \times 10^{16}$ GeV, where $\epsilon \equiv \frac{1}{2}M_{\text{Pl}}^2(V'/V)^2 \ll 1$ is the slow-roll parameter (see Ref. [3]). Such a low energy scale is a problem if one also wishes inflation to truly solve the horizon/flatness problem of the standard cosmology, since this low-energy scale corresponds to a lifetime several orders of magnitude larger than the Planck time, the “natural” time-scale for recollapse [1–3]. This is a generic problem of ‘low-energy’ inflationary models. A first phase of chaotic inflation which sets up appropriate initial conditions for a second phase of inflation, which in turn produces the required density fluctuations, is an alternative [120,121].

Of course this is not the only solution. The above objection cannot be made of compactified multi-dimensional models in which four-dimensional unification occurs around the GUT scale, since then GUT scale inflation occurs immediately upon exit from the Planck epoch. In the most extreme version of these models, in which unification occurs around the TeV scale [122,123], it is, however, not clear at present whether inflation can be implemented naturally or not [124–126].

8.2. Escape route 2: fermionic and instant preheating

Until now we have considered a model of the post-inflationary universe based solely on scalar fields. The case of preheating to gauge bosons is known to be qualitatively similar to the scalar case [127], though more complex. Fermionic preheating, on the other hand, is effectively crippled by the Pauli exclusion principle, which places the limit $n_k \leq 1/2$ on the occupation number, as opposed to the $n_k \gg 1$ possible in the scalar case [128–130].

A natural question then is how the metric resonances are affected when the inflaton can decay *only* into fermions at the end of inflation. Our aim here is to present evidence that the exclusion principle effectively stifles the resonant amplification of metric

perturbations at preheating, and thereby possibly offers a way out from the super-Hubble resonances that occur in the scalar case.

Consider a massless spin $\frac{1}{2}$ fermion, ψ , coupled to the inflaton via a $h\bar{\psi}\phi\psi$ interaction term. ψ satisfies the Dirac equation

$$[i\gamma^\mu \nabla_\mu - h\phi(t)]\psi = 0, \tag{60}$$

where γ^μ are the Dirac matrices satisfying the Clifford algebra relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}$. Making the ansatz $\psi = [i\gamma^\mu \nabla_\mu + h\phi(t)]X$ (see [129,130]), X satisfies the Klein–Gordon equation with a complex effective mass [cf. Eqs. (1) and (2)],

$$\Omega_k^2 = \omega_k^2 + qf - i\sqrt{qf}, \tag{61}$$

where $f = \phi^2/\varphi_0^2$ as in Section 2, and $q \propto h^2/m^2$ for a massive but not self-interacting inflaton, while $q \propto h^2/\lambda$ in the case of a massless, quartically self-coupled inflaton. The comoving number of created fermions is [130]

$$n_k = \frac{1}{2} - \frac{[2\omega_k^2 \text{Im}(X_k \dot{X}_k^*) + \sqrt{qf}]}{2\Omega_k} \tag{62}$$

with energy density

$$\rho_\psi = \frac{1}{2\pi^3} \int dk k^2 \Omega_k n_k. \tag{63}$$

Note that most of this energy density is in the form of inhomogeneity since $n_{k=0} \leq 1/2$ like all other modes, *unless* a (chiral-symmetry breaking) condensate forms with $\langle \bar{\psi}\psi \rangle \neq 0$. Assuming that such a condensate has not formed, one expects from Eq. (63) that the total energy radiated into the fermion field will be small, essentially due to the low occupation numbers forced by the exclusion principle.

This is borne out by a full closed-time-path analysis of the problem [128], which shows that the ensemble-averaged energy dissipated to ψ from a massive inflaton is

$$\langle\langle \rho_\psi \rangle\rangle = \frac{m^2 h^2 \varphi_0^2}{64\pi^2}, \tag{64}$$

which is rather small. This should be compared with the energy density in scalar χ particles at t_{end} , the end of the linear resonance, which one can estimate in the Hartree approximation [21,22] as $\rho_\chi(t_{\text{end}}) \sim g|\phi|n_\chi \sim m^2\varphi_0^2$. This second energy density is much larger than its fermionic counterpart unless $h^2 \geq 1$, and this is not allowed by the CMB, unless the theory is supersymmetric. Further, the energy pumped into the χ field predominantly goes into super-Hubble modes, whereas in the fermionic case, only a small fraction of the total fermionic energy can go into amplifying super-Hubble modes due to the exclusion principle. This is exacerbated by the small phase-space volume, $\propto k^3$, at small k .

The equation of state for ψ must satisfy $p_\psi \leq \rho_\psi$, and hence we are assured that $\delta T^{\mu\nu}$ is small relative to the energy density in the background. The perturbed Einstein field equations [4]

$$\nabla_i(\dot{\Phi} + H\Phi) \propto \delta T_i^0 \tag{65}$$

suggest that it is unlikely that any significant amplification of super-Hubble metric perturbations exists in the case of preheating only into fermions.

More interesting is the realistic case where scalar and fermionic fields coexist, coupled to the inflaton. Examples of fermion-stimulated decay of the inflaton into bosons exist [129], and it is not clear how metric perturbations evolve in this case.

A particular example is the recent model known as instant preheating [131], in which the sequence $\phi \rightarrow \chi \rightarrow \psi$ occurs, with the end point being small numbers of very massive fermions, with masses $m_\chi > 10^{16}$ GeV possible. While it is not our aim to study the evolution of Φ in this model, we expect that the resonances discussed here will not exist in their current form in instant preheating, since the χ bosons decay almost immediately (within one oscillation) in the simplest scenarios, while the resulting fermions are few in number, and the energy transferred to them is typically rather small. Still, this energy may grow to dominate the total energy density of the universe with time if the inflaton energy redshifts relativistically, $\propto a^{-4}$. It is difficult to say anything more quantitative, and these interesting issues are left for future work.

8.3. Escape route 3: χ mass and self-interaction

In our numerical analysis, we limited ourselves to a model of preheating with a massless scalar field χ . Two characteristics one might expect of χ are a non-zero mass ($m_\chi \neq 0$), and self-interaction ($\lambda_\chi \neq 0$). For the potential

$$V(\phi, \chi) = \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{2} g \phi^2 \chi^2 + \frac{1}{2} m_\chi^2 \chi^2 + \frac{1}{4} \lambda_\chi \chi^4, \quad (66)$$

the perturbed Klein–Gordon equation, Eq. (40), for $\delta\chi_k$ then becomes

$$\begin{aligned} (\delta\chi_k)'' + 3H(\delta\chi_k)' + \left[\frac{k^2}{a^2} + m_\chi^2 + g\varphi^2 + 3\lambda_\chi \chi^2 \right] \delta\chi_k \\ = 4\dot{\chi}\dot{\Phi}_k - 2 \left[g\varphi^2 \chi + m_\chi^2 \chi + \lambda_\chi \chi^3 \right] \Phi_k - 2g\varphi\chi\delta\varphi_k. \end{aligned} \quad (67)$$

From this it is clear that the effective mass of the χ field depends on m_χ , χ^2 and $g\varphi^2$ (whether one uses $[(\delta\chi)_{k=0}]^2$ or $\langle \delta\chi^2 \rangle$ for the background field χ is not very important at this qualitative level). In the simple preheating case in which metric perturbations are ignored, increasing m_χ leads to a sharp decrease in the strength of the resonance. (Effectively it increases the A parameter of the Mathieu equation ($A \propto k^2/a^2 + m_\chi^2/m_\phi^2 + 2q$), leading to a rapid decrease in μ_k .) Similarly, a positive λ_χ term tends to shut off the χ resonance because once the variance of the χ field becomes large, the $3\lambda_\chi \chi^2$ term acts like an effective mass term, hence damping the resonance [19,132]. However, in our case we have the extra driving term proportional to Φ_k , which also depends on m_χ^2 and λ_χ . Hence it is not obvious that adding these two parameters will definitely damp the resonance. This remains an interesting topic for further research.

8.4. Escape route 4: Suppression of super-Hubble initial conditions

The largest resonance effects occur when $q \gg 1$, that is, when the coupling g is much larger than $(m/M_{\text{Pl}})^2$. During inflation, when φ is slowly rolling, such a large

coupling g would lead to a large effective mass for the χ field, $m_\chi^2 \sim g\varphi^2 \gg m^2$. It has been drawn to our attention that this large effective m_χ^2 during inflation might suppress the value of the χ field at the beginning of preheating on $k/a \ll H$ scales, relative to modes in the $k \rightarrow \infty$ limit [133]. Such a suppression of the amplitude of the χ_k modes in the $k \rightarrow 0$ limit might suggest that the large- k modes would have a “head start” at preheating, and would therefore go non-linear before the super-Hubble modes would, effectively damping the resonance via backreaction *before* the super-Hubble modes experienced significant resonant growth.

This situation, however, is complicated by several factors, each of which deserves further attention:

1. The Floquet indices $\mu_k \rightarrow 0$ rapidly with growing k . During the first few inflaton oscillations, for multi-field models with large q , modes grow due to stochastic resonance. In this case, the amplification factor goes as [21,22] $e^{\mu_k} - 1 \simeq 2\exp[-\pi k^2/(2a^2 m^2 \sqrt{q_{\text{eff}}})]$, which obviously decays rapidly with k . This shows that large- k modes will not grow at a higher exponential rate during preheating, even if they had a large “head start.”

At later times, when the evolution is closer to ordinary Floquet theory, the above conclusion remains: μ_k will be larger for the lowest resonance band, at small- k , and drops off rapidly with increasing k .

2. Assuming that long-wavelength modes of interest first cross the Hubble scale 50 e -folds before the end of inflation, these modes would naively be suppressed by a factor of $a^{-3/2} \sim e^{-75} \sim 10^{-30}$. Even a suppression of $\delta\chi_k(t_0)$ initial conditions for the small- k modes of order 10^{-30} compared with large- k modes is not necessarily significant when one considers exponential instabilities at preheating: $\mu_k/m \sim O(1)$ for $k \rightarrow 0$ modes when $q \gg 1$, so that small- k modes could *still* go non-linear first if $\mu_{k \rightarrow 0} \geq 75\mu_{k \rightarrow \infty}$. Considering that μ_k falls off exponentially with k , this constraint can easily be satisfied over large regions of parameter space.
3. Not only are the μ_k *much smaller* for large k than for small k , but the *widths* of the resonance bands, Δk , are likewise much narrower for large k . This is crucial since backreaction is controlled by k -space integrals, such as $\Sigma_I \sim \int dk k^2 |\delta\varphi_{Ik}|^2$, for the field fluctuations $\delta\varphi_I$. The decrease in the resonance-band window at large k is partially compensated for by the larger phase-space volume, $k^2 dk$ factor, and therefore requires careful quantitative study.
4. This scenario for suppressing the amplitude of super-Hubble modes neglects the quantum-to-classical transition, which remains very subtle. The standard procedure in non-equilibrium quantum field theory is to *define* the initial conditions as $\delta\varphi_{Ik}(t_0) \equiv 1/\sqrt{2\Omega_k(t_0)}$ and $[d\delta\varphi_{Ik}/dt]_{t_0} \equiv -i\sqrt{\Omega_k(t_0)}/2$ for *all* k . [23–25,66,67,134]. This prescription has been used in every numerical and nearly every analytical study of preheating [16–20]. Note that these initial conditions for the start of preheating suppress large- k modes and favor small- k modes. By adopting our scale-invariant initial conditions for $\delta\varphi_{Ik}$ in our simulations in Section 6.1, we have thereby neglected this additional *enhancement* of super-Hubble modes.

Because of the many subtleties involved in the question of initial conditions, this obviously warrants further study. It is crucial to note, however, that even if super-Hub-

ble modes were suppressed by a large factor during inflation, some modes Φ_k are still very likely to go non-linear during preheating, at some scale k : if preheating happens to the matter-field modes at any particular scale k , then the metric perturbations Φ_k *must* necessarily experience similar exponential growth; this is fixed by the constraint equation, Eq. (41). And, as emphasized above in Section 7, since preheating only ends via non-linear mechanisms, the exponential amplification of these Φ_k modes will continue into the non-linear regime. Whether or not the strongest amplification and earliest transition to non-linearity occurs on super-Hubble or sub-Hubble scales, it is bound to happen in *some* region of k -space during preheating.

9. Hierarchical classification of inflationary models

Here we examine a variety of inflationary scenarios for possible sensitivity to strong preheating (see Fig. 16).

9.1. Simple chaotic inflation

The start of reheating in chaotic inflation is characterized by very large values of $\varphi_0 \approx M_{\text{Pl}}$.¹² In simple polynomial models $V = \lambda\phi^{2n}$, λ is forced to be very small due to the COBE CMB results, with $\lambda \approx 10^{-12}$ for $n = 2$, or $m \approx 10^{-6} M_{\text{Pl}}$ for $n = 1$. Hence we find the result for scalar fields coupled to the inflaton, as in Eq. (9),

$$q \sim g \times 10^{10}, \quad (68)$$

so that for natural couplings $g \in [10^{-6}, 10^{-2}]$ we have $q \in [10^4, 10^8]$. In other words, large resonance factors are generic in the simplest models of chaotic inflation, when scalar fields in addition to the inflaton are included.

A very important point to notice is that the large q arise due to the confluence of the largeness of φ_0 and the smallness of the inflaton mass *during reheating*. If either of these is given up, the resonances can be expected to be naturally much weaker.

Hence, in a chaotic inflation model based on a potential which exhibits a large change in the inflaton effective mass between the inflationary phase and the reheating phase, the q parameter might easily be several orders of magnitude smaller due to the strong curvature of the potential at reheating.

Finally, in the case where the potential exhibits self-interaction for the inflaton, e.g. $V = \lambda\phi^4/4$, it has been strongly argued that the value $\lambda \sim 10^{-12}$ is unnecessarily small, arising from a naive identification of classical perturbations (responsible for the CMB anisotropies) with quantum fluctuations, and that a value nearer $\lambda \sim 10^{-6}$ is more accurate [135]. If this is true, then for models with a *massless* quartically self-coupled inflaton coupled to *distinct* bosons, the resonance parameter q would be reduced by

¹² The exact moment when oscillations are said to start, and hence the value of the initial amplitude of oscillations, φ_0 , is rather subtle. Using the criterion that it is defined by the time when the first of the slow-roll parameters ϵ or η become unity, gives $\varphi_0 = \sqrt{2} M_{\text{Pl}}$ for the quadratic potential and $\varphi_0 = 2\sqrt{3} M_{\text{Pl}}$ for the quartic potential. Using such large values makes the resonance much stronger. To obtain a lower limit we use $\varphi_0 \approx 0.3 M_{\text{Pl}}$, as typical in the literature, see e.g. Refs. [16–22].

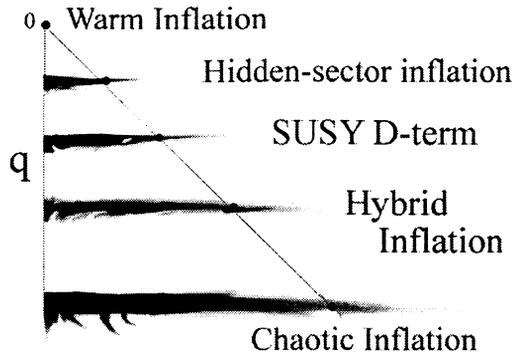


Fig. 16. A schematic representation of the hierarchy of inflationary models with regard to their strength of preheating. At the top, where $q = 0$, sits warm inflation. At the bottom, where typically $q \gg 1$, sit the simplest chaotic inflation models. Most other inflationary models lie in between and exhibit the effects discussed in this paper to varying degrees depending on precise details of the couplings between the inflaton and other fields.

around 10^6 , since the square of effective frequency of the inflaton’s oscillations, $\omega^2 \propto \lambda \phi_0^2$, appears in the denominator for q . Naturally, this would be a very large change, which, for moderate couplings $g < 10^{-3}$, would move the system into the mild or weak resonance regimes. This change in λ , however, would not necessarily weaken the resonances for chaotic inflation models with a massive inflaton coupled to distinct fields.

9.2. Hybrid and multiple inflation

General hybrid models can exhibit strong preheating with large q parameters [136], and to this extent they will suffer strong distortion of the metric perturbation spectrum during any oscillatory phase. What would be interesting would be to study models in which an oscillatory phase with large resonance parameters occurs between successive phases of inflation. Once the oscillations have finished, the amplified spectrum would be smoothed and stretched during the second inflationary phase leaving a highly non-trivial imprint on the CMB, assuming the second phase did not last too long.

The extra freedom in these models implies that it should be possible to construct models with very similar features to those of the broken spectral index genre, which are able to fit current CMB and large scale structure data better than standard inflationary models [118,119].

9.3. Supersymmetric (SUSY) inflation

Consider perhaps the simplest SUSY theory based on the superpotential [122,123]:

$$W = \alpha S \bar{\varphi} \varphi - \mu^2 S. \tag{69}$$

This is the most general form consistent with R -parity under which the inflaton singlet, S , transforms as $S \rightarrow e^{i\theta} S$, the superpotential transforms as $W \rightarrow e^{i\theta} W$, and the product

$\bar{\varphi}\varphi$ is invariant. The unbroken supersymmetric potential corresponding to this superpotential is

$$V = \alpha^2 |S|^2 (|\varphi|^2 + |\bar{\varphi}|^2) + |\alpha\varphi\bar{\varphi} - \mu^2|^2 + \text{D-terms}. \quad (70)$$

The D -terms vanish along the D -flat direction $|\varphi| = |\bar{\varphi}^*|$. Inflation occurs for $S \gg \mu/\sqrt{\alpha}$ at the minimum of the potential ($\langle \varphi \rangle = \langle \bar{\varphi} \rangle = 0$), which is not supersymmetric, with $V = \mu^4$. Quantum corrections break the flatness of the potential and cause slow-roll but as long as supersymmetry is broken softly the induced curvature will be small and the inflaton very light, as in chaotic inflation above. If this were to remain the situation, these theories would have the same problems as the simple chaotic inflation models. However, for $S \leq \mu/\sqrt{\alpha}$, V has a new minimum which is supersymmetric ($V = 0$) corresponding to $\langle S \rangle = 0, \langle \varphi \rangle = \mu/\sqrt{\alpha}$ [122,123]. Reheating is expected to complete during oscillations about this new minimum. The amplitude of both S and φ oscillations is of order $\mu/\sqrt{\alpha}$, while their masses are of order $m_S^2 \approx 4\mu^2$ and $m_\varphi^2 \approx 8\alpha\mu^2$. Hence, assuming no other fields, the resonance parameters at the final oscillation stage are

$$q_S \approx \frac{\alpha^2 \mu^2}{4\mu^2 \alpha} \sim \alpha \sim 10^{-2}, \quad (71)$$

$$q_\varphi \approx \frac{\alpha^2 \mu^2}{8\mu^2 \alpha^2} \sim 0.1, \quad (72)$$

since $\alpha \sim 10^{-2}$ to fit observations [122,123]. Hence, in this most minimal of supersymmetric models, there is little resonance in the primary oscillation phase and in the inflaton and Higgs superfields φ . In this sense it is closer to the single-field models since it contains only one energy scale (μ^2) and a single dimensionless coupling, α , which is forced to be small.

However, if we couple for example, another scalar field χ to S and φ , with coupling $g^2 \gg \alpha^2$, then the resonance parameters would be of the order $g^2/\alpha^2 \sim g^2 \times 10^4$, so that if g were near strong coupling, the strong resonances we have discussed here would occur. However, since these extra couplings would typically give mass to other fields, (e.g. the heavy right-handed neutrinos of the theory [140]) and would play a role in determining the baryon asymmetry, building a fully consistent model with $g^2 \sim 1$ appears technically very difficult.

9.4. Hidden sector inflation

In these string-inspired models [118], the inflaton in the hidden sector only has gravitational couplings to the fields of the visible sector and hence the q parameter is very small, being of order unity, as occurs for gravitational waves [55,56]. The amplification of metric perturbations is essentially negligible in these models, as it is in the quintessential inflation model of Vilenkin and Peebles [141].

9.5. Warm inflation

At the opposite end of the scale from simple chaotic inflation models is warm inflation and its variants [137–139], in which inflation occurs due to strong overdamping

of the inflaton due to couplings to other fields. Reheating does not occur due to coherent oscillations but rather occurs continuously during inflation with natural exit to a radiation-dominated FRW universe. Since there are no coherent oscillations to speak of, $q = 0$, and there is no resonant amplification of metric perturbations at all.

10. Conclusions and future issues

This paper continues the exploration of metric perturbations' evolution during the post-inflationary epoch known as preheating. In spirit it is a continuation and development of the study in Paper I [40], but with significant advances in conceptual and numerical analysis of the problem. We have identified a number of important issues that distinguish the evolution of metric perturbations during the violent preheating epoch ($q \gg 1$) from metric evolution in old theories of reheating. Primary among these is the resonant amplification of *both* super- and sub-Hubble modes of Φ , the gauge-invariant scalar metric perturbation.

We demonstrate explicitly that such amplification *does not* violate causality – it is merely a reflection of the perfection of the coherence of the inflaton condensate at the very end of inflation, which is correlated on vastly super-Hubble scales in standard models of inflation.

Such resonances depend heavily upon the presence of multiple matter-fields coupled via non-gravitational interactions. In the multi-field case, entropy perturbations are generic, and although they may be small during inflation, they are resonantly amplified during strong preheating. This leads to the destruction of the conservation of traditional quantities such as the Bardeen parameter ζ , traditionally used to transfer the metric perturbation Φ from inflation to photon decoupling. This implies that direct numerical integration of the governing equations must be used.

The wide resonances at $q \gg 1$ destroy the scale invariance of the inflationary power spectrum and could hugely over-produce large-angle anisotropies in the Cosmic Microwave Background if left undamped after preheating. The resonances can be so strong as to force the perturbations to go non-linear, requiring study of the full, nonlinearized Einstein field equations (see Ref. [62] for an exciting start in this direction in the single-field case). One issue that is likely to be very important in understanding the full extent of amplification of the super-Hubble modes is the *break-up* of the inflaton condensate. As second-order effects become important, we expect on general grounds that the perfect coherence of the oscillations of the inflaton condensate will be lost, and the coherence length of the homogeneous mode will be reduced. This will tend to shut off the super-Hubble resonances, following essentially from the induced super-Hubble inhomogeneity.

We wish to emphasize that this powerful growth in the metric perturbations is robust and cannot be swept away or removed by fine-tuning the initial value of Φ during inflation. The primary reason why fine-tuning fails is that the resonances of preheating end typically only through backreaction effects, which require at least study of second-order metric perturbation effects and a thorough understanding of mode-mode coupling. This is the first time inflationary models have been faced with this complication, bringing them nearer in spirit to the non-linear field theories used in topological defect

studies. This mode-mode coupling tends to re-establish the scale invariance of the power spectrum, but this time with a spectral index which is determined by the non-linear structure of the Einstein field equations, in analogy with the situation with turbulence and the Navier–Stokes equations. Memory of the initial spectral index is lost and the perturbations tend to an active, incoherent state, again mimicking topological defects. This may have profound implications for the small-scale CMB and in particular for the secondary Doppler peaks, as a means for distinguishing between inflation and defect theories, assuming the mode-mode coupling is preserved on relevant scales until decoupling.

As a means for quantifying the amplification of perturbations, we have introduced here the concept of the *time to non-linearity*, t_{nl} , which provides a robust (though non-unique) and useful generalization of the Floquet index μ_k for expanding universes. It quantifies the time at which the linearized equations break down.

The metric preheating we have developed here, despite being based on a simple model, yields very strong deviations from the old theory of perturbative evolution. These deviations depend crucially on a number of conceptual points, highlighted in Section 3, related to the existence of multiple, non-gravitationally interacting fields at preheating. While certain regions of preheating q -parameter space appear to be observationally ruled out by the CMB, there appear a number of interesting and even exciting mechanisms for making the strong resonances acceptable. Secondary phases of inflation are expected to damp the amplified modes and stretch them. This allows one to put more freedom into the original spectrum while bringing the overall amplitude down to an observationally acceptable level.

Alternatively, the resonance may be significantly weaker if the fields into which the inflaton is decaying have strong self-interaction, or are fermionic. The decay into fermions is particularly interesting since it is constrained by the Pauli exclusion principle, which limits the amplification of Φ . While we have discussed the basic mechanisms at work, both of these aspects deserve closer attention.

For the theory of preheating itself, one of the implications of studying the complete system of equations including metric perturbations is the appearance of rescattering – the production of quanta due to couplings between the fluctuations of different fields – even *at linear order*. Previously this had been thought [21,22] to be a purely non-linear process. On smaller scales, the amplification of scalar metric perturbations strongly enhances primordial black hole formation and gravitational wave emission via rescattering.

We note that all inflationary models are definitely not born equal with regard to these resonances. Simple chaotic inflation models, with their large amplitude oscillations and small effective mass, yield extremely strong resonances. Alternatively, models with big changes to the effective mass and/or small amplitude oscillations or weak couplings between the inflaton and other fields, exhibit much weaker resonances.

To conclude, we believe it is important to analyze the limitations of the work presented here. Since preheating is a problem involving multi-field, non-equilibrium, semi-classical quantum gravity on a dynamic background, it is clear that the essentially classical model we have used is not ideal. In addition, we have uncovered a serious ambiguity related to the definition of the homogeneous components of the non-inflaton fields. The standard definition in the metric perturbation literature ignores the contribution of $\Phi(k=0)$ to the evolution of homogeneous background quantities. Since the

energy in this mode grows very rapidly, we have argued that its influence on the background should be included as a first step to studying the full backreaction problem.

Other aspects of preheating that should be addressed in future work include:

(i) Can one formulate a consistent quantum and non-equilibrium framework which includes the metric perturbations? In particular, does the Schwinger–Keldysh closed-time-path (CTP) formalism [23–25,128,142,143] carry through to this more complex setting? Further, can one reconcile the standard relativistic definitions of background quantities with the order-parameter [135] and other field-theoretic definitions?

(ii) Backreaction issues: can one go beyond the simple linearized equations in a realistic, analytical manner? Is there a relatively simple and consistent Hartree or large- N approximation for Eqs. (39-41)?

(iii) The transfer function to decoupling and thermalization: how does thermalization proceed and how do metric perturbations evolve through this phase? How does the viscosity arising from finite temperature effects alter perturbation evolution [84,106,144–146]?

(iv) Quantum gravity corrections: stochastic backreaction effects are expected to be strong during preheating. This implies a stochastic evolution of background quantities such as the scale factor and zero-modes [142,143]. The evolution of the metric is determined by a Langevin-type equation which describes the non-equilibrium dynamics of the gravitational field. The effects of these stochastic fluctuations have already been analyzed phenomenologically in the context of preheating without metric perturbations [70,71,73,74], and found generically to enhance particle production. We have similar expectations on very general grounds for the case in which metric fluctuations are included. (See also [147].) Almost surely going to the (white-noise) stochastic limit will not help reduce the amplification of metric perturbations, though it will change the wavelength-dependence of the amplification.

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