

## Post-inflation reheating in an expanding universe

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An analytic means of studying the resonant decay of the inflaton field is developed for the case of background expansion  $H \neq 0$ . It is shown that the parametric resonance in the inflaton's decay need not disappear when the expansion of the universe is taken into account, although the total number of particles produced is fewer than that in the  $H \simeq 0$  case.

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### I. INTRODUCTION

Soon after the first models of inflation were published in 1981 and 1982 [1, 2], several papers were written to study how the universe would reheat following the supercooled inflationary state [3]. The process appeared to be straightforward: in models such as new inflation (and chaotic inflation [4]), which incorporate a second order phase transition to end inflation, the inflaton field would wind up oscillating around the minimum of its potential near the end of inflation. These oscillations would produce a sea of relativistic particles, if one added (by hand) interaction terms between the inflaton and these lighter species. The decay rates  $\Gamma_{\phi \rightarrow \chi, \psi}$  could then be calculated, where  $\phi$  is the (decaying) inflaton field,  $\chi$  is some light boson field, and  $\psi$  is some light fermion field. From these decay rates, the energy density ( $\rho_p$ ) of the produced particles could be calculated, and related to the final reheat temperature ( $T_{\text{rh}}$ ):  $\rho_p \simeq \pi^2 N_{\text{eff}} T_{\text{rh}}^4/30$ , where  $N_{\text{eff}}$  counts the effective number of massless spin degrees of freedom. This method was thought to estimate  $T_{\text{rh}}$  to within an order of magnitude.

Recently, much attention has been focused on an overlooked but dramatic feature of the production-via-oscillation model: the inflaton's oscillations should be *unstable*, and should exhibit *parametric resonance*. Certain modes  $\chi_k$  within small bands  $\epsilon \ll k$  should grow *exponentially*, swamping the production in other modes. This effect has been studied by Kofman, Linde, and Starobinsky, in [5] for both the broad and narrow resonance regimes, and, following different methods, by Shtanov, Traschen, and Brandenberger [6] for the narrow resonance regime. The importance of "induced amplification" in the inflaton's decay has been addressed recently with still different methods by Boyanovsky *et al.* in [7, 8]. (See also [9, 10]). Earlier work on some of these questions may be found in [11, 12]. The crucial feature of the new reheating scenario is that the equation of motion for the light boson field will not obey the simple harmonic oscillator equation. Instead, for an interaction of the form

$\mathcal{L}' = -\frac{1}{2}g^2\phi^2\chi^2$ , the mode functions of the boson field will obey

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + g^2\phi^2\right)\chi_k = 0, \quad (1)$$

where overdots denote derivatives with respect to cosmic time  $t$ ,  $a(t)$  is the scale factor for the (assumed) flat Friedmann-Robertson-Walker spacetime, and  $H \equiv \dot{a}/a$  is the Hubble parameter. Since the inflaton field  $[\phi(t)]$  is rapidly oscillating, for frequencies  $\omega_I \gg H$  the  $3H\dot{\chi}_k$  term may be neglected and  $a(t)$  treated as constant. Then this equation for the boson field mode functions will look like the well-known Mathieu equation (see, e.g., [13, 14])

$$\ddot{\chi}_k + \omega_k^2(1 + h \cos \omega_I t)\chi_k = 0. \quad (2)$$

As demonstrated in [5, 6, 11] and treated below, for  $\omega_I = 2\omega_k + \epsilon$ , with  $\epsilon \ll \omega_k$ , solutions grow like  $\chi_k(t) \propto \exp(\pm\mu_{\pm}t)$ , instead of like  $\chi_k(t) \propto \cos(\omega_k t)$ . This means that the number of  $\chi$  bosons produced goes like  $N_{\chi}^{\text{res}} \propto \exp(2\mu_{+}t)$ , so that the decay rate for  $\phi \rightarrow 2\chi$  is much larger than originally calculated:  $\Gamma_{\chi}^{\text{res}} \gg \Gamma_{\chi}$ . (Note that this parametric resonance is only effective for inflaton decay into bosons; the exponential increase in fermion modes is forbidden because of Fermi-Dirac statistics.) This large decay rate increases  $\rho_p$ , ultimately giving  $T_{\text{rh}}^{\text{res}} \gg T_{\text{rh}}$ .

In this paper, we develop an analytic means of studying the inflaton's decay into other particles for the case of  $H \neq 0$ . We find that the total number of particles produced from the parametric resonance effect when  $H \neq 0$  is fewer than that in the  $H \simeq 0$  case, but can still be exponentially greater than that when the resonance is neglected altogether. Rather than expansion of the universe, then, the true threat to resonant inflaton decays appears to be back reaction of the produced particles on the amplitude of the decaying inflaton field. While we do not study this back-reaction effect in detail here, some approximate schemes for including some of its effects are discussed in the Appendix.

We concentrate on the case of the inflaton field  $\phi$  decaying into further inflaton bosons, via the self-interaction potential  $V(\phi)$ . We restrict attention to the small-amplitude, narrow-resonance regime, as the inflaton field oscillates near the minimum of its potential,

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similar to that studied in [6]. In Sec. II, the number of particles produced per mode  $k$ ,  $N_k$ , is calculated for nonzero expansion, assuming that the decay is not resonant. Section III presents the calculation of  $N_k^{\text{res}}$  when the expansion of the universe is neglected. In Sec. IV we calculate  $N_k^{\text{res}}$ , taking into account the expansion of the universe, and demonstrate that the parametric resonance need not vanish. These values of  $N_k$  correspond to particle production during the narrow-resonance regime of the "preheating" epoch discussed in [5, 15]. Final reheating temperatures can then be calculated, as discussed in [5], using the methods developed in [3], where these methods should now be applied to the produced particles of the preheating epoch; this will not be included here. For more on the question of thermalization and the final reheat temperature, see [8]. Concluding remarks follow in Sec. V.

## II. NONRESONANT INFLATON DECAY IN AN EXPANDING BACKGROUND

We will study a very simple model of inflation: chaotic inflation with the potential  $V(\phi) = \frac{1}{4}\lambda\phi^4$ . We will assume that the metric may be written in the form of a flat Friedmann-Robertson-Walker line element,  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\vec{x}^2$ . Then, given the Lagrangian density

$$\mathcal{L} = \sqrt{-g} \left[ \frac{1}{16\pi G} R - \frac{1}{2} \phi_{;\mu} \phi^{;\mu} - V(\phi) \right], \quad (3)$$

the equations of motion take the familiar form

$$H^2 = \frac{8\pi G}{3} \left[ V(\phi) + \frac{1}{2} \dot{\phi}^2 \right],$$

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + \frac{dV}{d\phi} = 0. \quad (4)$$

Next we decompose the inflaton field into the sum of a classical background field and a quantum fluctuation:  $\phi(\vec{x}, t) = \varphi(t) + \delta\phi(\vec{x}, t)$ . (We will neglect the metric fluctuations which couple to these inflaton fluctuations for the present analysis; for more on these metric fluctuations, see, e.g., [16, 17].) In order to study the production of  $\phi$  particles when the inflaton field is oscillating near the minimum of its potential, we will need to calculate mode solutions for the quantum fluctuations  $\delta\phi$  before and after the phase transition, matching the solutions by means of a Bogolyubov transformation.

We will first concentrate on the modes for the fluctuations near the end of the inflationary period. In terms of conformal time,  $d\eta \equiv a^{-1}dt$ , the fluctuations are quantized as follows:

$$\delta\hat{\phi}(x) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \left[ \delta\phi_k(\eta) \hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \delta\phi_k^*(\eta) \hat{a}_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}} \right], \quad (5)$$

where carets denote quantum operators, and the creation and annihilation operators obey the canonical commutation relations

$$\begin{aligned} [\hat{a}_{\vec{k}}, \hat{a}_{\vec{\ell}}] &= [\hat{a}_{\vec{k}}^\dagger, \hat{a}_{\vec{\ell}}^\dagger] = 0, \\ [\hat{a}_{\vec{k}}, \hat{a}_{\vec{\ell}}^\dagger] &= \delta^3(\vec{k} - \vec{\ell}), \quad \text{with } \hat{a}_{\vec{k}}|0\rangle = 0. \end{aligned} \quad (6)$$

From Eq. (4), and denoting  $d/d\eta$  by a prime, the equation of motion for the mode functions  $\delta\phi_k(\eta)$  becomes

$$\delta\phi_k'' + 2\frac{a'}{a}\delta\phi_k' + (k^2 + 3\lambda\varphi^2 a^2)\delta\phi_k = 0. \quad (7)$$

Next we introduce a "conformal field,"  $\psi \equiv a\delta\phi$ , whose mode functions obey the equation of motion

$$\psi_k'' + \left[ k^2 - \frac{a''}{a} + 3\lambda\varphi^2 a^2 \right] \psi_k = 0. \quad (8)$$

Near the end of inflation, the scale factor  $a(t)$  will not in general retain its de Sitter (exponential) form. Instead, we may write  $a(t) \propto t^p$ , with  $1/2 \leq p \leq \infty$ , where  $p = 1/2$  corresponds to the radiation-dominated epoch, and  $p \rightarrow \infty$  recovers the de Sitter epoch. This scale factor corresponds to a conformal-time scale factor of  $a(\eta) \propto \eta^{p/(1-p)}$ . Because we are ignoring the parametric resonance from the classical  $\varphi$ -field's oscillations in this section, we may simply assume that near the minimum of its potential,  $\varphi^2 \ll k^2$ . If we lastly define a new field  $\chi \equiv \eta^{-1/2}\psi$ , and define a new variable  $z \equiv k\eta$ , then Eq. (8) becomes

$$\frac{d^2\chi_k}{dz^2} + \frac{1}{z} \frac{d\chi_k}{dz} + \left( 1 - \frac{1}{z^2} \left[ \frac{(3p-1)^2}{4(p-1)^2} \right] \right) \chi_k \simeq 0. \quad (9)$$

This is now in the form of Bessel's equation. Mode functions for the original field  $\delta\phi$  may then be written in terms of Hankel functions:

$$\begin{aligned} \delta\phi_k(\eta) &= \frac{\eta^{1/2}}{a(\eta)} \left[ A_k H_\nu^{(1)}(k\eta) + B_k H_\nu^{(2)}(k\eta) \right], \\ \nu &= \frac{3p-1}{2(p-1)}. \end{aligned} \quad (10)$$

We assume that during inflation, the inflaton field is in its vacuum state (see Sec. 5.2 of [19] for further discussion of this point). The coefficients  $A_k$  and  $B_k$  thus may be fixed by choosing an appropriate quantum vacuum state. The Bunch-Davies, or "adiabatic," vacuum requires that the fluctuations  $\psi_k(\eta)$  behave as Minkowski-spacetime mode functions far inside the horizon, that is,  $\psi_k \rightarrow (2k)^{-1/2} \exp(-ik\eta)$  for  $k\eta \gg 1$  [17, 18, 20]. From the asymptotic properties of the Hankel functions, this gives

$$A_k = 0, \quad B_k = \frac{\sqrt{\pi}}{2} \exp \left[ -i\frac{\pi}{2} \left( \nu + \frac{1}{2} \right) \right]. \quad (11)$$

For the problem of (nonresonant) reheating, we will be interested in the opposite asymptotic limit:  $k\eta \ll 1$ , corresponding to long-wavelength modes. In this limit, the fluctuations  $\psi_k(\eta)$  take the form

$$\psi_k(\eta) \rightarrow \frac{2^{\nu-1}}{\sqrt{\pi}} \exp\left[-i\frac{\pi}{2}\left(\nu - \frac{1}{2}\right)\right] \Gamma(\nu) \eta^{1/2} (k\eta)^{-\nu}. \quad (12)$$

In order to calculate the number ( $N_k$ ) of  $\phi$  particles produced per mode  $k$ , we match this solution to long-wavelength mode solutions following the phase transition into the radiation-dominated era.

To do this, we make use of a Bogolyubov transformation (see, e.g., [18]). For long-wavelength modes, we may approximate the phase transition as instantaneous, occurring at some time  $\eta_*$  [21]:

$$\begin{aligned} \eta < \eta_* : a(\eta) &= (a_o \eta)^{p/(1-p)}, \\ \eta > \eta_* : a(\eta) &= C(\eta - \bar{\eta}), \end{aligned} \quad (13)$$

where  $\bar{\eta} \equiv \eta_* - (a_o^2 \eta_*)^{-1}$ . It is convenient to set  $a(\eta_*) = 1$ , which sets  $\eta_* = a_o^{-1}$ , and thus  $\bar{\eta} = 0$  and  $C = a_o$ . (Note that in the limit  $p \rightarrow \infty$ , the reference scale  $a_o$  becomes  $-H_o$ , where  $H_o$  is the Hubble constant of the de Sitter spacetime.) Next we perform a Bogolyubov transformation to match the long-wavelength mode solutions in the radiation-dominated era:

$$\psi_k(\eta > \eta_*) = \frac{1}{\sqrt{2k}} [\alpha_k e^{-ik\eta} + \beta_k e^{ik\eta}]. \quad (14)$$

The Bogolyubov coefficients  $\alpha_k$  and  $\beta_k$  may be determined by requiring that both  $\psi_k$  and  $\psi'_k$  be continuous at  $\eta = \eta_*$ . Then, using  $N_k = |\beta_k|^2$ , this gives

$$N_k = |\beta_k|^2 = \frac{4^{\nu-3/2}}{\pi} \Gamma^2(\nu) \left(\nu - \frac{1}{2}\right)^2 \left(\frac{k}{a_o}\right)^{-2\nu-1}. \quad (15)$$

Equation (15) is the main result of this section, and it can now be compared with the cases in which the effects of the inflaton's parametric resonance are included.

### III. PARTICLE CREATION FROM PARAMETRIC RESONANCE, WITH $H \simeq 0$

We return to Eq. (8) for the fluctuations  $\psi_k(\eta)$ . Rather than neglect the  $3\lambda\varphi^2(\eta)a^2(\eta)$  term in this section, however, we study the effects of this term as the inflaton field  $\varphi$  oscillates near the minimum of its potential. If the frequency of these oscillations is large enough, then we may expand the fields  $\varphi(\eta)$  and  $\varphi^2(\eta)$  in conformal-time harmonics, analogous to the cosmic-time harmonic decomposition adopted in [6]. Keeping the lowest term, we may write

$$\varphi^2(\eta) \simeq \overline{\varphi^2} \cos(\gamma\eta), \quad (16)$$

where  $\overline{\varphi^2}$  is a slowly decreasing amplitude. This quasiperiodic approximation for  $\varphi^2(\eta)$  may be compared with the numerical results (for Minkowski spacetime) calculated in [7]. For this section, we will assume that  $\overline{\varphi^2} \simeq$  constant. (As emphasized in [5, 8], the chief contributor to the decrease of  $\overline{\varphi^2}$  over time is the back reaction from the created particles, and will be further discussed in the Appendix.) Substituting this ansatz into Eq. (8) yields

$$\psi_k'' + \omega_k^2 \left[1 + ga^2 \cos(\gamma\eta) - \frac{a''}{k^2 a}\right] \psi_k = 0, \quad (17)$$

where

$$\omega_k^2 \equiv k^2, \quad g \equiv \frac{3\lambda\overline{\varphi^2}}{\omega_k^2}. \quad (18)$$

Note that by working in terms of conformal time and the "conformal field"  $\psi$ , the frequency  $\omega_k$  equals the (constant) comoving wave number  $k$ , and does not redshift with increasing  $a(\eta)$ . All of the effects of expansion are included in the explicit factors of  $a(\eta)$  and  $a''(\eta)$ . When expansion is neglected (i.e., when we set  $a = 1$ ), Eq. (17) reduces to the Mathieu equation, the solutions of which are exponentially unstable when  $g \ll 1$  and  $\gamma = 2\omega_k + \epsilon$ , with  $\epsilon \ll \omega_k$ . To see this, we may proceed as follows. Our treatment follows the methods outlined in [14], and substantially reproduces the results from the alternative approach adopted in [6].

We introduce the trial solution

$$\begin{aligned} \psi_k(\eta) &= c(\eta) f[a(\eta)] \cos[(\omega_k + \frac{1}{2}\epsilon)\eta] \\ &\quad + d(\eta) f[a(\eta)] \sin[(\omega_k + \frac{1}{2}\epsilon)\eta]. \end{aligned} \quad (19)$$

Here we have introduced the "scaling" function  $f(a)$  to absorb effects from the expansion of the universe. We assume that the coefficients  $c(\eta)$  and  $d(\eta)$  are slowly varying with time, as compared with the frequency  $\gamma/2$ , although we make no such assumptions about the behavior of  $f(a)$ . Then it is self-consistent to put  $c', d' = O(\epsilon)$ , and to neglect higher derivatives. In general, resonant solutions can be found for frequencies  $n\gamma/2 \simeq \omega_k$ , but each integer  $n \geq 1$  corresponds to keeping terms of  $O(g^n)$  in the perturbation expansion [14]. Since we are only going to keep terms to  $O(g)$ , we may make the approximations

$$\begin{aligned} \cos(\gamma\eta/2) \cos(\gamma\eta) &= \frac{1}{2} \cos(\gamma\eta/2) + \frac{1}{2} \cos(3\gamma\eta/2) \\ &\simeq \frac{1}{2} \cos(\gamma\eta/2), \\ \sin(\gamma\eta/2) \cos(\gamma\eta) &= -\frac{1}{2} \sin(\gamma\eta/2) + \frac{1}{2} \sin(3\gamma\eta/2) \\ &\simeq -\frac{1}{2} \sin(\gamma\eta/2). \end{aligned} \quad (20)$$

Then the equation of motion (17) applied to the trial solution of Eq. (19), yields

$$\omega_k A(\eta) \sin(\gamma\eta/2) + \omega_k B(\eta) \cos(\gamma\eta/2) + O(\epsilon^2, g^2) = 0, \quad (21)$$

with the coefficients  $A$  and  $B$  given by

$$\begin{aligned} A &= -2c'f - cf' \left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k} (2d'f' + df'') \\ &\quad - df \left(\frac{1}{2}g\omega_k a^2 + \epsilon + \frac{1}{\omega_k} \frac{a''}{a}\right), \\ B &= 2d'f + df' \left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k} (2c'f' + cf'') \\ &\quad + cf \left(\frac{1}{2}g\omega_k a^2 - \epsilon - \frac{1}{\omega_k} \frac{a''}{a}\right). \end{aligned} \quad (22)$$

For Eq. (21) to be satisfied, we require that both of the coefficients ( $A, B$ ) of the trigonometric terms vanish

identically. This gives a pair of coupled differential equations involving the coefficients  $c(\eta)$ ,  $d(\eta)$ , and  $f(a)$ .

For the remainder of this section we will neglect all expansion of the universe, and set  $f(a) = a = 1$ . Then, if we set  $c(\eta) = C \exp(s\eta)$  and  $d(\eta) = D \exp(s\eta)$ , Eqs. (21) and (22) yield

$$\begin{aligned} sC + \frac{1}{2}(\frac{1}{2}g\omega_k + \epsilon)D &= 0, \\ sD + \frac{1}{2}(\frac{1}{2}g\omega_k - \epsilon)C &= 0. \end{aligned} \quad (23)$$

These equations may be solved with

$$\begin{aligned} \frac{D}{C} &= \mp \frac{1}{y_k} \equiv \mp \sqrt{\frac{\frac{1}{2}g\omega_k - \epsilon}{\frac{1}{2}g\omega_k + \epsilon}}, \\ s &= \pm \frac{1}{2} \sqrt{(\frac{1}{2}g\omega_k)^2 - \epsilon^2}. \end{aligned} \quad (24)$$

Thus there exists a growing solution and a decaying solution:

$$\psi_k^\pm(\eta) = \frac{\exp[\pm s\eta]}{\sqrt{\gamma y_k}} [y_k \cos(\gamma\eta/2) \mp \sin(\gamma\eta/2)]. \quad (25)$$

This normalization was chosen, following [11], so that  $\psi_k^+ \psi_k^- - \psi_k^- \psi_k^+ = -1$ .

From Eqs. (24) and (25), it is clear that the solutions  $\psi_k$  will be exponentially unstable whenever  $s$  is real. Thus, the parametric resonance will occur only within a small frequency band:

$$|\epsilon| < \frac{1}{2}g\omega_k. \quad (26)$$

It will be convenient to introduce the variable  $\ell$  as

$$\ell \equiv \frac{2\epsilon}{g\omega_k}, \quad (27)$$

so that resonance occurs when  $-1 < \ell < 1$ . In terms of  $\ell$ ,  $s$  and  $y_k$  may be rewritten

$$s = \frac{g\omega_k}{4} \sqrt{1 - \ell^2}, \quad y_k = \sqrt{\frac{1 + \ell}{1 - \ell}}. \quad (28)$$

Note that  $y_k \rightarrow 1$  as  $\ell \rightarrow 0$ , near the center of the resonance band. We further introduce the function  $S_\pm(\eta)$  as

$$S_\pm(\eta) \equiv y_k \cos(\gamma\eta/2) \mp \sin(\gamma\eta/2), \quad (29)$$

so that near the center of the resonance band, as  $\ell \rightarrow 0$ ,

$$|S_\pm(\eta)|^2 \rightarrow 1 \mp \sin(\gamma\eta), \quad (30)$$

or, averaging over a few oscillations near  $\ell \sim 0$ ,

$$\langle |S_\pm(\eta)|^2 \rangle \rightarrow 1. \quad (31)$$

Near the center of the resonance band, then, the solutions  $\psi_k^\pm(\eta)$  become

$$\psi_k^\pm(\eta) \rightarrow \frac{\exp[\pm s\eta]}{\sqrt{2k}} S_\pm(\eta) \simeq \frac{\exp[\pm s\eta]}{\sqrt{2k}}, \quad (32)$$

where the irrelevant phase from the  $S_\pm$  term has been dropped.

We can now make use of a Bogolyubov transformation to solve for the number of particles per mode,  $N_k^{\text{res}}$ , produced by the decaying inflaton field during its resonant oscillations. In Sec. II, we made the approximation of an instantaneous phase transition at some time  $\eta_*$ , which can only be appropriate for long-wavelength modes. In the present case, however, the resonant modes cannot have arbitrarily long wavelengths: from Eqs. (18) and (26), we require  $k \geq 2|\epsilon|/g$ . Instead, use may be made of the more elaborate time-dependent Bogolyubov transformation developed in Appendix B of [6], the results of which are

$$N_k = |\beta_k|^2 = \frac{1}{1 - \ell^2} \sinh^2(s\eta). \quad (33)$$

As explained in [6], when  $s$  is not exactly constant in time, but changes adiabatically (such that  $|s'| \ll s^2$ ), then this equation for  $N_k$  may be modified to

$$N_k \simeq \sinh^2 \left( \int_{\text{res. band}} s d\eta \right), \quad (34)$$

where the integral extends over the resonance band [Eq. (26)]. Following [6], the (divergent) coefficient  $(1 - \ell^2)^{-1}$  has been dropped, because near  $|\ell| \simeq 1$ , the exponent  $s$  no longer changes adiabatically.

Equation (34), with  $s$  given by Eq. (24), is the main result of this section. Clearly, if  $\int s d\eta$  is large enough within the resonance band, then the number of particles produced will be exponentially greater than the number produced from the nonresonant decay studied in Sec. II, given by Eq. (15). There is an important difference, however, between our result for  $N_k$  in Eq. (34) and the corresponding expression in [6]. The analysis in [6] was carried out in terms of cosmic time  $t$  and the fluctuations  $\delta\phi_k$ ; the equations of motion thus resemble Eqs. (1) and (2) above, with  $\omega_k \simeq k/a$ , and with the assumption  $\omega_k \gg H$ . As studied in [6], then, the resonant modes exit the resonance band due both to the redshift of the physical wave numbers  $k/a$ , as well as to the back reaction from produced particles. This is to be contrasted with the foregoing analysis: In our case, by working in terms of conformal time and the ‘‘conformal field’’  $\psi_k$ , the only time dependence of the exponent  $s$  comes from the slow time dependence of the decaying amplitude  $\varphi^2$ . That is, in our case, it is  $g$  which is slowly changing with time, and not  $\omega_k$ ; and it is this change of  $g$  which causes various modes  $\psi_k^+$  to slide outside of the resonance band. The time dependence of  $g$  is further addressed in the Appendix. Meanwhile, we will assume for the remainder of this paper that  $\overline{\varphi^2}(\eta)$ , and hence  $g(\eta)$  and  $s(\eta)$ , may be treated as slowly varying with time. With this assumption, we may now study what happens to the resonant modes when the expansion of the universe is included.

#### IV. RESONANT INFLATON DECAY IN AN EXPANDING BACKGROUND

We return to the full equation of motion for the fluctuations  $\psi_k$ , Eq. (17), with  $\omega_k$  and  $g$  as defined in Eq. (18).

We introduce the same trial solution, Eq. (19), and keep terms to  $O(\epsilon, g)$ . This leads to Eq. (21), with the full coefficients  $A(\eta)$  and  $B(\eta)$  given by Eq. (22). We saw in Sec. III that a consistent resonant solution may be found when  $f(a)$  and  $a$  are both set equal to unity. In this section, we keep these terms explicit and general, and show that consistent resonant solutions may be found when the expansion of the universe is included.

We split the coefficients  $A$  and  $B$  into two terms each:  $A = A_1 + A_2$ , and  $B = B_1 + B_2$ , with

$$\begin{aligned} A_1 &= -2c'f - df\left(\frac{1}{2}g\omega_k + \epsilon\right), \\ A_2 &= -cf' \left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k} (2d'f' + df'') \\ &\quad - df \left(\frac{1}{2}g\omega_k (a^2 - 1) + \frac{1}{\omega_k} \frac{a''}{a}\right), \\ B_1 &= 2d'f + cf\left(\frac{1}{2}g\omega_k - \epsilon\right), \\ B_2 &= df' \left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k} (2c'f' + cf'') \\ &\quad + cf \left(\frac{1}{2}g\omega_k (a^2 - 1) - \frac{1}{\omega_k} \frac{a''}{a}\right). \end{aligned} \quad (35)$$

In order for Eq. (21) to be satisfied, we again require that the coefficients vanish:  $A = B = 0$ . Because we are interested in resonant behavior, we will further look for solutions when we set each of the terms  $A_1$ ,  $A_2$ ,  $B_1$ , and  $B_2$  separately equal to zero. Then the coefficients  $A_1$  and  $B_1$  reduce to the same pair of linear differential equations as in Eq. (23), which may be solved with the same  $y_k$  and  $s$  as given in Eq. (24). The task now is to use the equations

$$\begin{aligned} -c \left(\frac{f'}{f}\right) \left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k} \left[2d' \left(\frac{f'}{f}\right) + d \left(\frac{f''}{f}\right)\right] \\ - d \left(\frac{1}{2}g\omega_k (a^2 - 1) + \frac{1}{\omega_k} \frac{a''}{a}\right) = 0, \end{aligned}$$

$$\begin{aligned} d \left(\frac{f'}{f}\right) \left(2 + \frac{\epsilon}{\omega_k}\right) + \frac{1}{\omega_k} \left[2c' \left(\frac{f'}{f}\right) + c \left(\frac{f''}{f}\right)\right] \\ + c \left(\frac{1}{2}g\omega_k (a^2 - 1) - \frac{1}{\omega_k} \frac{a''}{a}\right) = 0, \end{aligned} \quad (36)$$

to find a solution for the expansion factor,  $f[a(\eta)]$ .

The first step is to notice that several terms in Eq. (36) may be combined:

$$\left[2d' \left(\frac{f'}{f}\right) + d \left(\frac{f''}{f}\right)\right] = \frac{1}{(fd)} \frac{d}{d\eta} (d^2 f'), \quad (37)$$

and similarly for the corresponding  $cf$  term in the second line of Eq. (36). Then, because the coefficients  $c(\eta)$  and  $d(\eta)$  are determined by Eq. (23), the two equations in (36) may be combined to give

$$\begin{aligned} \frac{d}{d\eta} [e^{2s\eta} f'] \\ = \left[ \frac{a''}{a} + \frac{1}{2} \frac{(y_k^2 - 1)}{(y_k^2 + 1)} g\omega_k^2 [1 - a^2(\eta)] \right] e^{2s\eta} f(\eta) \\ = \left[ \alpha(p) \eta^{-2} + \frac{1}{2} g\ell\omega_k^2 [1 - a^2(\eta)] \right] e^{2s\eta} f(\eta) \\ \equiv E^2(\eta) e^{2s\eta} f(\eta). \end{aligned} \quad (38)$$

In the second line, we have used the definitions of  $y_k$  in Eq. (24) and  $\ell$  in Eq. (27), and have introduced the coefficient  $\alpha(p)$ . For a cosmic-time scale factor  $a(t) \propto t^p$ , or, equivalently, a conformal-time scale factor  $a(\eta) = (a_0\eta)^{p/(1-p)}$ , the coefficient  $\alpha(p)$  is defined as

$$\frac{a''}{a} = \frac{p(2p-1)}{(p-1)^2} \eta^{-2} \equiv \alpha(p) \eta^{-2}. \quad (39)$$

Note that  $\alpha(p) \geq 0$  for  $p \geq 1/2$ , and that for de Sitter expansion,  $\alpha(\infty) = 2$ .

From Eq. (38), we may write the second-order differential equation for  $f$ :

$$f'' + 2sf' - E^2(\eta)f = 0. \quad (40)$$

If we now use the fact that  $s = O(g, \epsilon)$ , and that  $g \ll 1$ , then we may approximate<sup>1</sup>  $E^2(\eta) \simeq \alpha(p) \eta^{-2}$ , and

$$f'' - \alpha(p) \eta^{-2} f \simeq 0. \quad (41)$$

This equation may be solved with the ansatz  $f(\eta) = \beta \eta^{\mu_{1,2}}$ , with

$$\mu_{1,2} = \frac{1}{2} \left[ 1 \pm \sqrt{1 + 4\alpha(p)} \right]. \quad (42)$$

The two possible solutions for  $\mu$ , labeled by subscript 1 and 2, correspond to the choice of  $\pm$  in Eq. (42); the choice of whether  $\mu_1$  or  $\mu_2$  is appropriate will be determined below. With this solution for  $f(\eta)$ , the resonant solutions  $\psi_k$  may be written:

$$\psi_k^\pm(\eta) = \frac{f(\eta) \exp[\pm s\eta]}{\sqrt{\gamma y_k}} S_\pm(\eta), \quad (43)$$

with  $S_\pm(\eta)$  given by Eq. (29).

We may now choose the appropriate exponent  $\mu_{1,2}$  for the function  $f$  by matching the  $\eta$  dependence of this resonant solution for  $\psi_k^\pm$  with the nonresonant solutions found in Sec. II. For fixed  $\eta$ , the (nonresonant) long-wavelength solutions behave as [see Eqs. (10) and (12)]

$$\psi_k^{\text{nr}} \propto \eta^{p/(1-p)}. \quad (44)$$

In the resonant case considered in this section,  $\psi_k^\pm \propto f \propto \eta^{\mu_{1,2}}$ . From the definition of  $\alpha(p)$  in Eq. (39), the two exponents may be written

$$\mu_1 = \frac{1-2p}{1-p}, \quad \mu_2 = \frac{p}{1-p}. \quad (45)$$

<sup>1</sup>This approximation for  $E^2(\eta)$  is further justified far from the edge of the resonance band, when  $|\ell| \ll 1$ . Also, we may assume, as in Sec. II, that the phase transition occurs near  $a(\eta_*) = 1$ .

Thus the appropriate exponent for the resonant case is  $\mu_2$ . Note that  $f(\eta)$  has the same  $\eta$  dependence as the conformal-time scale factor,  $a(\eta)$ . Because we require that  $f[a(\eta)] = 1$  when  $a(\eta) = 1$ , we may fix the normalization of  $f$  as  $f = a(\eta) = (a_o\eta)^{\mu_2}$ . In addition, given that  $\alpha(p) \geq 0$  for  $p \geq 1/2$ , we will write this exponent as

$$\mu_2 = -|\mu_2| = -\frac{1}{2} \left[ \sqrt{1 + 4\alpha(p)} - 1 \right], \quad (46)$$

and thus  $f = (a_o\eta)^{-|\mu_2|}$ .

With the "scaling" function  $f$  now determined, the full resonant solutions in an expanding background may be written

$$\psi_k^\pm(\eta) = \frac{\exp[\pm x_\pm \eta]}{\sqrt{\gamma y_k}} S_\pm(\eta), \quad (47)$$

with the exponent  $x_\pm(\eta)$  defined by

$$x_\pm(\eta) \equiv \pm s - \frac{|\mu_2|}{\eta} \ln(a_o\eta). \quad (48)$$

The  $s$  term is given by Eq. (24). As in the Sec. III, resonance occurs when  $2|\epsilon| < g\omega_k$ . With the fluctuations  $\psi_k^\pm(\eta)$  now in the form of Eq. (47), we may again use the result of Appendix B of [6], and solve for the number of  $\phi$  particles produced per mode  $k$ :

$$N_k \simeq \sinh^2 \left( \int_{\text{res. band}} x_+(\eta) d\eta \right). \quad (49)$$

From the form of  $\mu_2$  in Eq. (46), it is clear that the expansion of the universe decreases the exponential production:  $\int x_+ d\eta$  over the resonance band is less than  $\int s d\eta$ , corresponding to the nonexpanding case of Sec. III. Nevertheless, it is possible analytically to find self-consistent resonant solutions for the fluctuations  $\psi_k^\pm$  when the expansion of the universe is included.

## V. CONCLUSION

We have demonstrated that the resonance effects studied in [5-7] can indeed have dramatic consequences for post-inflationary reheating. In particular, it has been shown that solutions for the number of particles produced from the resonant decay of the inflaton field may be found given a general background expansion,  $a(t) \propto t^p$ . Although the total number of particles produced in this expanding case,  $N_k$  in Eq. (49), is less than the corresponding expression when  $H \simeq 0$  [Eq. 34], there still remain regions of parameter space for which this resonant production far outweighs the nonresonant production described in Sec. II. The potential trouble for the new resonant reheating scenario is therefore not expansion of the universe, but rather the decaying amplitude of the classical  $\varphi$  field. This question is discussed further below, in the Appendix.

As pointed out in [5], the case of inflaton decay into further inflatons (as studied above) may be of interest for dark matter searches. Following their production, the inflatons would decouple from the rest of matter. If the inflatons were given a tiny mass (which has been neglected

here), then these bosons could serve as a natural candidate for the missing dark matter. Furthermore, much of the formalism developed in this paper may be taken over, unchanged, for studying the case of inflaton decay into some distinct light species of boson, as studied in [5, 6]. In such decays, it is again expected that the parametric resonance would survive a nonzero expansion of the universe. The difference for these decays would reside solely in the form of  $\overline{\varphi^2}(\eta)$ .

The foregoing analysis for the simple model of a minimally-coupled scalar field  $\phi$  with an Einstein-Hilbert gravitational action can also be applied directly to many classes of Generalized Einstein Theory inflationary models, such as those studied in [22]. For these models, the conformal-transformation factor  $\Omega(x) \rightarrow 1$  as the inflaton field reaches the minimum of its potential. During the  $\phi$  field's oscillatory phase, the nonminimal  $\phi^2 R$  coupling takes the form of the standard  $(16\pi G)^{-1} R$  gravitational action of a minimally-coupled theory. Near the epoch of reheating, then, the effective Lagrangian takes the form of Eq. (3), from which the reheating analysis may proceed as above. It would be interesting to calculate  $N_k$  for the new models of open inflation [23], following the methods developed above, which might display qualitatively different reheating scenarios. This is the subject of further study.

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## APPENDIX

It is important to understand how  $\overline{\varphi^2}$ , and hence  $g$  and  $s$ , changes with time. A proper treatment would be to extend the Minkowski-spacetime approach adopted in [7, 8], and to work out the thermal Green's functions for the decaying inflaton field for the general background expansion considered here. (See also [24].) As a first approximation, we may posit the reasonable phenomenological ansatz, which may be compared with Fig. 1 in [7]:

$$\overline{\varphi^2}(\eta) \simeq \varphi_o^2 e^{-\eta/\tau}, \quad (A1)$$

where  $\tau$  is some damping time scale, most likely to be determined from numerical integration. In other words, we will assume here that the frequency of the  $\cos(\gamma\eta)$  oscillations may be treated as nearly constant, while the amplitude of the oscillations decreases. From Fig. 1 in [7], it is clear that we may assume that  $\tau^{-1} \ll \omega_k$ . If we then write  $g(\eta) = g_o \exp(-\eta/\tau)$ , and keep terms to first order in  $g_o$ ,  $(\epsilon/\omega_k)$ , and  $(\tau^{-1}/\omega_k)$ , we may repeat

the above analysis in Eqs. (17)–(25). We will restrict our attention to the nonexpanding case, since the expanding case examined in Sec. IV reveals the same dependence of  $N_k$  on  $g(\eta)$ .

We introduce the trial solution

$$\psi_k(\eta) = c(\eta)e^{-\eta/\tau} \cos(\gamma\eta/2) + d(\eta)e^{-\eta/\tau} \sin(\gamma\eta/2). \quad (\text{A2})$$

Again taking  $c', d' = O(\epsilon)$  leads to the coupled differential equations

$$\begin{aligned} -2c' + 2c\tau^{-1} - d\epsilon - \frac{1}{2}dg_o\omega_k e^{-\eta/\tau} &= 0, \\ 2d' - 2d\tau^{-1} - c\epsilon + \frac{1}{2}cg_o\omega_k e^{-\eta/\tau} &= 0. \end{aligned} \quad (\text{A3})$$

These equations may be solved approximately by writing  $c(\eta) = C \exp[\theta\eta]$  and  $d(\eta) = D \exp[\theta\eta]$ , with

$$\begin{aligned} \frac{D}{C} &= \mp \frac{1}{y_k(\eta)} \equiv \mp \sqrt{\frac{\frac{1}{2}g_o\omega_k \exp[-\eta/\tau] - \epsilon}{\frac{1}{2}g_o\omega_k \exp[-\eta/\tau] + \epsilon}}, \\ \theta(\eta) &= \pm \frac{1}{2} \sqrt{\left(\frac{1}{2}g_o\omega_k\right)^2 e^{-2\eta/\tau} - \epsilon^2} + \frac{1}{\tau} \\ &\equiv \pm s(\eta) + \frac{1}{\tau}. \end{aligned} \quad (\text{A4})$$

With these solutions, the growing and decaying mode functions take the form

$$\psi_k^\pm(\eta) = \frac{\exp[\pm s(\eta)\eta]}{\sqrt{\gamma y_k(\eta)}} [y_k(\eta) \cos(\gamma\eta/2) \mp \sin(\gamma\eta/2)]. \quad (\text{A5})$$

These solutions have the benefit over the corresponding solutions in Eq. (25) of ending the resonance in a natural way: unlike the solutions from Sec. III, these mode functions do not continue to grow exponentially forever. Now the resonance band is explicitly time dependent:

$$|\epsilon| < \frac{g_o\omega_k}{2} e^{-\eta/\tau}. \quad (\text{A6})$$

In addition, the slight time dependence of  $y_k(\eta)$  and  $s(\eta)$  still allows these mode functions to be normalized as:  $\psi_k^{+\prime}\psi_k^- - \psi_k^{-\prime}\psi_k^+ = -1 + O(\tau^{-1}/\omega_k)$ .

With this simple ansatz for the decaying inflaton field amplitude, the exponent  $s(\eta)$  changes in time as

$$\frac{|s'|}{s^2} = \frac{\tau^{-1}}{s(\eta)} \frac{1}{[1 - \ell^2(\eta)]}. \quad (\text{A7})$$

In analogy with the situation in Sec. III, we have defined the variable  $\ell(\eta) \equiv (2\epsilon)/[g(\eta)\omega_k]$ . In order to satisfy the adiabaticity requirement for  $s(\eta)$  (away from the edges of the resonance band), we must require  $\tau^{-1} < g_o\omega_k$ . Note that this is more stringent than the original assumption,  $\tau^{-1} \ll \omega_k$ . If this new constraint may be satisfied, then the number of  $\phi$  particles produced may be calculated from

$$\begin{aligned} \int_{\text{res. band}} s(\eta) d\eta &= \frac{\tau\epsilon}{2} \int_{-1}^{+1} d\ell \frac{\sqrt{1 - \ell^2}}{\ell^2} \\ &= \frac{\tau\epsilon}{2} \left[ -\frac{\sqrt{1 - \ell^2}}{\ell} - \arcsin(\ell) \right]_{\ell=-1}^{\ell=+1} \\ &= -\frac{\pi\tau\epsilon}{2}. \end{aligned} \quad (\text{A8})$$

This gives  $N_k \simeq \sinh^2(\int s d\eta) = \sinh^2(\pi\tau\epsilon/2)$ . Given the constraints on both  $\tau$  and  $\epsilon$ , we find  $N_k \gg 1$ .

As emphasized in [5, 8], the decay of the amplitude of the inflaton might instead take the form of a power law in time, rather than the exponential form adopted in Eq. (A1). If we write

$$\overline{\varphi^2}(\eta) \simeq \varphi_o^2 \left( \frac{\eta}{\eta_o} \right)^{-b}, \quad (\text{A9})$$

with  $b > 0$  a real constant, and thus

$$g(\eta) = g_o \left( \frac{\eta}{\eta_o} \right)^{-b}, \quad \ell(\eta) = \frac{2\epsilon}{g_o\omega_k} \left( \frac{\eta}{\eta_o} \right)^b, \quad (\text{A10})$$

then we may again repeat the steps in Eqs. (17)–(25). The solutions  $y_k$  and  $s$  again take the form

$$y_k(\eta) = \sqrt{\frac{\frac{1}{2}g(\eta)\omega_k + \epsilon}{\frac{1}{2}g(\eta)\omega_k - \epsilon}}, \quad s(\eta) = \frac{1}{2} \sqrt{\left(\frac{1}{2}g(\eta)\omega_k\right)^2 - \epsilon^2}, \quad (\text{A11})$$

and the growing and decaying mode functions again take the form of Eq. (A5), with  $y_k$  and  $s$  now given by Eq. (A11). Now the exponent  $s(\eta)$  changes in time as

$$\frac{|s'|}{s^2} = \frac{b\eta^{-1}}{s(\eta)} \frac{1}{[1 - \ell^2(\eta)]}. \quad (\text{A12})$$

In order to satisfy the adiabaticity condition, we therefore require  $b < g_o$ . Then the number of  $\phi$  particles produced may be calculated from

$$\begin{aligned} \int_{\text{res. band}} s(\eta) d\eta &= \frac{\epsilon\eta_o}{2b} \left( \frac{g_o\omega_k}{2\epsilon} \right)^{1/b} \int_{-1}^{+1} d\ell \ell^{1/b} \frac{\sqrt{1 - \ell^2}}{\ell^2} \\ &\simeq \text{const} \times \frac{\epsilon\eta_o}{2b} \ell_o^{-1/b}, \end{aligned} \quad (\text{A13})$$

where  $\ell_o \equiv \ell(\eta_o)$ . From Eq. (A10), we see that  $\ell_o < \ell_{\text{max}} = 1$ , so that given  $0 < b < g_o \ll 1$ ,  $\ell_o^{-1/b} \gg 1$ . This yields  $N_k \sim \sinh^2[\epsilon\eta_o\ell_o^{-1/b}/(2b)] \gg 1$ .

These two different forms for the decay of the inflaton field because of back reaction from produced particles, the exponential decay of Eq. (A1), and the power-law decay of Eq. (A9), both allow a large production of particles from the inflaton's decay, so long as the adiabaticity requirement on  $s(\eta)$  may be met. It is clear, however, that a more extended treatment of the effects of back reaction on the decaying amplitude  $\overline{\varphi^2}(\eta)$  is required, and is the subject of further study.

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