# Preheating in an expanding universe: Analytic results for the massless case

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Analytic results are presented for preheating in both flat and open models of chaotic inflation, for the case of massless inflaton decay into further inflaton quanta. It is demonstrated that preheating in both these cases closely resembles that in Minkowski spacetime. Furthermore, quantitative differences between preheating in spatially flat and open models of inflation remain of order $10^{-2}$ for the chaotic inflation initial conditions considered here. [S0556-2821(97)01314-3]

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## I. INTRODUCTION

Recently, a new view of the post-inflationary reheating period has been established [1,2]. In place of the original view, in which the inflaton decayed perturbatively [3], an inherently nonperturbative, highly efficient resonance has been investigated. The new theory of reheating now involves three distinct stages: an oscillating inflaton sets up a parametric resonance in its decay to some boson species; this explosive stage has been termed ''preheating'' [1]. (See also [2,4–14].) Next, these far-from-equilibrium decay products themselves interact and decay, as can be studied along the methods in [3]. And finally, the decay products thermalize, completing the reheating process. Such explosive preheating can radically change the thermal history of the universe following inflation; some nonstandard effects associated with the new preheating picture include the possibility for nonthermal symmetry restoration [15], grand unified theory (GUT) scale baryogenesis [16], and the amplification of gravitational radiation following the preheating period [17].

Because preheating is a nonlinear, nonequilibrium process, particle production in most specific models must be analyzed numerically, and several results have been reported for both Minkowski and spatially flat expanding spacetimes [4,7,9,10,18]. Much of the analytical literature to date has treated the expanding background case by means of two approximations: (1) that the oscillating zero mode of the inflaton oscillates as a cosine function, $\varphi_0 \propto \cos(mt)$, with $m$ the mass of the inflaton, and (2) that the expansion may be neglected for frequencies $m \gg H$, where $H$ is the Hubble parameter. When these approximations are made, the equation of motion for the quantum fluctuations (associated with particle production) can be cast in the form of a Mathieu equation. Solutions of the Mathieu equation generically exhibit an infinite hierarchy of resonance bands; for wave numbers $k$ within these resonance bands, the solutions grow exponentially in time, driving the explosive, resonant particle production. As discussed in [9], however, in the context of Minkowski spacetime, the first of these approximations can lead to large quantitative errors when a quartic self-coupling exists for the inflaton. In these cases, the zero mode evolves as an elliptic function in time, and solutions for the fluctuat-

ing fields reveal only one single resonance band in $k$.

In this paper, we extend this analytical study to the case of inflaton decay into inflaton bosons in an expanding Friedmann-Robertson-Walker spacetime. For a quartic interaction potential and a *massless* inflaton, the time evolution of the entire system (zero mode, quantum fluctuations, and background spacetime) can be studied consistently with analytical methods. Because in this case the observed spectrum of cosmic microwave background anisotropies places strict limits on the self-coupling, $\lambda \sim 10^{-12}$, we may study this nonperturbative, nonequilibrium system by means of analytic approximations to the full, nonlinear equations of motion. In [9], the large-$N$ approximation is employed to study preheating analytically in Minkowski spacetime. Here, we rely on a Hartree factorization to study preheating in an expanding spacetime. As demonstrated explicitly in [19], these two approximation schemes are closely related, and in the case of preheating amount only to the substitution $\lambda \to 3\lambda$ in the equation of motion for the fluctuating field. (See also [20].) Numerical results have indicated that the Hartree approximation can break down for quantitative results far from the weak-coupling range (that is, for inflaton decay into a distinct species of boson, with coupling constant $g \gg \lambda$) [7]. However, for the case studied here, of inflaton decay into other inflaton quanta, the Hartree approximation should remain quantitatively reliable. With this analytical framework, it is also easy to understand numerical results [7,18] which indicate that *no* parametric resonance can occur for a massive inflaton in expanding spacetime, if its only decay channel is into inflaton quanta.

We present analytic results for preheating for both ordinary chaotic inflation and for chaotic inflation within a model of open inflation [21]. The growth of the back reaction from the created quanta on the oscillating zero mode is calculated, as well as the maximum number of quanta produced during preheating. As discussed below, for preheating of a massless inflaton into massless inflaton quanta, the Ricci curvature scalar vanishes during preheating. Thus, the preheating dynamics for this model in an expanding spacetime remain conformally equivalent to the Minkowski spacetime case; it is not surprising, therefore, that the results presented here share the same form as many of the Minkowski results studied in [9]. The change from large-$N$ to Hartree methods, however, does change some of the details of the quantitative analysis. Also, as demonstrated below, the spectrum of pro-

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duced quanta in the open inflation scenario takes the same form as that for quanta produced in an expanding, spatially flat model; for chaotic inflation initial conditions, there are only very small numerical deviations between the two situations.

In Sec. II, we present the specific model to be analyzed here, discuss the approximation scheme to track the quantum back reaction, and choose an appropriate initial vacuum state with reference to which we can measure the resonant particle production. Section III includes analysis of the background spacetime dynamics during the preheating phase. In Sec. IV, we solve consistently for both the oscillating zero mode of the inflaton field and the coupled quantum fluctuations, for both the flat and open universe cases. In Sec. V, we compare numerically the preheating spectra for flat and open inflation, and provide further consistency checks on some of the approximations made throughout the analysis. Concluding remarks follow in Sec. VI.

## II. DYNAMICS OF THE MODEL

We only consider the case of inflaton decay into inflaton bosons, due to a nonlinear self-coupling. The Lagrangian density thus may be written

$$\mathcal{L} = \sqrt{-g} \left[ \frac{1}{16\pi G} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right],$$

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4. \tag{1}$$

We will consider the case of an additional nonminimal coupling between the inflaton and $R(t)$, the Ricci curvature scalar, in Sec. III. The line element for a general Friedmann-Robertson-Walker (FRW) spacetime may be written

$$ds^2 = -dt^2 + a^2(t) h_{ij} dx^i dx^j, \tag{2}$$

with

$$h_{ij} dx^i dx^j = d\mathbf{x}^2 \quad (K=0)$$

$$= dr^2 + \sinh^2 r (d\theta^2 + \sin^2\theta d\phi^2) \quad (K=-1), \tag{3}$$

and $K=0$ for a flat universe, $K=-1$ for an open universe. [Naturally, the angular coordinate $\phi$ should not be confused with the inflaton field $\phi(t,\mathbf{x})$.] In this metric, the equations of motion follow:

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right],$$

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + \frac{dV}{d\phi} = 0, \tag{4}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, $\nabla^2$ is the comoving Laplacian operator, and dots represent derivatives with respect to cosmic time, $t$.

Making the familiar decomposition between the "classical" and fluctuating portions of the inflaton field,

$$\phi(t,\mathbf{x}) = \varphi(t) + \delta\phi(t,\mathbf{x}), \tag{5}$$

we see that the potential of Eq. (1) contains a coupling between $\varphi$ and $\delta\phi$. It is this coupling which, under certain conditions, will allow for a highly efficient transfer of energy from the "classical" to the fluctuating portions of the inflaton field; this transfer of energy is manifested as a rapid production of out-of-equilibrium inflaton quanta. Because the self-coupling strength $\lambda$ is constrained to be very weak for this model ($\lambda \sim 10^{-12}$), based on the observed anisotropies in the cosmic microwave background radiation, we may employ the Hartree factorization to study the nonequilibrium dynamics of these coupled systems [22]. This factorization amounts to making a particular choice of vacuum state (as discussed below), and making the substitutions

$$(\delta\phi)^3 \rightarrow 3\langle(\delta\phi)^2\rangle(\delta\phi),$$

$$(\delta\phi)^4 \rightarrow 6\langle(\delta\phi)^2\rangle(\delta\phi)^2 - 3\langle(\delta\phi^2)\rangle^2, \tag{6}$$

where quantities in angular brackets indicate expectation values of the associated quantum operators (to be defined explicitly below); the tadpole condition further requires $\langle(\delta\phi)\rangle = 0$.

In an open universe, there exists a physical curvature length scale $a(t)/|K|$, with $K=-1$. The corresponding comoving curvature scale is thus simply $+1$. As usual, scalar fields may be expanded in eigenfunctions of the comoving Laplacian. Writing $\nabla^2 f_P(\mathbf{x}) = -P^2 f_P(\mathbf{x})$, then in an open universe, modes with eigenvalue $P^2 \geq 1$ vary on scales shorter than the comoving curvature scale, and hence may be labeled "subcurvature" modes, whereas modes with $0 \leq P^2 < 1$ correspond to "supercurvature" modes [23,24]. As first noted in [25], and as demonstrated below in Sec. V, preheating in an open universe will *only* populate subcurvature modes for potentials of the form in Eq. (1). For this reason, we may expand the fluctuating field $\delta\phi$ as follows:

$$\delta\phi(t,\mathbf{x}) = \int d\tilde{\mu}(k) \, \delta\phi_{kjm}(t) Z_{kjm}(\mathbf{x}), \tag{7}$$

where the eigenfunctions of the Laplacian obey

$$\nabla^2 Z_{kjm}(\mathbf{x}) = -(k^2 - K) Z_{kjm}(\mathbf{x}). \tag{8}$$

Here $k$ is the comoving *subcurvature* wave number, with $0 \leq k^2 < \infty$; it is related to the eigenvalue $P^2$ above by $k^2 = P^2 + K$, and only applies for $P^2 \geq 1$. The measure for the wave number integral is [23,24,26]

$$\int d\tilde{\mu}(k) = \int_0^\infty dk k^2 \quad (K=0)$$

$$= \int_0^\infty dk \sum_{j=0}^\infty \sum_{m=-j}^j \quad (K=-1). \tag{9}$$

With these definitions, the eigenfunctions of the Laplacian may be written

$$Z_{kjm}(\mathbf{x}) = (2\pi)^{-3/2} e^{i\mathbf{k}\cdot\mathbf{x}} \quad (K=0),$$

$$Z_{kjm}(r,\theta,\phi) = \sqrt{\frac{2}{\pi}} \prod_{s=0}^{j} (s^2+k^2)^{-1/2} Y_{jm}(\theta,\phi)$$

$$\times (\sinh r)^j \left( \frac{-1}{\sinh r} \frac{d}{dr} \right)^{j+1} \cos(kr)$$

$$(K=-1), \quad (10)$$

with $Y_{jm}(\theta,\phi)$ the usual spherical harmonics [24,26,27].

To study particle production during the preheating epoch, we may now promote the field $\delta\phi$ to a Heisenberg operator:

$$\hat{\delta\phi}(t,\mathbf{x}) = \int d\widetilde{\mu}(k) [\delta\phi_k(t) \hat{a}_{kjm} Z_{kjm}(\mathbf{x})$$

$$+ \delta\phi_k^*(t) \hat{a}_{kjm}^\dagger Z_{kjm}^*(\mathbf{x})], \quad (11)$$

with $\hat{a}_{kjm}$ and $\hat{a}_{kjm}^\dagger$ the canonical time-independent annihilation and creation operators, respectively. These are defined with respect to the initial Fock vacuum. With this, the expectation value $\langle \hat{\delta\phi}^2 \rangle$ becomes, for both $K=0$ and $K=-1$,

$$\langle \hat{\delta\phi}^2 \rangle = \int_0^\infty \frac{dk k^2}{2\pi^2} |\delta\phi_k(t)|^2, \quad (12)$$

where translational invariance allows $\langle \hat{\delta\phi}^2 \rangle$ to depend only on time; we will therefore write this quantity as $\langle \hat{\delta\phi}^2(t) \rangle$. Defining the constant

$$\mathcal{C}^2 \equiv \langle \hat{\delta\phi}^2(t_0) \rangle, \quad (13)$$

where $t_0$ is the beginning of the preheating epoch, and switching to conformal time, $d\eta \equiv a^{-1} dt$, we may introduce the dimensionless variables

$$\tau = \mathcal{C}\eta, \quad \ell = k/\mathcal{C}, \quad M = m/\mathcal{C},$$

$$\varphi(t) = \frac{\mathcal{C}}{\sqrt{\lambda}} \frac{\psi(\tau)}{a(\tau)}, \quad \delta\phi_k(\tau) = \frac{1}{\sqrt{\mathcal{C}}} \frac{\chi_\ell(\tau)}{a(\tau)},$$

$$\Sigma(\tau) = [\langle \chi^2(\tau) \rangle - \langle \chi^2(\tau_0) \rangle]. \quad (14)$$

Then $\langle \hat{\delta\phi}^2(t) \rangle = \mathcal{C}^2 a^{-2}(\tau) \langle \chi^2(\tau) \rangle$, and using the factorization of Eq. (6), the equations of motion for the $\psi$ and $\chi$ fields may be written

$$\left[ \frac{d^2}{d\tau^2} + a^2(\tau) M^2 - \frac{a''}{a} + \psi^2(\tau) + 3\lambda a^2(\tau_0) + 3\lambda \Sigma(\tau) \right] \psi(\tau) = 0,$$

$$\left[ \frac{d^2}{d\tau^2} + \ell^2 - \mathcal{K} + a^2(\tau) M^2 - \frac{a''}{a} + 3\psi^2(\tau) + 3\lambda a^2(\tau_0) + 3\lambda \Sigma(\tau) \right] \chi_\ell(\tau) = 0, \quad (15)$$

where primes denote derivatives with respect to $\tau$, and $\mathcal{K} \equiv \mathcal{C}^{-2} K$. It is convenient to define the frequencies $W_\ell(\tau)$ and $\omega_\ell(\tau)$ as

$$W_\ell^2(\tau) \equiv \ell^2 - \mathcal{K} + a^2(\tau) M^2 - \frac{a''}{a},$$

$$\omega_\ell^2(\tau) \equiv W_\ell^2(\tau) + 3\psi^2(\tau) + 3\lambda a^2(\tau_0) + 3\lambda \Sigma(\tau). \quad (16)$$

The quantity $\Sigma(\tau)$ measures the back reaction of created quanta on the evolution of the oscillating zero mode, $\psi(\tau)$, and $\Sigma(\tau_0) = 0$. We will study solutions of the coupled equations of Eq. (15) in Sec. IV, for early times when $\lambda \Sigma(\tau)$ may be neglected. We will also determine self-consistently in that section the time at which the approximation $\lambda \Sigma(\tau) \to 0$ breaks down.

It remains in this section to derive an expression for the particle number operator appropriate for the nonequilibrium dynamics. In terms of the rescaled field $\chi$, the quantum fluctuation operator $\hat{\delta\phi}$ may be written

$$\frac{1}{\sqrt{\mathcal{C}}} \hat{\delta\phi}(t,\mathbf{x}) = \int d\widetilde{\mu}(\ell) \frac{1}{a(\tau)} [\chi_\ell(\tau) \hat{a}_{\ell jm} Z_{\ell jm}(\mathbf{x})$$

$$+ \chi_\ell^*(\tau) \hat{a}_{\ell jm}^\dagger Z_{\ell jm}^*(\mathbf{x})], \quad (17)$$

and, from the Lagrangian density in Eq. (1), the Heisenberg operator for the conjugate field may be written:

$$\sqrt{\mathcal{C}} \hat{\Pi}(t,\mathbf{x}) = \int d\widetilde{\mu}(\ell) a(\tau) \sqrt{h(\mathbf{x})}$$

$$\times \{ [\chi_\ell'(\tau) - \mathcal{H}(\tau) \chi_\ell(\tau)]$$

$$\times \hat{a}_{\ell jm} Z_{\ell jm}(\mathbf{x}) + \text{H.c.} \}, \quad (18)$$

with $\mathcal{H}(\tau) \equiv a'/a$, and "H.c." denoting a Hermitian conjugate. Canonical quantization then gives the normalization condition for the mode functions $\chi_\ell(\tau)$:

$$\chi_\ell \chi_\ell^{*\prime} - \chi_\ell' \chi_\ell^* = i. \quad (19)$$

This normalization comes from quantizing the fields $\hat{\delta\phi}$ and $\hat{\Pi}$ on a Cauchy surface. Yet for models of open inflation, the interior of the nucleated bubble is *not* a Cauchy surface for the entire de Sitter space. It has been demonstrated, however, that for scalar fields expanded only in subcurvature modes, quantizing *as if* the interior of the nucleated bubble were a proper Cauchy surface reproduces exactly the same result as when the fields are quantized on a proper Cauchy surface and analytically continued inside the bubble [28]. For this reason, we may use the normalization in Eq. (19) for both the flat

and open inflation cases. In the context of preheating, this makes sense physically, since we are quantizing these fields after the volume of the interior of the bubble has expanded by a factor of $\sim (e^{65})^3$ (according to any ''observers'' on the interior of the bubble), so that all causal properties of these fields should be specifiable with reference to the interior of the bubble alone.

To calculate the particle number per mode, it is convenient to write

$$\hat{\delta\phi}(t,\mathbf{x}) = \int d\widetilde{\mu}(\ell)\hat{\delta\phi}_{\ell jm}(t)Z_{\ell jm}(\mathbf{x}),$$

$$\hat{\Pi}(t,\mathbf{x}) = \int d\widetilde{\mu}(\ell)\hat{\Pi}_{\ell jm}(t,\mathbf{x})Z_{\ell jm}(\mathbf{x}). \quad (20)$$

For $K=0$, we may thus write

$$\frac{1}{\sqrt{\mathcal{C}}}\hat{\delta\phi}_{\ell jm}(t) = \frac{1}{a(\tau)}[\chi_\ell \hat{a}_{\vec{\ell}} + \chi_{\vec{\ell}}^* \hat{a}_{-\vec{\ell}}^\dagger],$$

$$\sqrt{\mathcal{C}}\hat{\Pi}_{\ell jm}(t,\mathbf{x}) = a(\tau)\sqrt{h(\mathbf{x})}[(\chi_\ell' - \mathcal{H}\chi_\ell)\hat{a}_{\vec{\ell}}$$
$$+ (\chi_{\vec{\ell}}^{*\prime} - \mathcal{H}\chi_{\vec{\ell}}^*)\hat{a}_{-\vec{\ell}}^\dagger]. \quad (21)$$

As pointed out in [24] for the $K=-1$ case, we may use the purely real representation of the $Y_{jm}(\theta,\phi)$'s, which makes the eigenfunctions $Z_{\ell jm}(\mathbf{x})$ purely real for subcurvature modes. Thus, for $K=-1$ we also may write expressions as in Eq. (21), but with the arguments of the creation and annihilation operators changed to $\hat{a}_{\vec{\ell}} \to \hat{a}_{\ell jm}$ and $\hat{a}_{-\vec{\ell}}^\dagger \to \hat{a}_{\ell jm}^\dagger$.

Care must be taken when studying preheating in an expanding universe to choose an appropriate vacuum state with respect to which we can measure particle production. There are two independent concerns: first, preheating is a nonequilibrium process among interacting fields, so defining ''particle'' states is ambiguous even in Minkowski spacetime [9]; in addition, we have the usual ambiguity in defining ''free particle'' states in any non-Minkowskian spacetime [26]. At $\tau \to -\infty$ (where we take the bubble nucleation to have occurred at $\tau = -\infty$), the spacetime inside the nucleated bubble is a de Sitter spacetime, so in the absence of interactions, the vacuum state for the $\delta\phi$ field should be the Bunch-Davies vacuum. If the transition from pure de Sitter expansion to the different rate of expansion at the time of preheating occurs adiabatically, then at the onset of preheating the $\chi_\ell$ modes should obey the initial conditions $\chi_\ell(\tau_0) = [2W_\ell(\tau_0)]^{-1/2}$ and $\chi_\ell'(\tau_0) = -i[W_\ell(\tau_0)/2]^{1/2}$ in the absence of interactions. These modes would represent the free-particle ''adiabatic'' states [26]. Furthermore, if we consider the interaction strength to be turned on adiabatically beginning some time before $\tau_0$, then these initial conditions should be replaced by

$$\chi_\ell(\tau_0) = \frac{1}{\sqrt{2\omega_\ell(\tau_0)}}, \quad \left.\left(\frac{d\chi_\ell}{d\tau}\right)\right|_{\tau=\tau_0} = -i\sqrt{\frac{\omega_\ell(\tau_0)}{2}}. \quad (22)$$

Equation (22) gives the initial conditions for the ''adiabatic'' particle states for the nonequilibrium dynamics at the onset of preheating. We may, therefore, define adiabatic creation and annihilation operators $\hat{\alpha}_{\ell jm}(\tau)$ and $\hat{\alpha}_{\ell jm}^\dagger(\tau)$ through the relations

$$\frac{1}{\sqrt{\mathcal{C}}}\hat{\delta\phi}_{\ell jm}(\tau) \equiv \frac{1}{a(\tau)}\frac{1}{\sqrt{2\omega_\ell(\tau)}}\left[\hat{\alpha}_{\ell jm}(\tau)\exp\left(-i\int\omega_\ell(\tau)d\tau\right) + \hat{\alpha}_{\ell jm}^\dagger(\tau)\exp\left(i\int\omega_\ell(\tau)d\tau\right)\right],$$

$$\sqrt{\mathcal{C}}\hat{\Pi}_{\ell jm}(\tau,\mathbf{x}) \equiv -i\sqrt{\frac{\omega_\ell(\tau)}{2}}a(\tau)\sqrt{h(\mathbf{x})}\left[\left(1 - i\frac{\mathcal{H}(\tau)}{\omega_\ell(\tau)}\right)\hat{\alpha}_{\ell jm}(\tau)\exp\left(-i\int\omega_\ell(\tau)d\tau\right) + \text{H.c.}\right]. \quad (23)$$

The $\hat{\alpha}_{\ell jm}(\tau)$ annihilates the time-dependent adiabatic vacuum state $\hat{\alpha}_{\ell jm}(\tau)|0(\tau)\rangle = 0$ for all $\ell, j$, and $m$. These adiabatic creation and annihilation operators can be related to the time-independent operators $\hat{a}_{\ell jm}$ and $\hat{a}_{\ell jm}^\dagger$ by means of a Bogolyubov transformation. As above, we replace the arguments of these operators as $\hat{\alpha}_{\ell jm} \to \hat{\alpha}_{\vec{\ell}}$ and $\hat{\alpha}_{\ell jm}^\dagger \to \hat{\alpha}_{-\vec{\ell}}^\dagger$ in Eq. (23) for the $K=0$ case. Note that these expansions are only valid when the frequency $\omega_\ell(\tau)$ is purely real; as demonstrated in Sec. III, this will always be the case for the scenarios considered here.

From the expansions in Eqs. (21) and (23), we may solve for the adiabatic particle number per mode. The result is the same for both the $K=0$ and $K=-1$ cases:

$$N_\ell^{\text{ad}}(\tau) = \langle\hat{\alpha}_{\ell jm}^\dagger(\tau)\hat{\alpha}_{\ell jm}(\tau)\rangle$$
$$= \frac{\omega_\ell(\tau)}{2}\left[|\chi_\ell(\tau)|^2 + \frac{1}{\omega_\ell^2(\tau)}|\chi_\ell'(\tau)|^2\right] - \frac{1}{2}. \quad (24)$$

This yields the number of ''adiabatic''-state quanta produced relative to the initial Fock vacuum, $|0\rangle = |0(\tau_0)\rangle$. With this choice of vacuum state and initial conditions, the particle number for inflaton quanta does indeed vanish at the onset of preheating. As discussed in Sec. III, for the models of interest here, the background spacetime will evolve as if it were radiation dominated for the entire period of preheating, so

that there will be no further Bogolyubov transformations needed to relate the particle number at the beginning to that at the end of preheating.

With the choice of initial conditions for the $\chi$ field, Eq. (22), the quantity $\mathcal{C}^2 = \langle \hat{\delta\phi}^2(\tau_0) \rangle$ is formally quadratically divergent, and would have to be renormalized with some regularization scheme. (This is essentially the zero-point energy divergence.) For here, we will simply note that to remain consistent, we require the energy density of the "classical" portion of the inflaton field to exceed that of the quantum fluctuation portion at the beginning of preheating; that is, $\varphi^2(\tau_0) \gg \langle \hat{\delta\phi}^2(\tau_0) \rangle$. This is equivalent to the requirement

$$\psi^2(\tau_0) \gg \lambda a^2(\tau_0), \tag{25}$$

using the definition of $\psi(\tau)$ in Eq. (14). We will make use of this relation below.

### III. EVOLUTION OF THE BACKGROUND SPACETIME

In this section we study the background spacetime dynamics during the period of preheating. Up until the time $\tau_0$, the fluctuations of the field $\delta\phi$ are in their vacuum state, and we need only consider the energy density from the "classical" part of the inflaton field, $\varphi$. In terms of cosmic time $t$, this may be written

$$\rho_\varphi(t) = \tfrac{1}{2} \dot{\varphi}^2 + \tfrac{1}{2} m^2 \varphi^2 + \tfrac{1}{4} \lambda \varphi^4. \tag{26}$$

In terms of dimensionless conformal time $\tau$ and the rescaled field $\psi$, the energy density is, thus,

$$\rho_\varphi(\tau) = \frac{\mathcal{C}^4}{2\lambda a^4(\tau)} \left[ \frac{1}{2} \psi^4 + \psi'^2 + a^2 M^2 \psi^2 + \mathcal{H}\psi(\mathcal{H}\psi - 2\psi') \right]. \tag{27}$$

At the onset of preheating, the field $\psi$ begins to oscillate (as treated explicitly in Sec. IV), so that averaging over a period of its oscillations, both $\psi^2$ and $\psi'^2$ will remain nearly constant. For the chaotic inflation scenario we consider here, then, when the mass $m$ vanishes, the energy density at the onset of preheating will be dominated by the $a^{-4}$ terms. Note that by working in terms of conformal time, we do not have to make any assumptions about the magnitude or rate of change of the Hubble parameter $H(t)$, as we would if we worked in terms of cosmic time, $t$. Instead, we only need to use the chaotic inflation initial conditions, $\psi^4(\tau_0) \gg \psi^2(\tau_0)$. We will confirm in Sec. V that when $m = 0$, these initial conditions ensure that $\mathcal{H} \ll \psi$ during preheating.

Thus, when $m = 0$ (and only then), we may approximate the time evolution of the background spacetime as that of a radiation-dominated FRW metric at the very onset of preheating. (This point is also noted in [14].) Furthermore, for the case of inflaton decay into inflaton bosons, the produced quanta will also be massless when $m = 0$, so that over the entire preheating period we may keep the approximation $\rho(\tau) \propto a^{-4}(\tau)$. Note that this behavior of the energy density holds even though the quanta are far from thermal equilibrium when produced.

Using the Friedmann equation [see Eq. (4)], and writing $\rho(\eta) = \rho_0 [a(\eta)/a_0]^{-4g}$, we may solve for the behavior of the scale factor in terms of conformal time $\eta$ for both the $K = 0$ and $K = -1$ cases. For a flat universe, the scale factor evolves as

$$a(\eta) = a_0 \left( \frac{\eta}{\eta_0} \right)^{1/(2g-1)}, \quad g \neq \frac{1}{2}, \tag{28}$$

and for $K = -1$,

$$a(\eta) = a_0 \sinh^d \left( \frac{\eta}{|d|} + X \right), \quad d \equiv \frac{2}{4g-2}, \quad g \neq \frac{1}{2}. \tag{29}$$

In this case, by defining $a_0 = a(\eta_0)$, the Friedmann equation further implies the relation $H_0^2 = 2/a_0^2$. In terms of the dimensionless conformal time $\tau$, setting $g = 1$ (for $m = 0$) yields the very simple result for $K = 0, 1$:

$$\frac{a''}{a} = -\mathcal{K}. \tag{30}$$

This result, valid only for $m = 0$, means that the addition of a nonminimal coupling between the inflaton and the Ricci curvature scalar of the form $\tfrac{1}{2} \xi R \phi^2$ would not affect any of the preheating dynamics. To see this, consider the Ricci curvature scalar in terms of cosmic time, $t$:

$$R(t) = \frac{6}{a^2(t)} [\ddot{a}a + \dot{a}^2 + K]. \tag{31}$$

Rewriting this in terms of the dimensionless conformal time $\tau$,

$$R(\tau) = \frac{6\mathcal{C}^2}{a^2(\tau)} \left[ \frac{a''}{a} + \mathcal{K} \right], \tag{32}$$

we see from Eq. (30) that a consistent solution of the modified Friedmann equation would include $R = 0$ for both $K = 0$ and $K = -1$ when $m = 0$. Thus, for the massless case, a nonminimal coupling would not affect the preheating dynamics.

Using Eq. (30), we may further confirm that the frequency $\omega_\ell(\tau)$ will always remain real [see Eq. (16)], which allows us to employ the adiabatic creation and annihilation operators of Eq. (23). In fact, when $m = 0$, this frequency is the same for both the $K = 0$ and $K = -1$ cases:

$$\omega_\ell^2(\tau) = \ell^2 + 3\psi^2(\tau) + 3\lambda a^2(\tau_0) + 3\lambda \Sigma(\tau). \tag{33}$$

The condition of Eq. (25) further allows us to neglect the $3\lambda a^2(\tau_0)$ term relative to the $3\psi^2$ term during preheating. With these expressions for the evolution of the background spacetime, we may now study the dynamics of the $\psi$ and $\chi$ fields during preheating.

### IV. EVOLUTION OF THE FIELDS DURING PREHEATING

Using Eqs. (25) and (30), we may rewrite the equation of motion for the $\psi$ field in Eq. (15) as

$$\left[\frac{d^2}{d\tau^2}+\psi^2(\tau)+\mathcal{K}+3\lambda\Sigma(\tau)\right]\psi(\tau)=0. \qquad (34)$$

In this section, we will shift the time of the onset of preheating to $\tau_0=0$. With the initial conditions, $\psi(0)=\psi_0$ and $(d\psi/d\tau)_{|\tau=0}=0$, the equation of motion for $\psi(\tau)$ may be solved in terms of a Jacobian elliptic cosine function for early times, when the back reaction $3\lambda\Sigma(\tau)$ is negligible. In general, the Jacobian cosine function $\mathrm{cn}(u,\bar{k})$ obeys the differential equation [29]:

$$\left[\frac{d^2}{du^2}+(1-2\bar{k}^2+2\bar{k}^2c^2)\right]c=0, \qquad (35)$$

where we have used the abbreviation $c\equiv\mathrm{cn}(u,\bar{k})$. For early times, then, the $\psi$ field oscillates as

$$\psi(\tau)=\psi_0\mathrm{cn}\left(\sqrt{\psi_0^2+\mathcal{K}}\,\tau,\frac{\psi_0}{\sqrt{2(\psi_0^2+\mathcal{K})}}\right), \qquad (36)$$

for $\psi_0^2\gg-2\mathcal{K}$, appropriate for a chaotic inflation scenario.

In terms of the timelike variable $u=\sqrt{\psi_0^2+\mathcal{K}}\,\tau$, the equation of motion for the fluctuations becomes

$$\left[\gamma^2\frac{d^2}{du^2}+\ell^2+3\psi^2(u)\right]\chi_\ell(u)=0, \qquad (37)$$

for early times, when we may neglect $\lambda\Sigma(\tau)$. Here we have defined the constant $\gamma^2\equiv\psi_0^2+\mathcal{K}$. We may rewrite the Jacobian cosine function in terms of the doubly periodic Weierstrass function $\wp(z)$ by noting both that $\mathrm{cn}^2(u,\bar{k})+\mathrm{sn}^2(u,\bar{k})=1$, and that

$$\mathrm{sn}^2(u,\bar{k})=\frac{1}{\bar{k}^2(e_1-e_3)}\left[\wp\left(\frac{u+iK'(\bar{k})}{\sqrt{e_1-e_3}}\right)-e_3\right], \quad (38)$$

where $K'(\bar{k})$ is the complementary complete elliptic integral of the first kind, and the $e_i$ are the three constants associated with the Weierstrass function $\wp(z)$ [29]. These constants sum to zero: $e_1+e_2+e_3=0$; in terms of them, the modulus $\bar{k}$ may be written $\bar{k}^2=(e_2-e_3)/(e_1-e_3)$. The function $\wp(z)$ is periodic with respect to the two periods, $2\omega$ and $2\omega'$, as follows [29,30]:

$$\wp(z+2M\omega+2N\omega')=\wp(z),$$

$$\omega=\frac{K(\bar{k})}{\sqrt{e_1-e_3}}, \qquad \omega'=i\frac{K'(\bar{k})}{\sqrt{e_1-e_3}}, \qquad (39)$$

for integers $M$ and $N$. We may set $(e_1-e_3)=1$, which yields

$$e_1=\frac{3\gamma^2+\mathcal{K}}{6\gamma^2}, \quad e_2=-\frac{\mathcal{K}}{3\gamma^2}, \quad e_3=-\frac{3\gamma^2-\mathcal{K}}{6\gamma^2}. \quad (40)$$

With these substitutions, Eq. (37) becomes

$$\left[\frac{d^2}{du^2}+p^2-6\wp(u+\omega')\right]\chi_p(u)=0,$$

$$p^2(k)\equiv\frac{\ell^2-2\mathcal{K}}{\gamma^2}. \qquad (41)$$

This is now in the form of a Lamé equation. We may solve it explicitly by introducing two dimensionless constants $a$ and $b$ by means of the transcendental relations

$$3\wp(a)+3\wp(b)=-p^2,$$

$$\wp'(a)=-\wp'(b), \qquad (42)$$

where primes in this section will denote derivatives with respect to $z\equiv u/\sqrt{e_1-e_3}=u$. Using the differential equation [29,30]

$$\wp'^2(z)=4\wp^3(z)-g_2\wp(z)-g_3,$$

$$g_2\equiv2(e_1^2+e_2^2+e_3^2), \quad g_3\equiv4e_1e_2e_3, \qquad (43)$$

the relations in Eq. (42) imply

$$\wp(b)=-\frac{1}{6}p^2+\frac{1}{2}\sqrt{1+\frac{\mathcal{K}^2}{3\gamma^4}-\frac{1}{3}p^4},$$

$$\wp'^2(b)=\frac{4}{27}p^6-\frac{1}{3}\left(1+\frac{\mathcal{K}^2}{3\gamma^4}\right)p^2-\frac{\mathcal{K}}{3\gamma^2}\left(1-\frac{\mathcal{K}^2}{9\gamma^4}\right). \quad (44)$$

With these relations, solutions of Eq. (41) may be written

$$U_p(u)=\frac{\sigma(u+\omega'+a)\sigma(u+\omega'+b)\sigma^2(\omega')}{\sigma^2(u+\omega')\sigma(\omega'+a)\sigma(\omega'+b)}$$

$$\times\exp\{-u[\zeta(a)+\zeta(b)]\}, \qquad (45)$$

where $\sigma(z)$ and $\zeta(z)$ are the quasiperiodic Weierstrass functions, defined by the relations [29,30] $\zeta'(z)\equiv-\wp(z)$ and $\sigma'(z)/\sigma(z)\equiv\zeta(z)$. The solution $U_p(u)$ is normalized as $U_p(0)=1$, and the linearly independent solution is $U_p(-u)$. Using the initial conditions of Eq. (22), the fluctuations $\chi_\ell(\tau)$ can be written as a linear combination of $U_p(u)$ and $U_p(-u)$:

$$\chi_\ell(u)=\frac{1}{2\sqrt{2\omega_\ell(0)}}\left[\left(1+i\frac{\omega_\ell(0)}{\gamma U_p'(0)}\right)U_p(-u)\right.$$

$$\left.+\left(1-i\frac{\omega_\ell(0)}{\gamma U_p'(0)}\right)U_p(u)\right], \qquad (46)$$

where the primes here denote $d/du=\gamma^{-1}d/d\tau$. From the form of the number operator, Eq. (24), solutions $U_p(\pm u)$ which grow in time will contribute to particle production.

In general, solutions to second-order differential equations with periodic coefficients will obey Floquet's theorem [31]; that is, for a periodic ''potential'' with period $2\omega$ [as is the case for Eq. (41)], solutions behave as

$$U_p(u+2\omega)=U_p(u)e^{iF(p)}, \qquad (47)$$

where the Floquet index $F(p)$ is independent of time. The solutions $U_p(-u)$ have the Floquet index $-F(p)$. The quasiperiodicity of the $\sigma(z)$ functions [30],

$$\sigma(z+2\omega) = -\sigma(z)\exp[2\zeta(\omega)(z+\omega)], \qquad (48)$$

yields, for the solutions in Eq. (45),

$$F(p) = 2i\{\omega[\zeta(a)+\zeta(b)]-(a+b)\zeta(\omega)\}. \qquad (49)$$

The solutions $U_p(u)$ will reveal exponential instabilities, then, whenever $F(p)$ has a nonzero imaginary component. Equations (24) and (46) reveal that it is these instabilities which are responsible for the highly efficient, resonant particle production during preheating. As is clear from the relations of Eq. (42), together with series expansions of $\wp(z), \wp'(z),$ and $\zeta(z)$ (see [30]), such instabilities will occur only when *both* $\wp(b)$ and $\wp'(b)$ are real. From Eq. (44), we see that for the physical wave numbers $0 \leq k^2 < \infty$, there exists only *one* band in which these exponential instabilities may occur:

$$\frac{3}{2}\sqrt{1+\frac{\mathcal{K}^2}{3\gamma^4}} \leq p^2 \leq \sqrt{3+\frac{\mathcal{K}^2}{\gamma^4}}, \qquad (50)$$

or, in terms of $\psi_0$ and the dimensionless wave number $\ell$:

$$\frac{3}{2}\sqrt{\psi_0^4+2\mathcal{K}\psi_0^2+\frac{4}{3}\mathcal{K}^2}+2\mathcal{K} \leq \ell^2$$
$$\leq \sqrt{3\psi_0^4+6\mathcal{K}\psi_0^2+4\mathcal{K}^2}+2\mathcal{K}. \qquad (51)$$

The existence of only one single resonance band for physical wave numbers in this case matches the results found for Minkowski spacetime in [9]. This resonance band is in terms of *comoving* wave number $k$; because we have neglected interactions amongst the final-state bosons, the only way a mode $\chi_\ell(\tau)$ will slide out of the resonance band in this system is when the back reaction $\lambda\Sigma(\tau)$ grows so large that it damps the oscillations of the zero mode, halting the parametric resonance. We will demonstrate in Sec. V that for preheating in an expanding open FRW universe, this range only includes subcurvature modes.

In order to evaluate quantitatively values of the solutions $U_p(\pm u)$ in this resonance band, it is helpful to rewrite the solutions in terms of the quasiperiodic Jacobian $\vartheta$ functions, which possess very rapidly converging series expansions. Using the relations [29]

$$\sigma(z) = 2\omega\frac{\vartheta_1(v)}{\vartheta_1'(0)}\exp\left[\frac{\zeta(\omega)z^2}{2\omega}\right],$$

$$\vartheta_1\left(v+\frac{\omega'}{2\omega}\right) = i\vartheta_4(v)\exp\left[-i\pi\left(v+\frac{\omega'}{4\omega}\right)\right], \qquad (52)$$

the solutions $U_p(u)$ may be written

$$U_p(\pm v) = \frac{\vartheta_4(v+\alpha)\vartheta_4(v+\beta)\vartheta_4^2(0)}{\vartheta_4^2(v)\vartheta_4(\alpha)\vartheta_4(\beta)}\exp[\pm ivF(p)], \qquad (53)$$

where we have used $2\omega v \equiv u, 2\omega\alpha \equiv a,$ and $2\omega\beta \equiv b$. The relation [29]

$$\zeta(z) = \frac{\zeta(\omega)}{\omega}z+\frac{1}{2\omega}\frac{\vartheta_1'(v)}{\vartheta_1(v)} \qquad (54)$$

further allows us to rewrite the Floquet index $F(p)$ as

$$F(p) = i\left[\frac{\vartheta_1'(\alpha)}{\vartheta_1(\alpha)}+\frac{\vartheta_1'(\beta)}{\vartheta_1(\beta)}\right]. \qquad (55)$$

With the constraints on $a$ and $b$ in Eq. (42), the Floquet index will reach a maximum in the resonance band for $a \to b$. We may evaluate the Floquet index at this maximum by means of the series expansion for $\vartheta_1(v)$ [29]:

$$\frac{\vartheta_1'(v)}{\vartheta_1(v)} = \pi\cot(\pi v)+4\pi\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})}\sin(2n\pi v),$$
$$= \pi\cot(\pi v)+4\pi q^2\sin(2\pi v)+O(q^4), \qquad (56)$$

where the elliptic nome $q \equiv \exp[-iK'(\bar{k})/K(\bar{k})]$ satisfies $0 \leq q \leq 0.0432$. Writing $F(p) \simeq 2i[\vartheta_1'(\beta_{max})/\vartheta_1(\beta_{max})]$ near its maximum in the resonance band yields

$$\text{Re}\left(\frac{\vartheta_1'(\beta_{max})}{\vartheta_1(\beta_{max})}\right) = 4\pi q+O(q^3),$$

$$\text{Re}\left(\frac{\partial^2}{\partial v^2}\left(\frac{\vartheta_1'(v)}{\vartheta_1(v)}\right)\bigg|_{\beta_{max}}\right) = -16\pi^3 q+O(q^3), \qquad (57)$$

where the maximum of the resonance occurs at $\beta_{max}$. From the relations in Eqs. (42) and (44), $\beta_{max}$ corresponds to the wave number

$$p_{max}^2 \to \sqrt{3+\frac{\mathcal{K}^2}{\gamma^4}}. \qquad (58)$$

In other words, the maximum resonance will occur for quanta with wave numbers near the top of the resonance band. This behavior also matches that found for Minkowski spacetime in [9].

With these expressions, we may determine the growth of the back reaction due to created quanta, as well as the number of quanta produced during the preheating epoch. From the sign of $F(p)$ at its maximum, it is clear that the $U_p(-u)$ modes will grow exponentially in time for modes within the resonance band, while the resonant $U_p(u)$ modes will decrease exponentially. Keeping only the growing modes, then, we may approximate

$$|\chi_\ell(\tau)|^2 \simeq \frac{1}{8\omega_\ell(0)}|U_p(-u)|^2\left[1+\frac{\omega_\ell^2(0)}{\gamma^2 U_p'^2(0)}\right] \qquad (59)$$

and

$$\Sigma(\tau) \simeq \int_0^\infty \frac{d\ell\,\ell^2}{2\pi^2} |\chi_\ell(\tau)|^2$$

$$\simeq \int_0^\infty \frac{d\ell\,\ell^2}{16\pi^2 \omega_\ell(0)} |A_p(u)|^2 \left[ 1 + \frac{\omega_\ell^2(0)}{\gamma^2 U_p'^2(0)} \right]$$

$$\times \exp\left[ \frac{u}{\omega} \mathrm{Re}[F(p)] \right] \tag{60}$$

inside the resonance band. Here we have written the combination of oscillating $\vartheta$ functions as the single function, $A_p(u)$. We will evaluate this integral by means of a saddle-point approximation. Using Eq. (41), we may substitute

$$d\ell\,\ell^2 = \gamma^3 dp\, p \sqrt{p^2 - \frac{2\mathcal{K}}{\gamma^2}}, \tag{61}$$

and using Eq. (57) we may set

$$\mathrm{Re}[F(p)] \simeq 8\pi q - 16\pi^3 q (p - p_{\max})^2 + O(q^3). \tag{62}$$

Furthermore, the integral will reach its maximum value when the oscillating term $A_p(u) = 1$. This yields

$$\Sigma(\tau) \simeq \left[ \frac{\gamma^3 p \sqrt{p^2 - 2\mathcal{K}/\gamma^2}}{128\pi^3 \omega_\ell(0)} \left( 1 + \frac{\omega_\ell^2(0)}{\gamma^2 U_p'^2(0)} \right) \right] \Bigg|_{p_{\max}}$$

$$\times \frac{1}{\sqrt{q\gamma\tau/\omega}} \exp\left[ \frac{8\pi\gamma q}{\omega} \tau + O(q^3) \right]. \tag{63}$$

Following [9], we may then write the back reaction in the form

$$\Sigma(\tau) \equiv \frac{\gamma^{3/2}}{N(\mathcal{K})\sqrt{\tau}} \exp[B(\mathcal{K})\gamma\tau]. \tag{64}$$

In this form, we may solve for the time $\tau$ when the back reaction $3\lambda\Sigma(\tau)$ grows to be of the same magnitude as the tree-level terms. In our case, this occurs when $3\lambda\Sigma(\tau) \simeq 3\psi^2(\tau)$.

For the entire range $-2\mathcal{K} \leq \psi_0^2 \leq \infty$, the average of the square of the zero mode over a period of its oscillations gives, to a good approximation [9],

$$3\overline{\psi^2(\tau)} \simeq \frac{3}{2}\psi_0^2 = \frac{3}{2}(\gamma^2 - \mathcal{K}). \tag{65}$$

Setting the back reaction equal to this quantity yields

$$\tau_{\mathrm{end}} \simeq \frac{1}{B(\tilde{\mathcal{K}})\gamma} \ln\left( \frac{N(\tilde{\mathcal{K}})(1 - 3\tilde{\mathcal{K}})}{2\lambda\sqrt{B(\tilde{\mathcal{K}})}} \right), \tag{66}$$

where we have defined $\tilde{\mathcal{K}} \equiv \mathcal{K}/(3\gamma^2)$. Once the back reaction grows to equal the magnitude of the tree-level terms, the parametric amplification of the fluctuation modes ends. This is the end of the preheating epoch, and hence we label the time at which this occurs $\tau_{\mathrm{end}}$.

In order to evaluate $\Sigma(\tau)$, we must calculate $U_p'^2(0)$ at $p_{\max}$. Using the relations [30]

$$\zeta(z_1 + z_2) = \zeta(z_1) + \zeta(z_2) + \frac{1}{2} \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)},$$

$$\wp'(\omega') = 0, \quad \wp(\omega') = e_3, \tag{67}$$

then Eqs. (40) and (45) yield

$$U_p'^2(0)\big|_{p_{\max}} \simeq \frac{\wp'^2(b_{\max})}{[\wp(b_{\max}) - e_3]^2}$$

$$= \frac{2(3 + 9\tilde{\mathcal{K}}^2)^{3/2} - 18\tilde{\mathcal{K}}(1 - \tilde{\mathcal{K}}^2)}{[6 - (3 - 3\tilde{\mathcal{K}})\sqrt{3 + 9\tilde{\mathcal{K}}^2} - 9\tilde{\mathcal{K}}(1 - \tilde{\mathcal{K}})]}. \tag{68}$$

Similarly, the quantity

$$\omega_{\ell_{\max}}^2(0) = \ell_{\max}^2 + 3\psi_0^2 = \gamma^2[3 + \sqrt{3 + 9\tilde{\mathcal{K}}^2} - 3\tilde{\mathcal{K}}]. \tag{69}$$

The back reaction may thus be written

$$\Sigma(\tau) = \frac{3\gamma^{3/2}}{256\pi^3} G^{1/2}(\tilde{\mathcal{K}}) J(\tilde{\mathcal{K}}) \sqrt{\frac{K(\bar{k})}{q}} \frac{1}{\sqrt{\tau}}$$

$$\times \exp\left[ \frac{8\pi\gamma q}{K(\bar{k})}\tau + O(q^3) \right], \tag{70}$$

where we have defined

$$G(\tilde{\mathcal{K}}) \equiv \frac{3 + 9\tilde{\mathcal{K}}^2 - 6\tilde{\mathcal{K}}\sqrt{3 + 9\tilde{\mathcal{K}}^2}}{3 + \sqrt{3 + 9\tilde{\mathcal{K}}^2} - 3\tilde{\mathcal{K}}},$$

$$J(\tilde{\mathcal{K}}) \equiv \frac{3 + \sqrt{3 + 9\tilde{\mathcal{K}}^2}(1 + 3\tilde{\mathcal{K}} + 6\tilde{\mathcal{K}}^2) - 3\tilde{\mathcal{K}}(6 - 3\tilde{\mathcal{K}} - 2\tilde{\mathcal{K}}^2)}{(3 + 9\tilde{\mathcal{K}}^2)^{3/2} - 9\tilde{\mathcal{K}}(1 - \tilde{\mathcal{K}}^2)}. \tag{71}$$

When $K = 0$, this reduces to the more simple form

$$\Sigma(\tau)\big|_{K=0} = \frac{\psi_0^{3/2}}{256\pi^3}\sqrt{3 + \sqrt{3}} \sqrt{\frac{K(1/\sqrt{2})}{q_{\max}}} \frac{1}{\sqrt{\tau}}$$

$$\times \exp\left[ \frac{8\pi\psi_0 q_{\max}}{K(1/\sqrt{2})}\tau \right]. \tag{72}$$

The numerical values $K(1/\sqrt{2}) = 1.854$, $q_{\max} = 0.0432$, and $\lambda = 10^{-12}$ yield, for the quantities in Eq. (64),

$$N(0) = \frac{256\pi^3}{\sqrt{3 + \sqrt{3}}} \sqrt{\frac{q_{\max}}{K(1/\sqrt{2})}} = 556.995,$$

$$B(0) = \frac{8\pi q_{\max}}{K(1/\sqrt{2})} = 0.586,$$

$$\tau_{\mathrm{end}}(0) = \frac{1}{B(0)\psi_0} \ln\left( \frac{N(0)}{2\lambda\sqrt{B(0)}} \right) = \frac{57.266}{\psi_0}. \tag{73}$$

The quantities $N(\widetilde{\mathcal{K}})$, $B(\widetilde{\mathcal{K}})$, and $\tau_{\text{end}}(\widetilde{\mathcal{K}})$ will all approach these values in the limit $\psi_0 \to \infty$. We will calculate the maximum quantitative deviation from the flat-space results in Sec. V.

Using the same saddle-point approximation, we may solve for the total number of particles produced during preheating. Within the resonance band, we will approximate

$$\frac{d\chi_\ell}{d\tau} \simeq \left[\frac{-i\gamma}{2\omega} F(p)\right]\chi_\ell(\tau), \tag{74}$$

so that near the center of the resonance band [see Eq. (24)],

$$N_\ell^{\text{ad}}(\tau) \simeq \frac{\omega_\ell(\tau)}{2}|\chi_\ell(\tau)|^2\left[1 + \frac{16\pi^2 q^2 \gamma^2}{\omega^2 \omega_\ell^2(\tau)}\right]. \tag{75}$$

The total number of particles is

$$N_{\text{total}}^{\text{ad}}(\tau) = \int \frac{d\ell\,\ell^2}{2\pi^2} N_\ell^{\text{ad}}(\tau). \tag{76}$$

From Eq. (60), using $3\lambda\Sigma(\tau_{\text{end}}) \simeq 3\psi_0^2/2$, and approximating $\omega_\ell(\tau_{\text{end}}) \simeq \omega_\ell(0)$, this may be written

$$N_{\text{total}}^{\text{ad}}(\tau_{\text{end}}) \simeq \frac{\gamma^3(1-3\widetilde{\mathcal{K}})}{4\lambda}[3 + \sqrt{3 + 9\widetilde{\mathcal{K}}^2} - 3\widetilde{\mathcal{K}}]^{1/2}$$
$$\times \left[1 + \frac{16\pi^2 q^2}{K^2(\bar{k})(3 + \sqrt{3 + 9\widetilde{\mathcal{K}}^2} - 3\widetilde{\mathcal{K}})}\right]. \tag{77}$$

Equation (77) confirms that at the end of the preheating epoch, the number of particles produced is nonperturbatively large, $N_{\text{total}} \propto \lambda^{-1}$. Thus, for a massless inflaton decaying strictly into massless inflaton quanta, resonant preheating in an expanding FRW metric closely resembles the situation for Minkowski spacetime.

## V. COMPARISON OF OPEN AND FLAT CASES

In this section, we will demonstrate that two key approximations made above are consistent for the case of preheating with a massless inflaton, and then compare the numerical results for the $K=-1$ and $K=0$ cases. First consider the approximation made in Sec. III that $\mathcal{H} \ll \psi$ during preheating. Taking $\mathcal{H}(0) = 0$, we may write the average of this Hubble parameter over the duration of preheating. For $K=0$,

$$\bar{\mathcal{H}} = \frac{1}{2}\mathcal{H}(\tau_{\text{end}}) = \frac{1}{2\tau_{\text{end}}} \simeq 10^{-2}\psi_0, \tag{78}$$

where the last step comes from using Eq. (73). As demonstrated below, for the $K=-1$ case, $\tau_{\text{end}}$ remains greater than or equal to its numerical value in the $K=0$ case, so that

$$\bar{\mathcal{H}} = \tfrac{1}{2}\coth(\tau_{\text{end}}) < \psi_0 \tag{79}$$

for all $\psi_0^2 > -2\mathcal{K}$, as long as $\mathcal{C}^2 \lesssim 10$ after renormalization. Over the course of preheating, then, it remains a good approximation to neglect the $\mathcal{H}^2\psi^2$ term relative to the $\psi^4$ term

in the energy density of the zero mode, Eq. (27); when this is done, the energy density remains proportional to $a^{-4}$ when $m=0$.

The other main assumption to check concerns the expansion of the fluctuating field $\chi$ in subcurvature modes only, when $K=-1$. For the decay process to be resonant, we require that the period of the Floquet solutions remain less than the Hubble time. In terms of the timelike variable $u$, this period is $2\omega = 2K(\bar{k})$; in terms of conformal time $\eta$, then, this requirement becomes

$$a_0\frac{2K(\bar{k})}{\gamma\mathcal{C}} < H_0^{-1}. \tag{80}$$

When $K=-1$, $a_0 = \sqrt{2}H_0^{-1}$, so this requirement becomes

$$\gamma\mathcal{C} > 2\sqrt{2}K(\bar{k}). \tag{81}$$

The modulus $\bar{k}$ is related to $\gamma$ as [see Eq. (36)]

$$\bar{k}^2 = \frac{\gamma^2\mathcal{C}^2 - K}{2\gamma^2\mathcal{C}^2}. \tag{82}$$

Equation (81) is therefore satisfied for $\gamma\mathcal{C} \gtrsim 5.3$. From Eqs. (41) and (50), and using $k = \mathcal{C}\ell$, this means that

$$k_{\text{min}}^2 = \frac{3}{2}\sqrt{\gamma^4\mathcal{C}^4 + \frac{K^2\mathcal{C}^2}{3} + 2K} \geq \frac{3}{2}(5.3)^2 - 2 = 40.1$$

$$\text{or} \quad k_{\text{min}} \geq 6.3. \tag{83}$$

Thus, over the entire range of allowable initial conditions for $\psi_0$, preheating with $K=-1$ will only populate subcurvature modes, with $k_{\text{min}} > 1$.

With this restriction on the combination $\gamma\mathcal{C}$, we may further calculate the quantitative deviation for $K=-1$ in the growth of the back reaction and total number of particles produced from the $K=0$ case. Numerical differences from the $K=0$ case arise both from the dependence of $\bar{k}$ [and hence of $K(\bar{k})$ and $q$] on $\mathcal{K}$, as well as from the explicit factors of $\widetilde{\mathcal{K}} = K/(3\gamma^2\mathcal{C}^2)$ in such quantities as $N$, $B$, $\tau_{\text{end}}$, and $N_{\text{total}}^{\text{ad}}$. At the minimum allowable value of $\gamma\mathcal{C} = 5.3$, we have $\bar{k} = 0.72$ (near the flat case of $\bar{k} = 0.707$), or $q = 0.0396$ (near the flat case of $q = 0.0432$). With the constants at these values,

$$|\widetilde{\mathcal{K}}| \leq \frac{1}{3\gamma^2\mathcal{C}^2} = 1.2 \times 10^{-2}. \tag{84}$$

From the definition of $\gamma^2$, we also have the relation

$$\frac{\gamma}{\psi_0} = (1 - 3\widetilde{\mathcal{K}})^{-1/2}. \tag{85}$$

This means that the deviation from the $K=0$ case remains on the order of $10^{-2}$. For example, comparing Eqs. (71) and (73),

$$G_{\max}(\widetilde{\mathcal{K}}) = G(0) + 2.1 \times 10^{-2},$$

$$J_{\max}(\widetilde{\mathcal{K}}) = J(0) + 1.0 \times 10^{-2},$$

$$\frac{N_{\max}(\widetilde{\mathcal{K}})}{N(0)} = 0.93,$$

$$\frac{B_{\max}(\widetilde{\mathcal{K}})}{B(0)} = 0.89. \tag{86}$$

With these values, we see that the duration of preheating also remains nearly the same in the $K = -1$ and $K = 0$ cases:

$$\frac{\tau_{\mathrm{end}}(\widetilde{\mathcal{K}})}{\tau_{\mathrm{end}}(0)} = 1.14. \tag{87}$$

Finally, the total number of particles produced in the two cases obeys

$$\frac{N_{\mathrm{total}}^{\mathrm{ad}}(\widetilde{\mathcal{K}})}{N_{\mathrm{total}}^{\mathrm{ad}}(0)} = 0.98. \tag{88}$$

For $\gamma\mathcal{C} > (\gamma\mathcal{C})_{\min} = 5.3$, $\bar{k} \to 1/\sqrt{2}$ and $\widetilde{\mathcal{K}} \to 0$, so all numerical quantities further approach their flat-space values.

Thus, the requirement that the period of the Floquet solutions, $2\omega$, remain less than one Hubble time greatly restricts the numerical deviations between quantities in the $K = -1$ and $K = 0$ cases, when we consider chaotic inflation initial conditions. For all dynamically consistent initial values of $\psi_0$ in the open inflation case, the total number of particles produced during preheating remains within 2% of the flat-space results.

## VI. CONCLUSIONS

For a massless inflaton decaying resonantly into massless inflaton quanta, preheating in expanding FRW spacetimes closely resembles the Minkowski spacetime case. With chaotic inflation initial conditions, furthermore, quantitative differences in the spectra of produced particles remain small when comparing spatially flat with open models of inflation. In both cases, a single narrow resonance band for comoving wave number $k$ exists. Preheating ends at $\tau_{\mathrm{end}}$, when the back reaction $\lambda\Sigma(\tau)$ due to produced quanta damps the oscillations of the zero mode. Because the Hartree factorization employed here neglects interactions amongst the final-state bosons, such as rescatterings between each other and with the zero mode, the quantitative values for the spectra derived here likely represent an overestimate of their true magnitudes, though with $\lambda \sim 10^{-12}$, such rescatterings should re-

main subdominant effects. Furthermore, analytic study of the dynamics is useful for comparing preheating in the $K = 0$ and $K = -1$ cases.

The analytic study also clearly reveals the limits for preheating with a massive inflaton, when the expansion of the Universe is taken into account. From the form of the equations of motion in terms of the rescaled fields and conformal time, it becomes easy to understand why numerical studies of preheating in an expanding flat-FRW spacetime reveal a lack of resonant decays for massive inflatons and quanta, for many values of their couplings [7,18]. A nonzero mass for either the inflaton or the quanta would break conformal invariance. For chaotic inflation initial conditions as considered here, solutions for the zero mode of a massive inflaton would no longer be simply periodic or elliptic functions. Furthermore, if the mass of the zero mode were zero but that of the decay products nonzero, then the equation of motion for the fluctuating field would include the nonperiodic term, $a^2(\eta)m^2$. Either of these situations would mean that the ''potential'' for the fluctuating field would no longer be periodic. Yet the existence of Floquet solutions, with one or more bands of exponential instabilities, depends upon a periodic potential in the equation of motion of the fluctuating field. Only in the limit $g\varphi_0^2 \gg a^2(\eta)m^2$, where $m$ is the mass of the produced quanta and $g$ is the coupling strength between the inflaton and fluctuating field, will the potential approximate a periodic form. For the case of inflaton decay into inflaton bosons, as studied here, limits both on $\lambda$ and on $m_\varphi$ from observed cosmic microwave background anisotropies make it impossible to satisfy this limit, and *no* preheating may occur when $m_\varphi \neq 0$ for either $K = 0$ or $K = -1$, at least for chaotic inflation initial conditions.

Finally, as demonstrated here, the $K = 0$ and $K = -1$ cases agree in the limit $\psi_0 \gg |\mathcal{K}|$, which is not surprising considering the equations of motion in Eqs. (34) and (41). For this reason, it would be interesting to compare the preheating scenario for a symmetry-breaking potential with new inflation initial conditions, $\psi_0 \to 0$, for both the $K = 0$ and $K = -1$ cases. The question of a nonthermal restoration of symmetry with such a potential and initial conditions has been raised for preheating in both Minkowski and flat-FRW spacetimes [15] (though see also [9,10]). Because the preheating dynamics for such a model in an expanding open universe differ from those in both of these spatially flat cases, such a scenario warrants attention. This is the subject of further research.

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