Resonance structure for preheating with massless fields

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We extend recent work on the resonance structure for post-inflation reheating, providing an analytic treatment for models in which both the inflaton and the fields into which it decays are massless. Solutions are derived which are valid for either a spatially-flat or spatially-open metric. Closed-form solutions are given for the characteristic exponent, which measures the rate of particle production during preheating. It is demonstrated that in certain ranges of parameter space, the maximum values of the characteristic exponent in an open universe are several times greater than the maximum values in a spatially-flat universe. It is further demonstrated that the solutions found here by means of a simple algebraic construction match the two previously-known exact solutions, which were derived in terms of special functions. [S0556-2821(98)00904-7]

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I. INTRODUCTION

In this paper we extend recent work by Greene, Kofman, Linde, and Starobinsky on the resonance structure for models of postinflation reheating in which both the decaying inflaton and the fields coupled to it are massless [1]. Their work treats the case of massless fields in a spatially-flat background spacetime. Yet there is increasing evidence to suggest that we live in an open universe, and the recent models of open inflation present an interesting way to reconcile an open universe with an early phase of cosmological inflation [2]. The formalism of [1] cannot be used directly to study the preheating dynamics following a phase of open inflation. Here we extend the work of [1] to treat preheating in an open universe, and find both qualitative and quantitative changes in the spectra of produced quanta between spatially-open and spatially-flat models. In particular, when the initial value of the oscillating inflaton field is of the same order of magnitude as the associated curvature scale, there will be *no* resonant amplification of any modes—a result with no correlate in the corresponding spectra for preheating in a spatially-flat universe. Furthermore, characteristic exponents can reach larger values in an open universe than in a spatially-flat one, signalling a quicker rate of resonant particle production. Because of the conformal invariance of the massless theory, the extension provided here also allows study of preheating in Minkowski spacetime with massive fields, with or without explicit symmetry-breaking terms in the potential. With these extensions, the elegant and powerful methods introduced in [1] may be applied to a much wider class of interesting models.

The case of massless fields in an expanding spacetime admits a self-consistent analytic treatment for all three of the time-dependent quantities involved: the oscillating inflaton, the resonantly-amplified decay-product field, and the scale-factor for the background spacetime. Whereas previous authors have found closed-form solutions for such preheating systems in terms of special functions for two particular ratios of the relevant couplings [3,4], the authors of [1] produced a method for finding solutions for a far broader class of couplings by means of a much more simple, algebraic construction. In addition to extending their work to treat preheating in an open universe, we also demonstrate here that the Floquet index, which measures the rate of resonant production of decay-product quanta, can often be written in closed form, rather than in terms of the ''auxiliary functions'' found in [1], which are defined as definite integrals and evaluated numerically. Finally, we demonstrate explicitly that the solutions found by the algebraic construction of [1], with only minor modifications, match the exact results found previously in terms of special functions.

As is by now well known, linear second-order differential equations with purely periodic coefficients have solutions which obey Floquet's theorem: the solutions behave as $X(t+2\omega)=X(t)\exp(iFt)$, where 2ω is the period of the coefficients in the governing equation of motion, and F is the Floquet index (also known as the characteristic exponent). While the form of F will depend on model parameters, it will be independent of t. These solutions can always be rewritten in the form $X(t)=P(t)\exp(\mu_\kappa t)$, where $P(t)$ is a periodic function with period 2ω, and μ_κ is simply related to F (see, e.g., [5]). Obviously, when μ_κ has nonzero real parts, the solutions develop exponential instabilities. Such solutions are said to lie within ''resonance bands'' or ''instability regions,'' the dependence of which on model parameters can often be written explicitly based on an analysis of the form of μ_κ. Floquet's theorem lies at the heart of the study of preheating with massless fields, since, for early times after the inflaton has begun oscillating around the minimum of its potential, the equations of motion for any fields coupled to it assume the form of second-order differential equations with periodic coefficients. It is therefore imperative to develop efficient and accurate methods for evaluating μ_κ for a broad class of interesting models. This is the main goal of this paper.

In Sec. II, we present the models to be studied and their dynamics. We also discuss the question of appropriate vacuum states with respect to which the preheating production of quanta should be measured. Section III presents the extension of the method of [1] for the new cases, and includes explicit solutions of the equations of motion valid for early times (before back-reaction becomes significant). In Sec. IV we demonstrate how to rewrite the μ_κ parameters in closed form for many cases, which facilitates both their numerical evaluation and their comparison with known solutions. Appendix A includes an explicit demonstration of the equivalence of the solutions $X(t)$ found in Sec. III with solutions found by other means. In Appendix B it is shown that the μ_κ parameters calculated in Sec. IV likewise match the exact solutions found previously.

II. DYNAMICS OF THE MODEL

The effective Lagrangian density we will study involves two fields: an inflaton ϕ and a decay-product field χ, both of which are massless and have minimal couplings to the Ricci curvature scalar. For the field content of the model, we may then write

$$\mathcal{L} = -\sqrt{-g}\left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}(\partial_\mu\chi)^2 + \frac{\lambda}{4}\phi^4 + \frac{g^2}{2}\phi^2\chi^2\right]. \tag{1}$$

The spacetime near the beginning of the preheating epoch will assume the form of a Friedmann-Robertson-Walker (FRW) metric; for the usual models of inflation, this will be a spatially-flat metric (with $K=0$), while for the newer models of open inflation, this will be a spatially-open metric (with $K=-1$):

$$ds^2 = -dt^2 + a^2(t)h_{ij}dx^i dx^j,$$

$$h_{ij}dx^i dx^j = d\mathbf{x}^2 \quad (K=0),$$

$$= dr^2 + \sinh^2 r(d\theta^2 + \sin^2\theta d\phi^2) \quad (K=-1). \tag{2}$$

For either case, the equations of motion for the fields become

$$\ddot{\phi} + 3H\dot{\phi} + \lambda\phi^3 + g^2\chi^2\phi = 0$$

$$\ddot{\chi} + 3H\dot{\chi} - \frac{1}{a^2(t)}\mathbf{L}^2\chi + g^2\phi^2\chi = 0, \tag{3}$$

where dots denote $\partial/\partial t$, $H \equiv \dot{a}/a$, and $\mathbf{L}^2$ is the comoving spatial Laplacian operator. We have assumed that the slow-roll field ϕ is spatially homogenous; that is, we will ignore the spectrum of inflaton fluctuations for now. As demonstrated in [1], the preheating production of inflaton quanta from a massless inflaton with $\lambda\phi^4$ coupling is anomalously inefficient, as compared with the preheating production of a distinct (massless) boson field. The spectrum of inflaton fluctuations may also be studied in this formalism by associating χ with $\delta\phi$, and assigning $g^2=\lambda$ in the large N limit of the $O(N)$ approximation, or $g^2 = 3\lambda$ in the Hartree approximation. (See, e.g., [1,3].)

We may now promote the field χ to a Heisenberg operator. Details of this expansion, and of the canonical quantization of the resulting operators for both the cases $K=0$ and $K=-1$, may be found in [4]. For the $K=-1$ case, we will only track subcurvature modes here; the possibility of amplifying supercurvature modes during preheating in an open universe will be treated elsewhere [6]. The quantum χ field may then be expressed as a sum over modes and associated creation and annihilation operators, with

$$\hat{\chi}(t,r,\Omega) = \int d\tilde{\mu}[\chi_{p\ell m}(t,r,\Omega)\hat{a}_{p\ell m} + \text{H.c.}]. \tag{4}$$

Here $d\tilde{\mu}$ is the appropriate measure for the integral, including relevant sums over ℓ and m (the specific form depends on whether $K=0$ or $K=-1$, see [4]), $\hat{a}_{p\ell m}$ is a canonical time-independent annihilation operator, and "H.c." denotes a Hermitian conjugate.

In order to study the nonlinear dynamics, we will make use of a Hartree factorization. Because we will be studying the evolution of the fields before any back-reaction becomes significant, the details of the resonance structures for the model of equation (1) will in fact remain independent of the specific approximation scheme invoked to study the nonlinearities (though details concerning how long the preheating phase will last do depend on the choice of such schemes). In our case, the Hartree approximation entails replacing the $g^2\chi^2$ term in the equation of motion for ϕ with its vacuum expectation value, $g^2\langle\hat{\chi}^2\rangle$. This quantity gives a measure of the growth of the back-reaction upon the oscillating ϕ field due to the resonant shift of energy into χ modes. Because of spatial translation invariance, this quantity can only depend upon time. We will hence study the coupled system for early times, when this back-reaction term may be neglected relative to the tree-level terms.

The equations of motion become much more simple if we rewrite them in terms of conformal time, $d\eta = a^{-1}dt$, and in terms of the rescaled fields:

$$\phi(t) = \frac{1}{\sqrt{\lambda}a(\eta)}\varphi(\eta),$$

$$\chi_{p\ell m}(t,r,\Omega) = \frac{1}{a(\eta)}X_p(\eta)Y_{p\ell m}(r,\Omega),$$

$$\Sigma(\eta) = \langle X^2(\eta)\rangle. \tag{5}$$

The quantity $g^2\Sigma(\eta)$ measures the growth of the back-reaction upon $\varphi(\eta)$, and the spatial harmonics $Y_{p\ell m}$ obey

$$\mathbf{L}^2 Y_{p\ell m} = -(p^2-K)Y_{p\ell m}. \tag{6}$$

In this paper, we will only be concerned with the continuous spectrum of modes in both the $K=0$ and $K=-1$ cases, with the eigenvalue p^2 in the range $0 \leqslant p^2 < \infty$. With these definitions, the equations of motion may be rewritten

$$\left[\frac{d^2}{d\eta^2} - \frac{1}{a}\frac{d^2a}{d\eta^2} + \varphi^2 + g^2\Sigma\right]\varphi(\eta) = 0,$$

$$\left[\frac{d^2}{d\eta^2} - \frac{1}{a}\frac{d^2a}{d\eta^2} + p^2 - K + \frac{g^2}{\lambda}\varphi^2\right]X_p(\eta) = 0. \tag{7}$$

It is convenient to define the frequencies $W_p(\eta)$ and $\omega_p(\eta)$ as

$$W_p^2(\eta) \equiv p^2 - K - \frac{1}{a(\eta)}\frac{d^2a(\eta)}{d\eta^2},$$

$$\omega_p^2(\eta) \equiv W_p^2(\eta) + \frac{g^2}{\lambda}\varphi^2(\eta). \tag{8}$$

In this way, W_p is the (time-dependent) ''natural frequency'' for the X_p modes in the absence of interactions, and ω_p is the (time-dependent) ''natural frequency'' for these modes when subject to the $g^2\chi^2\phi^2$ coupling.

One further simplification may be made for the case of preheating with massless fields. As demonstrated in [4], the scale factor $a(\eta)$, averaged over a period of the inflaton field's oscillations, behaves as if the universe were radiation-dominated (even though all the fields are far from thermal equilibrium). Thus, to a good approximation, we have the relation

$$\frac{1}{a(\eta)}\frac{d^2a(\eta)}{d\eta^2}=-K \tag{9}$$

for the entire preheating epoch. (We thus ignore for now the possibility of additional resonance effects arising from the oscillation of $a(t)$; see [7].) This means that the Ricci curvature scalar vanishes during preheating:

$$R(\eta)=\frac{6}{a^2(\eta)}\left[\frac{1}{a}\frac{d^2a}{d\eta^2}+K\right]\rightarrow 0. \tag{10}$$

Models of preheating in an expanding universe with massless fields are thus conformal to models of preheating in Minkowski spacetime, even though the fields are minimally-coupled to R rather than conformally-coupled to R.

It remains to consider appropriate initial conditions for the modes $X_p(\eta)$; this is crucial to any study of preheating, since the initial conditions for X_p come from canonically quantizing the field χ on a proper Cauchy surface and choosing an appropriate vacuum state around which the quantized field may be expanded. As discussed in [4], when treating only the continuum of modes with $p^2\geqslant 0$, the proper initial conditions for $X_p(\eta)$ take the same form for both $K=0$ and $K=-1$. There are two independent concerns for choosing an appropriate vacuum state: the nonequilibrium interactions amongst the fields (relevant even for discussions of preheating in Minkowski spacetime), and the usual ambiguity regarding physical vacua in general-relativistic settings. We choose to study preheating as measured against the ''adiabatic'' vacuum: both the transition from de Sitter spacetime and the ''turning on'' of the interaction between the fields are assumed to occur adiabatically. In the absence of interactions, the transition from de Sitter spacetime to the more general FRW spacetime at the beginning of preheating would yield the following initial condition for the X_p modes at the onset of preheating (taken to be the time $\eta=\eta_0$): $X_p(\eta_0)=[2W_p(\eta_0)]^{-1/2}$ and $(dX_p/d\eta)_{\eta=\eta_0}=-i[W_p(\eta_0)/2]^{1/2}$. These modes would represent the free-particle ''adiabatic'' states [8]. If the interactions were also turned on adiabatically beginning some time before η_0, then these initial conditions would be replaced by

$$X_p(\eta_0)=\frac{1}{\sqrt{2\omega_p(\eta_0)}},\quad \left(\frac{dX_p}{d\eta}\right)_{\Big|\eta=\eta_0}=-i\sqrt{\frac{\omega_p(\eta_0)}{2}}. \tag{11}$$

These give the initial conditions for the ''adiabatic'' particle states for the nonequilibrium dynamics at the onset of preheating, at $\eta=\eta_0$. With this choice of vacuum state, the number of produced quanta per mode may similarly be written in the same form for both $K=0$ and $K=-1$ [4]:

$$N_p=\frac{\omega_p(\eta)}{2}\left[|X_p|^2+\frac{1}{\omega_p^2}\left|\frac{dX_p}{d\eta}\right|^2\right]-\frac{1}{2}. \tag{12}$$

Only when the χ field is properly quantized will this yield the number of ''adiabatic''-state quanta produced per mode relative to the initial Fock space vacuum. It is clear that when p lies in a resonance band and $X_p\sim e^{\mu_\kappa\eta}$, then $N_p\sim e^{2\mu_\kappa\eta}$, giving rise to the very efficient transfer of energy from the oscillating φ field into χ quanta.

Finally, there is the issue of renormalization. With our choice of vacuum (and hence of initial conditions for X_p), the term $\langle X^2(\eta_0)\rangle$ is formally quadratically divergent. This ultraviolet divergence may be treated in either of two ways: we may undertake a formal renormalization, as done, for example, in [9] (see also [10]), or we may use the ''physical'' criterion that the energy density contained in the vacuum fluctuations at the beginning of preheating must be less than that contained in the classical inflaton field, ϕ, as done in [4]. In practice, this amounts to setting $g^2\Sigma(\eta_0)\ll\varphi_0^2(\eta_0)$. Calculationally, the divergence will not be a problem, because preheating generically produces many quanta in low-p resonance bands, and we will only be ''following'' the evolution of these exponentially-growing unstable modes, so that in practice we will not have to evaluate a full integral over $0\leqslant p<\infty$. We may now turn to the evolution of the fields during preheating.

III. EVOLUTION OF THE FIELDS DURING PREHEATING

Using equation (9), the equation of motion for φ in equation (7) may be solved for early times, when $g^2\Sigma(\eta)$ may be neglected. Setting $\eta_0=0$, the solution is [3,4,1]

$$\varphi(\eta)=\varphi_0\mathrm{cn}(\sqrt{\varphi_0^2+K}\,\eta,\varphi_0/\sqrt{2(\varphi_0^2+K)}), \tag{13}$$

where $\mathrm{cn}(u,\nu)$ is the Jacobian cosine function, and this solution is valid for $\varphi_0^2\geqslant 2|K|$, appropriate for chaotic inflation initial conditions. Note that for nonzero K, the opposite limit, $\varphi_0\ll|K|$, will not produce any resonant preheating effects: in this case, $\varphi(\eta)$ reduces to the nonperiodic form $\varphi(\eta)\simeq\varphi_0\cosh(|K|\eta)$.

The solution in equation (13) also describes the early-time evolution of a massive inflaton for models in Minkowski spacetime: in this case, $K<0$ corresponds to preheating with an explicit symmetry-breaking potential, while $K>0$ corresponds to preheating with a massive inflaton but without symmetry breaking. If we scale the fields by the inflaton mass, then these reduce to $K=\pm 1$ [3]. For the remainder of this paper, however, we will restrict attention to preheating in an expanding universe with massless fields, and hence K will equal either 0 or -1 depending on the spatial curvature of the metric.

At this point, we could follow the procedure of [3,4] and substitute this solution for $\varphi(\eta)$ into the equation of motion for the modes $X_p(\eta)$. After much calculation, explicit forms for the modes may then be found in terms of special functions for certain values of the ratio of couplings g^2/λ. Instead, we will adopt the methods developed in [1] to produce exact (early-time) solutions for X_p by means of algebraic construction. This approach may be applied whenever the couplings satisfy $g^2/\lambda = n(n+1)/2$, with n a positive integer. With this ratio of the couplings, the equation of motion for X_p in equation (7) takes the form of a Lamé equation of order n.

The authors of [1] study the case of $K=0$. With the help of the following definitions, we may expand their method to the case of nonzero K:

$$\gamma \equiv \sqrt{\varphi_0^2 + K}, \quad u \equiv \gamma\eta, \quad z \equiv \mathrm{cn}^2(u,\nu), \tag{14}$$

where $\nu = \varphi_0/\sqrt{2(\varphi_0^2+K)} = \varphi_0/\sqrt{2\gamma^2}$. The Jacobian elliptic functions obey the relations (see, e.g., [11])

$$\mathrm{sn}^2(u,\nu) + \mathrm{cn}^2(u,\nu) = 1, \quad \mathrm{dn}^2(u,\nu) + \nu^2 \mathrm{sn}^2(u,\nu) = 1,$$

$$\frac{d}{du}\mathrm{cn}(u,\nu) = -\mathrm{sn}(u,\nu)\mathrm{dn}(u,\nu), \tag{15}$$

so that

$$\frac{dz}{du} = -2[(1-\nu^2)z + (2\nu^2-1)z^2 - \nu^2 z^3]^{1/2}. \tag{16}$$

Given the bound $\varphi_0^2 \geqslant 2|K|$, the modulus ν obeys

$$\frac{1}{2} \leqslant \nu^2 \leqslant 1. \tag{17}$$

Deviations of ν^2 from 1/2 indicate effects from the nonzero spatial curvature; the limit $\nu^2 \to 1/2$ is the same as the limit $\varphi_0^2 \gg |K|$, that is, the limit when the initial amplitude of the oscillating inflaton is much greater than the associated curvature scale.

The Jacobian cosine function $\mathrm{cn}(u,\nu)$ is periodic with period $4K(\nu)$ (as measured in "u" units of time), where $K(\nu)$ is the complete elliptic integral of the first kind. This means that $z(u)$ $[\propto \varphi^2(u)]$ has period $2K(\nu) \equiv 2\omega$, and hence that Floquet's theorem applies to the equation of motion for X_p in equation (7). In terms of z, the equation of motion for X_p becomes

$$4f(z)X_p'' + 2f'(z)X_p' + \left(\kappa^2 + 2\nu^2\frac{g^2}{\lambda}z\right)X_p = 0, \tag{18}$$

where primes denote d/dz, and we have defined

$$\kappa^2 \equiv p^2/\gamma^2. \tag{19}$$

Finally, we have defined the function

$$f(z) \equiv (1-\nu^2)z + (2\nu^2-1)z^2 - \nu^2 z^3. \tag{20}$$

With these definitions, we may study Eq. (18) for X_p for any value of the couplings $g^2/\lambda = n(n+1)/2$, following the general approach of [1].

Because the differential equation for X_p is second-order, there will exist two linearly-independent solutions, which we can label U_1 and U_2 (suppressing the index p for the moment). Consider the product $M(z) \equiv U_1(z)U_2(z)$. After straightforward algebra, and making use of Eq. (18), one can show that this function obeys the third-order differential equation:

$$2f(z)M'''(z) + 3f'(z)M''(z) + \left[f''(z) + 2\left(\kappa^2 + 2\nu^2\frac{g^2}{\lambda}z\right)\right] \times M'(z) + 2\nu^2\frac{g^2}{\lambda}M(z) = 0. \tag{21}$$

When $g^2/\lambda = n(n+1)/2$, this equation may be solved in terms of polynomials in z of order n, which we will label $M_{(n)}(z)$:

$$M_{(n)}(z) = \sum_{i=0}^{n} a_i^{(n)} z^{(n-i)}, \tag{22}$$

and we will set $a_0^{(n)} = 1$ for all n. Because $z(u)$ is periodic in u with period $2\omega = 2K(\nu)$, $M_{(n)}(z)$ will also be periodic in u. We will give some explicit examples below for $n=1$ and $n=2$. With these solutions for $M_{(n)}(z)$ in hand, one can then find solutions for $U_1(z)$ and $U_2(z)$, using the equation for X_p and the Wronskian relation between U_1 and U_2.

In general, for a differential equation of the form

$$\frac{d}{dx}\left[A(x)\frac{dy}{dx}\right] + B(x)y = 0, \tag{23}$$

the Wronskian for the two linearly-independent solutions will be proportional to $1/A(x)$. In our case, we have

$$\frac{d}{dz}\left[\sqrt{f(z)}\frac{dX_p}{dz}\right] + B(z)X_p = 0, \tag{24}$$

and hence

$$U_1'U_2 - U_1U_2' = \frac{C_{(n)}}{\sqrt{f(z)}}, \tag{25}$$

where the constant $C_{(n)}$ will be a function of κ and ν and will depend on the order n. Using this Wronskian and the fact that $U_1(z)U_2(z) = M(z)$, we may write solutions for the two mode functions:

$$U_{1,2}(z) = N\sqrt{|M_{(n)}(z)|}\exp\left(\pm\frac{C_{(n)}}{2}\int\frac{dz}{\sqrt{f(z)}M_{(n)}(z)}\right). \tag{26}$$

We have inserted a normalization factor N, which is set to unity in [1]. Instead, we will set

$$N = |M_{(n)}(z=1)|^{-1/2}, \tag{27}$$

since $z=1$ at $u=0$. With this normalization, when the solutions are written as functions of u, they satisfy $U_{1,2}(u=0) = 1$; and, as demonstrated in Appendix A for the case $n=1$, they match the exact solutions found previously, by very different means.

With $U_{1,2}(u=0)=1$, the normalized mode solutions $U_{1,2}$ do not obey the proper initial conditions at $u=\eta=0$, Eq. (11). We must instead take linear combinations of these solutions [3,4]:

$$X_p(u)=\frac{1}{2\sqrt{2\omega_p(0)}}\left[\left(1+i\frac{\omega_p(0)}{\gamma U_1'(u=0)}\right)U_1(u)+\left(1-i\frac{\omega_p(0)}{\gamma U_1'(u=0)}\right)U_2(u)\right], \tag{28}$$

where primes here denote d/du. Unless such linear combinations of $U_{1,2}$ are taken, the number operator N_p in Eq. (12) will *not* measure particle production with respect to the physically-relevant initial vacuum state. Numerically, however, if one makes the approximation of only following the exponentially-growing unstable modes, then N_p will only shift by a constant as one shifts the normalization of $U_{1,2}$ and the linear combination for X_p.

The solutions $U_{1,2}$ will be exponentially unstable whenever the exponent in equation (26) has a nonzero real part; we will use this to determine the bounds on the resonance bands in terms of κ and ν. Furthermore, given the periodicity of $\varphi(u)$ and the quasiperiodicity of $U_{1,2}(u)$, the authors of [1] demonstrate that the characteristic exponent may be written as a definite integral:

$$\mu_\kappa=-\frac{C_{(n)}}{2\omega}\int_0^1\frac{dz}{\sqrt{f(z)}M_{(n)}(z)}. \tag{29}$$

We will present means for further evaluating μ_κ given this expression in Sec. IV; in particular, we will provide a transparent demonstration that the definite integral in equation (29) is always purely real, so that all resonance structure comes from the behavior of $C_{(n)}$. We will demonstrate explicitly in Appendix B that this form for μ_κ does indeed match the exact solutions found earlier in [3,4] for the two specific cases already studied. We will show in Appendix A that the solutions in equation (26) obey Floquet's theorem, and can be written as $U_{1,2}(u)=P(\pm u)\exp(\mp\mu_\kappa u)$, with $P(u+2\omega)=P(u)$.

By plugging the solutions for $U_{1,2}$ in equation (26) back into the equation of motion for X_p, one finds [dropping the subscript (n) for the moment]

$$C^2=M'^2f-2M''Mf-M'Mf'-\kappa^2M^2-\nu^2n(n+1)M^2z. \tag{30}$$

It is straightforward to confirm that all of the z dependence in this expression cancels exactly for any order n, leaving the simpler relation:

$$C_{(n)}^2=-\kappa^2(a_n^{(n)})^2-a_{n-1}^{(n)}a_n^{(n)}(1-\nu^2). \tag{31}$$

Note that equations (18), (20), (21), (25), and (26) all reduce to the forms in [1] when $K=0$ $(\nu^2=1/2)$.

For $g^2=\lambda$ (that is, $n=1$), equation (21) may be solved, yielding

$$M_{(1)}(z)=z-\frac{\kappa^2}{\nu^2}+\frac{1}{\nu^2}-2,$$

$$\nu^4C_{(1)}^2=-\kappa^6+2\kappa^4(1-2\nu^2)-\kappa^2(1-5\nu^2+5\nu^4)-\nu^2(1-3\nu^2+2\nu^4). \tag{32}$$

When $K=0$ $(\nu^2=1/2)$, these become $M_{(1)}(z)\to z-2\kappa^2$ and $C_{(1)}^2\to\kappa^2(1-4\kappa^4)$. Thus, in this limit, there will exist one single resonance band for positive κ^2, determined by when $C_{(1)}^2>0$, and given by

$$0\leqslant\kappa^2\leqslant\frac{1}{2}. \tag{33}$$

This matches the resonance structure found in both [3,1]. It is interesting that in the opposite limit, with nonzero K and $\varphi_0^2\sim\mathcal{O}(|K|)$ $(\nu^2\to1)$, $C_{(1)}^2\to-\kappa^6-2\kappa^4-\kappa^2$, and there will not exist *any* resonance bands for $\kappa^2\geqslant0$. In fact, in an open universe, there will exist a minimum value of φ_0 for any given ratio g^2/λ below which no modes with $\kappa^2\geqslant0$ will be resonantly amplified. This issue will be treated further in [6].

For the next-simplest case, $g^2=3\lambda$ $(n=2)$, it will be easier to write the solutions in terms of the coefficients of the expansion in equation (22) (recalling that $a_0^{(2)}=1$):

$$M_{(2)}(z)=z^2+a_1^{(2)}z+a_2^{(2)},$$

$$a_1^{(2)}=-\frac{1}{3\nu^2}[\kappa^2-4(1-2\nu^2)],$$

$$a_2^{(2)}=\frac{1}{9\nu^4}[\kappa^4-5\kappa^2(1-2\nu^2)+4-25\nu^2+25\nu^4],$$

$$C_{(2)}^2=-\kappa^2(a_2^{(2)})^2-a_1^{(2)}a_2^{(2)}(1-\nu^2). \tag{34}$$

When $K=0$, this reduces to

$$C_{(2)}^2\to\frac{16}{81}\kappa^2\left(\kappa^4-\frac{9}{4}\right)(3-\kappa^4), \tag{35}$$

revealing the presence of a single resonance band for positive κ^2:

$$\frac{3}{2}\leqslant\kappa^2\leqslant\sqrt{3}, \tag{36}$$

matching the resonance structure found in [4,1]. Note that in the opposite limit, $\nu^2\to1$,

$$C_{(2)}^2\to-\frac{1}{81}\kappa^2(\kappa^4+4)^2, \tag{37}$$

again revealing *no* resonance bands for $\kappa^2\geqslant0$.

We now turn to the treatment of the characteristic exponents, μ_κ.

IV. EVALUATING THE CHARACTERISTIC EXPONENTS

As noted above, the authors of [1] found a form for the characteristic exponent, μ_κ:

$$\mu_\kappa = -\frac{C_{(n)}}{2\omega}\int_0^1 \frac{dz}{\sqrt{f(z)}M_{(n)}(z)}. \tag{38}$$

In this section, we will rewrite the definite integral in a way which demonstrates explicitly that the integral is real for all n and ν, so that the resonance bands are determined entirely by where the constant $C_{(n)}$ has nonzero real parts. In addition, the integral may be evaluated in closed form in many cases. Such closed-form solutions for μ_κ may then be evaluated without numerical integration, and will facilitate comparison with known solutions, as discussed in the appendices.

With the substitution

$$z = 1 - \sin^2\theta, \tag{39}$$

the indefinite integral in equation (26) becomes

$$-\int \frac{dz}{\sqrt{f(z)}M_{(n)}(z)} = 2\int \frac{d\theta}{[1-\nu^2\sin^2\theta]^{1/2}M_{(n)}(\sin^2\theta)}. \tag{40}$$

Thus, given $1/2 \leqslant \nu^2 \leqslant 1$, both the integral over dz and the integral over $d\theta$ will be manifestly real for all n and ν.

The form of the integral over $d\theta$ in equation (40) further encourages comparison with the incomplete elliptic integral of the third kind (see, e.g., [12]):

$$\Pi(n;\varphi\backslash\alpha) = \int^{\varphi} d\theta[1-n\sin^2\theta]^{-1}[1-\sin^2\alpha\sin^2\theta]^{-1/2}. \tag{41}$$

Whenever the function $M_{(n)}(\sin^2\theta)$ may be factored into n real roots, the integral over dz may be written explicitly as a sum over n distinct Π functions. In the cases when $M_{(n)}(\sin^2\theta)$ cannot be so factored, then the (principal value of the) integral over $d\theta$ in equation (40) will still be well defined over the limits of integration $(z\!:\!0,1)\rightarrow(\theta\!:\!\pi/2,0)$.

Consider the case when $M_{(n)}(\sin^2\theta)$ can be factored into n real roots; writing $x \equiv \sin^2\theta$, we have

$$\frac{1}{M_{(n)}(x)} = \frac{1}{\Pi_{i=1}^{n}(\beta_i - x)} = \sum_{i=1}^{n}\frac{D_i}{(1-\beta_i^{-1}x)}, \tag{42}$$

where the n constant coefficients D_i may be determined by the set of n equations:

$$\sum_{i=1}^{n} D_i\left(\prod_{j=1}^{n}\beta_j\right) = 1,$$

$$D_1(\beta_2^{-1}+\beta_3^{-1}+\cdots+\beta_n^{-1}) + \text{c.p.} = 0,$$

$$D_1(\beta_2^{-1}\beta_3^{-1}+\beta_2^{-1}\beta_4^{-1}+\cdots+\beta_{n-1}^{-1}\beta_n^{-1}) + \text{c.p.} = 0,$$

$$\cdots$$

$$D_1(\beta_2^{-1}\beta_3^{-1}\times\cdots\times\beta_n^{-1}) + \text{c.p.} = 0, \tag{43}$$

where c.p. denotes all cyclic permutations. Note that when the constant portion of $M_{(n)}(\sin^2\theta)$ (that is, $a_n^{(n)}$) is nonzero, all of the roots β_i will be nonzero, and hence their inverses β_i^{-1} will always be well defined. In general, both the constants D_i and β_i will be functions of n, κ^2, and ν. We will give explicit examples of the factorization of equation (42) for $n=1$ and $n=2$ below.

In the cases when $M_{(n)}(\sin^2\theta)$ admits such a factorization, then we may write the indefinite integral in the exponent of $U_{1,2}$ as

$$-\int \frac{dz}{\sqrt{f(z)}M_{(n)}(z)} = 2\sum_{i=1}^{n} D_i\Pi(\beta_i^{-1};\arcsin\sqrt{1-z}\backslash\arcsin\nu). \tag{44}$$

Using the fact that $\Pi(n;0\backslash\alpha)=0$ and $\Pi(n;\pi/2\backslash\alpha) = \Pi(n\backslash\alpha)$ (the complete elliptic integral of the third kind), we may further evaluate the characteristic exponent, μ_κ:

$$\mu_\kappa = -\frac{C_{(n)}}{2\omega}\int_0^1\frac{dz}{\sqrt{f(z)}M_{(n)}(z)} = -\frac{C_{(n)}}{\omega}\sum_{i=1}^{n} D_i\Pi(\beta_i^{-1}\backslash\arcsin\nu). \tag{45}$$

For cases in which a given $\beta_j^{-1}>1$, one may always rewrite $\Pi(\beta_j^{-1}\backslash\arcsin\nu)$ in terms of a sum of complete elliptic integrals, each of which is purely real (see [12]). This closed-form solution for μ_κ, valid whenever $M_{(n)}(\sin^2\theta)$ has the n nonzero, real roots β_i, may thus be evaluated in terms of well-known functions, without any need for numerical integration.

Consider the simplest case, $n=1$. In this case, $M_{(1)}(\sin^2\theta)$ may always be factored as needed:

$$M_{(1)}(\sin^2\theta) = -\sin^2\theta - \frac{\kappa^2}{\nu^2} + \frac{1}{\nu^2}(1-\nu^2), \tag{46}$$

or,

$$\beta_1^{-1} = D_1 = \frac{\nu^2}{1-\nu^2-\kappa^2}. \tag{47}$$

This yields

$$\mu_\kappa = -\frac{C_{(1)}}{\omega}\left(\frac{\nu^2}{1-\nu^2-\kappa^2}\right)\Pi\left(\frac{\nu^2}{1-\nu^2-\kappa^2}\backslash \arcsin\nu\right). \tag{48}$$

Equation (48) reveals that preheating in an open universe can be more efficient than in a spatially-flat one. The maximum for μ_κ when $K=0$ ($\nu^2=1/2$) is $\mu_\kappa=0.147$ at $\kappa^2=0.228$ (matching the result found in [1]). Yet μ_κ grows in the range $1/2<\nu^2<0.76$, reaching a maximum of $\mu_\kappa=0.224$ at $\kappa^2=0$ and $\nu^2=0.76$. This quantitative difference in the spectra will be treated further in [6]. We will demonstrate below in Appendix B that the analytic solution for μ_κ in equation (48) matches the exact solution found previously for $n=1$, with $\nu^2=1/2$.

For $n=2$, we have (again using $x\equiv\sin^2\theta$)

$$M_{(2)}(x)=x^2-x(2+a_1^{(2)})+(1+a_1^{(2)}+a_2^{(2)}), \tag{49}$$

or [dropping the superscript (2)]

$$\beta_{1,2}=\frac{1}{2}[2+a_1\mp\sqrt{(a_1)^2-4a_2}]$$

$$D_{1,2}=\frac{1}{2}\frac{1}{(1+a_1+a_2)\sqrt{(a_1)^2-4a_2}}\times[\pm 2\pm a_1+\sqrt{(a_1)^2-4a_2}]. \tag{50}$$

Note that when $(a_1)^2<4a_2$, the definite integral in μ_κ will still be purely real; it simply will not be expressible as a sum of complete elliptic integrals. In the limit $\nu^2\to 1/2$, however, the condition $(a_1)^2\geqslant 4a_2$ is equivalent to $\kappa^4\leqslant 3$ [see equation (34)]. And in this limit, the single resonance band extends only to $\kappa^4\leqslant 3$, so for the entire resonance band, the roots and coefficients in equation (50) will all be purely real.

As with the $n=1$ case, preheating in an open universe with $n=2$ can be more efficient than in a flat one. In this case, the enhancement of the resonance can be far more dramatic: whereas μ_κ reaches a maximum value when $K=0$ ($\nu^2=1/2$) of $\mu_\kappa=0.036$ (at $\kappa^2=1.615$), μ_κ rises sharply over the range $1/2<\nu^2\leqslant 0.90$. At $\nu^2=0.72$, μ_κ reaches a maximum of 0.097 (at $\kappa^2=0.64$), while at $\nu^2=0.90$, μ_κ reaches a maximum of 0.193 (at $\kappa^2=0$). This yields a characteristic exponent over five times greater than the maximum reached in a spatially-flat universe. We will demonstrate analytically in Appendix B that for $n=2$ and in the limit $\nu^2\to 1/2$, the closed-form solution obtained from equations (45) and (50) again matches the exact solution.

V. CONCLUSIONS

The methods developed in [1] for the analytic study of the resonance structure of preheating with massless fields are powerful and highly efficient. We have extended their work here to cover both models with nonzero spatial curvature (relevant to studies of open inflation), and models in Minkowski spacetime with or without explicit symmetry breaking. By considering a physically well-motivated choice of initial vacuum state, we have also fixed the overall normalization of the mode functions, which provides a simple form for the number of χ quanta produced per mode during preheating. With this choice of normalization, the solutions for the mode functions found here by simple algebraic construction match the solutions found previously in terms of special functions (see Appendix A).

Beyond the relative ease with which solutions for the mode functions may be found using the methods of [1], there is an added benefit of this approach over that taken in, e.g., [3,4]: the characteristic exponent, μ_κ, which determines the rate at which particles are resonantly produced during preheating (with $\ln N_p\simeq 2\mu_\kappa u$), may be evaluated independently of finding the full solutions for the mode functions, $U_{1,2}(z)$. We have shown here that in many cases of interest, these μ_κ parameters may in fact be solved for exactly in closed form, in terms of well-known functions. Written in this form, the characteristic exponents, calculated directly from equation (45), match the solutions found previously (see Appendix B).

Of course, this method, though simple and efficient, is still limited to the cases in which the ratio of the couplings satisfies $g^2/\lambda=n(n+1)/2$, with n a positive integer. The authors of [1] show, by means of numerical integration for arbitrary positive g^2/λ, that μ_κ is not a monotonic function of the couplings; for slightly different values of g^2/λ, the resonance can be much stronger than it is with some of the integer-valued ratios studied here. Still, it is useful in general to have analytic solutions in hand, especially if one wants to compare different models, such as $K=0$ versus $K=-1$, or, for Minkowski spacetime, $m_\phi^2>0$ versus $m_\phi^2<0$. Such analytic comparisons have revealed here, for example, that when $K=-1$ and the initial amplitude of the inflaton's oscillations falls below a minimum value, *no* modes with $\kappa^2\geqslant 0$ will be resonantly amplified, even though there is no associated threshold in the $K=0$ case. Moreover, when $K=-1$, certain regions of parameter space yield rates of resonant particle production over five times greater than the corresponding rates in a $K=0$ universe. These topics will be treated further in [6].

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APPENDIX A

In this appendix, we demonstrate that the solutions $U_{1,2}[z(u)]$ in equation (26), with the normalization of equation (27), match the exact solutions found in terms of special functions. We will examine the simplest case, $n=1$, which was studied in [3]. In terms of the parametrization of equation (14), and setting $\nu^2=1/2$, we may follow the same steps as in [3] to arrive at the solution

$$U_\kappa(u)=\frac{\vartheta_1\left(\frac{u}{2\omega}+v\right)}{\vartheta_1(v)}\frac{\vartheta_4(0)}{\vartheta_4\left(\frac{u}{2\omega}\right)}\exp[-uZ(2\omega v)], \tag{A1}$$

and $U_{1,2}(u)=U_\kappa(\pm u)$. Here $\vartheta_i(x)$ are the Jacobian theta functions, $\omega=K(\nu)$ is the complete elliptic integral of the first kind, and $Z(x)$ is the Jacobian zeta function (see, e.g., [11,12]). The quantity v is related to κ^2 by [3]

$$\wp(2\omega v+\omega')=-\kappa^2, \tag{A2}$$

where $\wp(x)$ is the doubly-periodic Weierstrass function. We have also used $\omega'\equiv iK'(\nu)$, where $K'(\nu)=K(\nu')$, the complete elliptic integral for the complementary modulus, $\nu'\equiv(1-\nu^2)^{1/2}$ [11,12]. We may rewrite the v dependence in terms of Jacobian elliptic functions, using

$$\mathrm{sn}^2(u,\nu)=\frac{1}{\nu^2\mathrm{sn}^2(u+\omega',\nu)}=2\wp(u+\omega')+1, \tag{A3}$$

where the last expression holds for $\nu^2=1/2$. Making these substitutions, Eq. (A2) may be rewritten:

$$1-2\kappa^2=\mathrm{sn}^2(2\omega v,\nu). \tag{A4}$$

Inside the resonance band, with $0\leqslant\kappa^2\leqslant1/2$, we thus have $0\leqslant v\leqslant1/2$.

In order to demonstrate the equivalence between the solutions in equations (26) and (A1), we will first examine the term outside of the exponent in equation (26):

$$\sqrt{|M_{(1)}(z)|}=\sqrt{z-2\kappa^2}=[\mathrm{sn}^2(2\omega v,\nu)-\mathrm{sn}^2(u,\nu)]^{1/2}. \tag{A5}$$

Again using equation (A3), the following relations between the theta functions [11]

$$\vartheta_1\left(x+\frac{1}{2}\tau\right)=i\exp\left[-i\pi\left(x+\frac{1}{4}\tau\right)\right]\vartheta_4(x),$$

$$\vartheta_4\left(x+\frac{1}{2}\tau\right)=i\exp\left[-i\pi\left(x+\frac{1}{4}\tau\right)\right]\vartheta_1(x) \tag{A6}$$

(where $\tau\equiv\omega'/\omega$), and the relation (for $\nu^2=1/2$)

$$\wp(u)=-\frac{1}{2}+\frac{1}{4\omega^2}\left[\frac{\vartheta_1'(0)}{\vartheta_4(0)}\frac{\vartheta_4\left(\frac{u}{2\omega}\right)}{\vartheta_1\left(\frac{u}{2\omega}\right)}\right]^2, \tag{A7}$$

we may rewrite equation (A5) in terms of theta functions:

$$\sqrt{|M_{(1)}(z)|}=\frac{1}{\sqrt{2}\omega}\frac{\vartheta_1'(0)}{\vartheta_4(0)}\frac{1}{\vartheta_4(v)\vartheta_4\left(\frac{u}{2\omega}\right)}\left[\vartheta_1^2(v)\vartheta_4^2\left(\frac{u}{2\omega}\right)-\vartheta_1^2\left(\frac{u}{2\omega}\right)\vartheta_4^2(v)\right]^{1/2}. \tag{A8}$$

The normalization of equation (27) may be similarly reexpressed:

$$N=\frac{1}{\sqrt{1-2\kappa^2}}=\frac{1}{\mathrm{sn}(2\omega v,\nu)}. \tag{A9}$$

Finally, using the relation between the squares of the theta functions [13],

$$\vartheta_1^2(y)\vartheta_4^2(z)-\vartheta_1^2(z)\vartheta_4^2(y)=\vartheta_1(y+z)\vartheta_1(y-z)\vartheta_4^2(0), \tag{A10}$$

we find

$$N\sqrt{|M_{(1)}(z)|}=\frac{\vartheta_4(0)}{\vartheta_1(v)\vartheta_4\left(\frac{u}{2\omega}\right)}\left[\vartheta_1\left(v+\frac{u}{2\omega}\right)\times\vartheta_1\left(v-\frac{u}{2\omega}\right)\right]^{1/2}. \tag{A11}$$

Now we may turn to the exponent in equation (26).

Using equations (44), (47), (A3), and (A4), with $\nu^2=1/2$, we may write the exponent of equation (26) as

$$\frac{C_{(1)}}{2}\int\frac{dz}{\sqrt{f(z)}M_{(1)}(z)}=-C_{(1)}D_1\Pi\left(\frac{1}{2}\mathrm{sn}^2(2\omega v+\omega');\arcsin\sqrt{1-z}\backslash\arcsin(\nu)\right). \tag{A12}$$

Because $0<\frac{1}{2}\mathrm{sn}^2(2\omega v+\omega')\leqslant\nu^2$, we may rewrite the incomplete elliptic integral as follows [12]:

$$\Pi(n;\varphi\backslash\alpha)=\delta_1\left[-\frac{1}{2}\ln\left(\frac{\vartheta_4(y+\beta)}{\vartheta_4(y-\beta)}\right)+y\frac{\vartheta_1'(\beta)}{\vartheta_1(\beta)}\right], \tag{A13}$$

with

$$\epsilon=\arcsin(n/\sin^2\alpha)^{1/2},\quad \beta=F(\epsilon\backslash\alpha)/2\omega,$$

$$y=F(\varphi\backslash\alpha)/2\omega,$$

$$\delta_1=[n(1-n)^{-1}(\sin^2\alpha-n)^{-1}]^{1/2}, \tag{A14}$$

where $F(\varphi\backslash\alpha)$ is the incomplete elliptic integral of the first kind. Here we have used the ''π'' conventions for the arguments of the theta functions of [11]. After some straightforward algebra, equation (A12) may then be rewritten:

$$\frac{C_{(1)}}{2}\int\frac{dz}{\sqrt{f(z)}M_{(1)}(z)}=\frac{1}{2}\ln\left(\frac{\vartheta_1\left(v+\frac{u}{2\omega}\right)}{\vartheta_1\left(v-\frac{u}{2\omega}\right)}\right)-\frac{u}{2\omega}\frac{\vartheta_4'(v)}{\vartheta_4(v)}, \tag{A15}$$

after use has been made of equation (A6) and the fact that $\vartheta_4(-x)=\vartheta_4(x)$. Finally, using [11]

$$\frac{\vartheta_4'(x)}{\vartheta_4(x)}=2\omega Z(2\omega x), \tag{A16}$$

and combining equations (A11) and (A15), we find that the solution for $U_1[z(u)]$ found in Sec. III, equation (26), may be rewritten as

$$U_1(u)=\frac{\vartheta_1\left(\frac{u}{2\omega}+v\right)}{\vartheta_1(v)}\frac{\vartheta_4(0)}{\vartheta_4\left(\frac{u}{2\omega}\right)}\exp[-uZ(2\omega v)]. \tag{A17}$$

Given the periodicity of the theta functions, this is now in the form $U_1(u)=P(u)\exp[-\mu_\kappa u]$, with $P(u+2\omega)=P(u)$. The second independent solution can then be written $U_2(u)=P(-u)\exp[\mu_\kappa u]$, and represents the growing solution.

This completes our demonstration that the solutions (for $n=1$) found here by means of very simple algebraic construction, following the method of [1], match the exact solutions found earlier by much more difficult means [3]. The comparison for $n=2$ between the solutions found in Sec. III and those found earlier in [4] follows very similarly.

APPENDIX B

Whereas in Appendix A we demonstrated that the solutions $U_{1,2}(z)$ for the mode functions match the earlier known solutions, in this appendix we will demonstrate that the crucial quantities, μ_κ, as calculated directly from equation (45) also agree with earlier known solutions. The difference between these two demonstrations lies in the fact that the solutions $U_{1,2}(z)$ are defined in terms of an *indefinite* integral over z, as in equation (26); the characteristic exponents μ_κ, however, are defined as *definite* integrals. As noted in the conclusions, one benefit (among many) of following the methods of [1] is that the μ_κ parameters may be evaluated independently of the mode functions $U_{1,2}(z)$, which is not possible if proceeding as in [3,4].

Consider the case $n=1$ with $\nu^2\to 1/2$. Equation (48) then becomes

$$\mu_\kappa=-\frac{1}{\omega}\sqrt{\kappa^2\left(\frac{1+2\kappa^2}{1-2\kappa^2}\right)}\,\Pi\left(\frac{1}{1-2\kappa^2}\backslash\arcsin(\nu)\right). \tag{B1}$$

Written in this form, $\beta_1^{-1}>1$, so we may rewrite the elliptic integral as follows [12]:

$$\Pi(\beta_1^{-1}\backslash\alpha)=\omega-\Pi(N\backslash\alpha)=-\delta_1\omega Z(\epsilon\backslash\alpha), \tag{B2}$$

with $N=\beta_1\sin^2\alpha$, and δ_1 and ϵ defined as in equation (A14), now in terms of N instead of n. Then, using equation (A4), we have

$$\epsilon=\arcsin\sqrt{1-2\kappa^2}=\arcsin(\mathrm{sn}(2\omega v)). \tag{B3}$$

Noting that when $\varphi=\arcsin(\mathrm{sn}(x,\nu))$ [12],

$$Z(\varphi\backslash\arcsin(\nu))=Z(x|\nu^2)=Z(x), \tag{B4}$$

we find that

$$\mu_\kappa=Z(2\omega v) \tag{B5}$$

for $n=1$ and $\nu^2\to 1/2$. This matches the result found in [3].

For $n=2$ and $\nu^2\to 1/2$, we may compare with the solution found in [4]:

$$\mu_\kappa=\frac{1}{2\omega}\left(\frac{\vartheta_1'\left(\frac{a}{2\omega}\right)}{\vartheta_1\left(\frac{a}{2\omega}\right)}+\frac{\vartheta_1'\left(\frac{b}{2\omega}\right)}{\vartheta_1\left(\frac{b}{2\omega}\right)}\right), \tag{B6}$$

where the constants a and b are defined by the relations

$$\wp(a)=-\frac{1}{6}\kappa^2-\frac{1}{2}\sqrt{1-\frac{1}{3}\kappa^4},$$
$$\wp(b)=-\frac{1}{6}\kappa^2+\frac{1}{2}\sqrt{1-\frac{1}{3}\kappa^4}. \tag{B7}$$

Writing these in terms of Jacobian elliptic functions, using the identities employed in Appendix A, we have

$$\mathrm{sn}^2(a-\omega')=1-\frac{1}{3}\kappa^2-\sqrt{1-\frac{1}{3}\kappa^4},$$
$$\mathrm{sn}^2(b-\omega')=1-\frac{1}{3}\kappa^2+\sqrt{1-\frac{1}{3}\kappa^4}. \tag{B8}$$

Examination of equations (34) and (50) in the limit $\nu^2\to 1/2$ reveals that

$$\beta_1^{-1}=\frac{1}{\mathrm{sn}^2(a-\omega')}=\frac{1}{2}\mathrm{sn}^2(a),$$
$$\beta_2^{-1}=\frac{1}{\mathrm{sn}^2(b-\omega')}=\frac{1}{2}\mathrm{sn}^2(b). \tag{B9}$$

Then

$$\mu_\kappa=\sqrt{\frac{3}{4\kappa^2}\left(\frac{\kappa^2+\frac{3}{2}}{\kappa^2-\frac{3}{2}}\right)}\Bigg[\left(-1+\frac{1}{3}\kappa^2-\sqrt{1-\frac{1}{3}\kappa^4}\right)\times\Pi\left(\frac{1}{2}\mathrm{sn}^2(a)\backslash\arcsin(\nu)\right)+\left(1-\frac{1}{3}\kappa^2-\sqrt{1-\frac{1}{3}\kappa^4}\right)\times\Pi\left(\frac{1}{2}\mathrm{sn}^2(b)\backslash\arcsin(\nu)\right)\Bigg]. \tag{B10}$$

Since these β_i^{-1} each satisfy $0\leqslant\beta_i^{-1}\leqslant 1/2$, we may again use equation (A13) for each of the two complete elliptic integrals. In each case, the $\ln(\vartheta_4(y+\beta)/\vartheta_4(y-\beta))$ term vanishes, leaving, after much algebra,

$$\mu_\kappa=\frac{1}{2\omega}\left(\frac{\vartheta_1'\left(\frac{a}{2\omega}\right)}{\vartheta_1\left(\frac{a}{2\omega}\right)}+\frac{\vartheta_1'\left(\frac{b}{2\omega}\right)}{\vartheta_1\left(\frac{b}{2\omega}\right)}\right). \tag{B11}$$

Thus, the closed-form solution for μ_κ found in Sec. IV, equation (45), matches the solution found previously in [4] for $n=2$.

[1] P. B. Greene, L. Kofman, A. Linde, and A. A. Starobinsky, Phys. Rev. D **56**, 6175 (1997).
[2] J. Gott, Nature (London) **295**, 304 (1982); J. Gott and T. Statler, Phys. Lett. **136B**, 157 (1984); M. Bucher, A. Goldhaber, and N. Turok, Phys. Rev. D **52**, 3314 (1995); A. Linde, Phys. Lett. B **351**, 99 (1995); A. Linde and A. Mezhlumian, Phys. Rev. D **52**, 6789 (1995); K. Yamamoto, M. Sasaki, and T. Tanaka, Astrophys. J. **455**, 412 (1995).
[3] D. Boyanovsky, H. J. de Vega, R. Holman, and J. F. J. Salgado, Phys. Rev. D **54**, 7570 (1996).
[4] D. I. Kaiser, Phys. Rev. D **56**, 706 (1997).
[5] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).
[6] D. I. Kaiser (in preparation).
[7] B. A. Bassett and S. Liberati, ''Geometric reheating after inflation,'' Report No. hep-ph/9709417.
[8] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, New York, 1982).
[9] D. Boyanovsky, D. Cormier, H. J. de Vega, R. Holman, A. Singh, and M. Srednicki, Phys. Rev. D **56**, 1939 (1997).
[10] J. Baacke, K. Heitmann, and C. Pätzold, Phys. Rev. D **56**, 6556 (1997).
[11] *Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdelyi *et al.* (McGraw-Hill, New York, 1953), Vol. 2.
[12] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
[13] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1927).