PHYSICAL REVIEW D VOLUME 52, NUMBER 8 15 OCTOBER 1995

Primordial spectral indices from generalized Einstein theories

David I. Kaiser*
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 17 August 1994; revised manuscript received 3 May 1995)

Primordial spectral indices are calculated to second order in slow-roll parameters for three closely related models of inflation, all of which contain a scalar field nonminimally coupled to the Ricci curvature scalar. In most cases, n_s may be written as a function of the nonminimal curvature coupling strength ξ alone, with $n_s(\xi) \leq 1$, although the constraints on ξ differ greatly between "new inflation" and "chaotic inflation" initial conditions. Under "new inflation" initial conditions, there are discrepancies between the values of n_s as calculated in the Einstein frame and the Jordan frame. The sources for these discrepancies are addressed, and shown to have negligible effects on the numerical predictions for n_s. No such discrepancies affect the calculations under "chaotic inflation" initial conditions.

PACS number(s): 98.80.Cq, 04.50.+h

I. INTRODUCTION

In many models of the very early Universe, the canonical Einstein-Hilbert gravitational action emerges only as a low-energy effective theory, rather than being assumed from the start [1]. A large class of these generalized Einstein theories (GET's) involves scalar fields nonminimally coupled to the Ricci curvature scalar. Such Brans-Dicke-like couplings [2] arise, for example, in models of superstring compactification [3] and Kaluza-Klein theories [4], and are related, via conformal transformation, to quantum-gravitational counterterms, which are proportional to the square of the Ricci scalar [5, 6].

Recent experimental determinations of the power spectrum of density perturbations [7], modeled as $\mathcal{P} \propto k^{n_s}$ [8], offer a rare glimpse of such Planck-scale physics. The spectral index for this scalar perturbation, n_s, functions as a test for models of the very early Universe, independently of the familiar test based on the magnitude of perturbations. It has been shown, for example, that one well-known GET model of inflation, extended inflation [9], cannot produce the observed nearly scale-invariant (Harrison-Zel'dovich) spectrum: extended inflation predicts $n_s \leq 0.76$, instead of $n_s = 1.00$ [10]. The constraints on n_s for extended inflation come from that model's incorporation of a first-order phase transition to exit inflation (see [11] for more on this so-called "ω problem"). As discussed in [12], this pitfall can be avoided in GET models of inflation which undergo a second-order phase transition to exit the inflationary phase. In this paper, three cousin models of extended inflation are considered, all of which fare much better in comparisons with the observed values of n_s.

The analysis is carried out to second order in the potential-slow-roll approximation (PSRA) parameters identified by Liddle, Parsons, and Barrow [13], who have recently amended earlier work by several authors [14, 15]. These papers are based on the Hamilton-Jacobi equations of motion for a theory with a scalar field minimally coupled to the curvature scalar; before they can be applied to the nonminimally coupled GETs considered here, use must be made of a conformal transformation [5, 16], which, via field redefinitions, puts the GET equations of motion into the "Einstein frame" form of an Einstein-Hilbert gravitational action with a minimally coupled scalar field.

In this connection, it is important to keep the cautionary note of Fakir and Habib in mind. In [17] they have demonstrated that ambiguities arise when studying the quantum fluctuations of scalar fields in GET's in various frames: the scalar two-point correlation function evaluated in the "Jordan" or "physical" frame, in which the nonminimal $\phi^2 R$ coupling is explicit, differs from the two-point correlation function evaluated after the field redefinitions, in the Einstein frame. Yet, as discussed below, when the inflationary expansion is quasi–de Sitter–type, $a(t) \propto \exp(Ht)$, with $\dot{H} \simeq 0$, the ambiguities isolated in [17] affect the *magnitude* of the correlation function only, and not the k dependence (and hence not n_s; see Eq. (58) in [17]). All three of the models considered below display such quasi–de Sitter expansion under "chaotic inflation" initial conditions, and thus the Einstein frame formalism employed here for n_s should remain unproblematic.

However, under "new inflation" initial conditions, two of the models evolve as a quasi power law, $a(t) \propto t^p$. In these cases, ambiguities similar to those discussed in [17] *do* affect the form of n_s. As discussed below, in Sec. III B, the discrepancy between values of n_s as calculated in the Jordan and Einstein frames arises because the curvature perturbation upon which the PSRA formalism is based, $\mathcal{R} = (H/\dot{\phi})\,\delta\phi$ [8, 14], is not invariant with respect to the conformal transformation. (This discrepancy can be resolved by choosing a suitable generalization of $\mathcal{R}$; see [18].) Still, it can be shown that even in these cases of quasi-power-law expansion, the numerical results for n_s in the Jordan frame, as calculated with the PSRA

*Electronic address: dkaiser@fas.harvard.edu

formalism, differ negligibly from the Einstein frame results.

The specific method for calculating n_s is developed in Sec. II. In Sec. III, the formalism is applied to induced-gravity inflation, for which we can compare the Einstein-frame results with Jordan-frame calculations. In Secs. IV and V, the analysis is presented for two models with a different nonminimal $\phi^2 R$ coupling and two different potentials. Concluding remarks follow in Sec. VI.

II. EINSTEIN-FRAME FORMALISM

The calculation of n_s for these GET models of inflation involves two distinct tasks: calculating the PSRA parameters, which consist of various combinations of $d^nU/d\varphi^n$, where U is the scalar field potential following the conformal transformation, and φ is the newly defined scalar field following the conformal transformation; and calculating $\varphi_{\rm HC}$, the value of φ when the scales of interest crossed outside of the horizon during inflation. In general the first of these tasks is straightforward, while the second can become quite tricky.

The action for the three models studied below can be written in the general form

$$S = \int d^4x\sqrt{-g}\left[f(\phi)R - \frac{1}{2}\phi_{;\mu}\phi^{;\mu} - V(\phi)\right], \qquad (1)$$

where $f(\phi)$ gives rise to the nonminimal coupling $\phi^2 R$. This action yields the coupled field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = f^{-1}(\phi)\left[\frac{1}{2}\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\lambda}\phi^{;\lambda}\right) + f(\phi)_{;\mu;\nu} - \Box f(\phi)g_{\mu\nu} - \frac{1}{2}V(\phi)g_{\mu\nu}\right],$$
$$\Box\phi \quad -V'(\phi) + f'(\phi)f^{-1}(\phi)\left[3\Box f(\phi) + \frac{1}{2}\phi_{;\lambda}\phi^{;\lambda} + 2V(\phi)\right] = 0. \qquad (2)$$

In Eq. (2), a prime indicates $d/d\phi$.

These complicated field equations can be simplified by making a particular conformal transformation (see, e.g., [5]):

$$\hat{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu},$$
$$\Omega^2(x) = 2\kappa^2 f(\phi), \qquad (3)$$

where quantities in the new frame are marked by a caret. The quantity $\kappa^2 = 8\pi M_{\rm Pl}^{-2}$, where $M_{\rm Pl} \simeq 1.22\times10^{19}$ GeV is the present value of the Planck mass. [We thus require that $f(\phi)$ remain positive definite, to ensure that $M_{\rm Pl}$ does not change sign.] If we further define a new scalar field φ and scalar potential U by

$$\frac{d\varphi}{d\phi} \equiv \kappa^{-1}\sqrt{\frac{f(\phi) + 3[f'(\phi)]^2}{2f^2(\phi)}},$$
$$U(\varphi) \equiv [2\kappa^2 f(\phi)]^{-2}\,V(\phi) = \Omega^{-4}\,V(\phi), \qquad (4)$$

then the action in the new frame may be written in the canonical Einstein-Hilbert form

$$S = \int d^4x\sqrt{-\hat{g}}\left[\frac{1}{2\kappa^2}\hat{R} - \frac{1}{2}\varphi_{;\lambda}\varphi^{;\lambda} - U(\varphi)\right]. \qquad (5)$$

The action in Eq. (5) now yields the familiar equations of motion,

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R} = \kappa^2\left[\varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\varphi_{;\lambda}\varphi^{;\lambda} - U(\varphi)\hat{g}_{\mu\nu}\right],$$
$$\Box\varphi - U'(\varphi) = 0, \qquad (6)$$

where derivatives are now taken with respect to the metric $\hat{g}_{\mu\nu}$, and a prime indicates $d/d\varphi$.

When evaluating the field equations, we will assume that the background spacetime can be written in the form of a flat ($k = 0$) Robertson-Walker line element:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\vec{x}^2$$
$$= \Omega^{-2}(x)[-d\hat{t}^2 + \hat{a}^2(\hat{t})d\vec{x}^2], \qquad (7)$$

from which we can see that $d\hat{t} = \Omega(x)dt$ and $\hat{a}(\hat{t}) = \Omega(x)a(t)$. These relationships will become important when evaluating $\varphi_{\rm HC}$.

The spectral index (n_s) is determined by [8]

$$n_s - 1 \equiv \frac{d\ln\delta_H^2}{d\ln k}, \qquad (8)$$

where $\delta_H^2 = |\delta\tilde{\rho}/\rho|^2$. An exactly scale-invariant (Harrison-Zel'dovich) spectrum of perturbations corresponds to $n_s = 1.00$. For inflationary models, one can relate deviations from this exactly scale-invariant spectrum directly to changes in the Hubble parameter $\hat{H}(\varphi)$ and its derivatives during the time that various scales were crossing outside of the horizon [13–15]. Such a parametrization corresponds to the "Hubble-slow-roll approximation" (HSRA) scheme of [13]. Using the inflationary equations of motion, these deviations in terms of $\hat{H}(\varphi)$ can then be rewritten as changes in the inflaton's potential $U(\varphi)$ and its derivatives. This parametrization corresponds to the "potential-slow-roll approximation" (PSRA) scheme of [13]. This is the approach adopted here.

To second order in PSRA parameters, the spectral index n_s depends only on three parameters, ϵ, η, and ζ.[1]

[1]To avoid confusion between the third PSRA parameter and the nonminimal coupling strength, we will use ζ to denote the PSRA parameter and ξ to denote the coupling strength. In [13], the third PSRA parameter is labeled ξ, instead of ζ.

These three functions of φ are defined by [13]:

$$\epsilon \equiv \frac{1}{2\kappa^2}\left(\frac{U'(\varphi)}{U(\varphi)}\right)^2,$$
$$\eta \equiv \frac{1}{\kappa^2}\left(\frac{U''(\varphi)}{U(\varphi)}\right),$$
$$\zeta \equiv \frac{1}{\kappa^2}\left(\frac{U'(\varphi)U'''(\varphi)}{U^2(\varphi)}\right)^{1/2}, \tag{9}$$

where, again, a prime denotes $d/d\varphi$. To second order, then, the spectral index is given by [13, 14]

$$n_s = 1 - 6\epsilon + 2\eta + \frac{1}{3}(44-18c)\epsilon^2 + (4c-14)\epsilon\eta + \frac{2}{3}\eta^2 + \frac{1}{6}(13-3c)\zeta^2, \tag{10}$$

where $c \equiv 4(\ln 2 + \gamma) \simeq 5.081$ ($\gamma \simeq 0.577$ is Euler's constant). During inflation, each of these PSRA parameters remains less than unity, and hence the deviations of the spectrum of perturbations from the scale-invariant spectrum should indeed remain small.

The PSRA parameters in Eq. (10) are to be evaluated at $\varphi_{\rm HC}$. Yet for two of the models considered below, $\varphi(\phi)$ cannot be written in closed form. Instead, the PSRA parameters can be written as functions of the Jordan-frame scalar field ϕ, by using Eq. (4) and

$$U' = \frac{dU}{d\varphi} = \frac{d\phi}{d\varphi}\frac{dU}{d\phi}, \tag{11}$$

and so on for the higher derivatives. From Eq. (4), it is clear that U and all of its derivatives can always be written in closed form in terms of ϕ. We can thus derive ϵ, η, and ζ as functions of ϕ alone. This leaves the task of calculating the value of ϕ, which corresponds to $\varphi_{\rm HC}$.

Solving for the value of the field at the time of horizon crossing is difficult in either frame; but, following [19], we can use the fact that scales of interest to us crossed outside of the horizon approximately 60 e folds before the end of inflation:

$$e^{\alpha} \equiv \frac{\hat{a}(\hat{t}_{\rm end})}{\hat{a}(\hat{t}_{\rm HC})} = \frac{\Omega(x_{\rm end})}{\Omega(x_{\rm HC})}\frac{a(t_{\rm end})}{a(t_{\rm HC})} \sim e^{60}. \tag{12}$$

We can check how sensitively this assumption affects the calculation of n_s; this is treated below, in Sec. III A. In each of the models considered below, $a(t)$ can be solved in closed form as a function of $\phi(t)$ during the period of slow roll, and since $\Omega(x)$ is also defined as a function of $\phi(t)$, we may find an approximate value for $\phi_{\rm HC}$ in each case, where $\phi_{\rm HC}$ is the value of the Jordan-frame scalar field at the time when the scales of interest crossed outside of the horizon *in the Einstein frame.* (See [20] for a similar discussion in the context of extended inflation.) This last step allows the PSRA parameters to be evaluated at the correct time.

Finally, it should be noted that by using the PSRA parameters instead of the HSRA parameters, we have necessarily made an additional assumption, referred to as the "inflationary attractor" assumption in [13]. That is, we have assumed that near $\hat{t}_{\rm HC}$, the full Einstein-frame field equations

$$\hat{H}^2 + \frac{\hat{k}}{\hat{a}^2} = \frac{\kappa^2}{3}\left[U(\varphi) + \frac{1}{2}\left(\frac{d\varphi}{d\hat{t}}\right)^2\right],$$
$$\frac{d^2\varphi}{d\hat{t}^2} + 3\hat{H}\frac{d\varphi}{d\hat{t}} = -U'(\varphi) \tag{13}$$

may be approximated as

$$\hat{H}^2 \simeq \frac{\kappa^2}{3}U(\varphi),$$
$$3\hat{H}\frac{d\varphi}{d\hat{t}} \simeq -U'(\varphi), \tag{14}$$

where $\hat{H} \equiv \hat{a}^{-1}d\hat{a}/d\hat{t}$. In other words, we have assumed that the standard Einstein-frame "slow-roll" approximations may be made. As discussed in [13, 21], inflationary solutions of the full equations of motion, Eq. (13), approach the "slow-roll" attractor situation, Eq. (14), at least exponentially quickly (provided that the sign of $d\varphi/d\hat{t}$, and hence of $\dot{\phi}$, does not change—we will assume this here). Thus, by using the PSRA parameters to study n_s, we assume that $\hat{t}_{\rm HC}$ occurs sufficiently late in the time evolution of the inflationary phase to allow the dynamics to converge on Eq. (14). It is this assumption that enables the Jordan-frame scale factor $a(t)$ to be solved in terms of $\phi(t)$ during the slow-roll period.

III. INDUCED-GRAVITY INFLATION

A. Einstein-frame results

The first model to be considered here is induced-gravity inflation [19, 22]. In this model, an extended inflationlike Brans-Dicke coupling is combined with a Ginzburg-Landau potential:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\xi\phi^2 R - \frac{1}{2}\phi_{;\mu}\phi^{;\mu} - V(\phi)\right],$$
$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2, \tag{15}$$

where ξ (> 0) is the nonminimal coupling strength and is related to the Brans-Dicke parameter ω by $\xi = (4\omega)^{-1}$. The nonminimal coupling turns the Planck mass into a dynamical quantity; the present value of the Planck mass is related to the vacuum expectation value of the potential v by $M_{\rm Pl} = \sqrt{8\pi\xi}\, v$. In a flat Friedmann universe, the Jordan-frame field equations are

$$H^2 = \frac{1}{3\xi\phi^2}V(\phi) + \frac{1}{6\xi}\left(\frac{\dot{\phi}}{\phi}\right)^2 - 2H\frac{\dot{\phi}}{\phi},$$
$$\ddot{\phi} + 3H\dot{\phi} + \frac{\dot{\phi}^2}{\phi} = \frac{1}{1+6\xi}\frac{1}{\phi}\left[4V(\phi) - \phi V'(\phi)\right], \tag{16}$$

where overdots denote time derivatives and primes denote $d/d\phi$; we have assumed that the classical background field ϕ is sufficiently homogenous, so that all spatial derivatives become negligible. These equations correspond exactly to the Einstein-frame equations (13). The Einstein-frame inflationary attractor field equations (14) may then be rewritten as

$$\left(H + \frac{\dot{\phi}}{\phi}\right)^2 \simeq \frac{1}{3\xi\phi^2} V(\phi),$$
$$3\left(H\dot{\phi} + \frac{\dot{\phi}^2}{\phi}\right) \simeq \frac{1}{(1+6\xi)} \frac{1}{\phi} \left[4V(\phi) - \phi V'(\phi)\right]. \tag{17}$$

Yet the assumption $U(\varphi) \gg \frac{1}{2}(d\varphi/d\hat{t}\,)^2$ is equivalent to $V(\phi) \gg \frac{1}{2}\dot{\phi}^2(1+6\xi)$, and thus it remains consistent further to simplify the field equations during slow roll as

$$H^2 \simeq \frac{1}{3\xi\phi^2} V(\phi),$$
$$3H\dot{\phi} \simeq \frac{1}{(1+6\xi)} \frac{1}{\phi} \left[4V(\phi) - \phi V'(\phi)\right]. \tag{18}$$

These approximate equations may be integrated to yield the solutions

$$\phi(t) = \phi_o \pm \sqrt{\frac{4\lambda\xi}{3(1+6\xi)^2}}\, v^2\, t,$$
$$\frac{a(t)}{a_B} = \left(\frac{\phi(t)}{\phi_o}\right)^{(1+6\xi)/4\xi} \times \exp\left[\frac{(1+6\xi)}{8\xi v^2}\left[\phi_o^2 - \phi^2(t)\right]\right]. \tag{19}$$

In Eq. (19), ϕ_0 and a_B are values at the beginning of the inflationary epoch. The $\pm$ in $\phi(t)$ is determined by the initial conditions: for a chaotic inflation scenario $\phi_0 \gg v$, and the $-$ should be used in the solution of $\phi(t)$; in a new inflation scenario $\phi_0 \ll v$, so the $+$ should be used in the solution for $\phi(t)$. Thus we can see that with the chaotic inflation initial condition, $a(t)$ is dominated by a quasi–de Sitter expansion for early times $[a(t) \propto \exp(\phi_o\sqrt{\lambda/3\xi}\, t)]$, whereas with the new inflation initial condition $a(t)$ is dominated by a quasi-power-law expansion at early times $[a(t) \propto t^{(1+6\xi)/4\xi}]$.

We may now make the conformal transformation of Eqs. (3) and (4), in order to calculate the PSRA parameters. The conformal factor $\Omega(x)$ for induced-gravity inflation is simply proportional to the Jordan-frame field, $\Omega(x) = \sqrt{\kappa^2\xi}\,\phi(t)$, and the new scalar field potential, written as a function of the Jordan-frame field, becomes $U(\phi) = (\kappa^2\xi\phi^2)^{-2}\, V(\phi)$. Finally, the Einstein-frame scalar field is defined by $d\varphi/d\phi = \sqrt{(1+6\xi)/\kappa^2\xi\phi^2}$. Using these relationships, the PSRA parameters of Eq. (9) become

$$\epsilon = \frac{8\xi}{(1+6\xi)} \frac{v^4}{(\phi^2 - v^2)^2},$$
$$\eta = \frac{8\xi}{(1+6\xi)} \frac{v^2\,(2v^2 - \phi^2)}{(\phi^2 - v^2)^2},$$
$$\zeta = \frac{4\sqrt{2}\xi}{(1+6\xi)} \sqrt{\frac{v^4\,(\phi^2 - 4v^2)}{(\phi^2 - v^2)^3}}. \tag{20}$$

Before we may evaluate n_s, we must calculate $\phi_{\rm HC}$ using Eq. (12), for which we need $\phi_{\rm end}$, the value of the Jordan-frame field at the time inflation ends in the Einstein frame.[2] Inflation ends (in the Einstein frame) once $d^2\hat{a}/d\hat{t}^2 = 0$ (instead of being < 0). To first order, this is determined by $\epsilon = 1$ [13]. If we write $\phi_{\rm end} = \beta(\xi)\, v$, then we may solve for β:

$$\beta \simeq \sqrt{\left|1 \pm \sqrt{\frac{8\xi}{(1+6\xi)}}\right|}, \tag{21}$$

where, again, the $\pm$ is determined by the initial conditions: $+$ for a chaotic inflation scenario and $-$ for a new inflation scenario. Note that $8\xi/(1+6\xi) \leq 4/3$, so in both the chaotic and new inflation scenarios the end of inflation occurs close to $\phi = v$, as expected.

If we next write $\phi_{\rm HC} = m(\xi)\, v$, then Eq. (12) becomes

$$e^\alpha = \left(\frac{\beta}{m}\right)^{(1+10\xi)/4\xi} \exp\left[\frac{(1+6\xi)}{8\xi}\left(m^2 - \beta^2\right)\right]. \tag{22}$$

In order to solve for $m(\xi)$ under chaotic inflation conditions, it is helpful to rewrite Eq. (22) as

$$\frac{m}{\beta} = \exp\left[\frac{(1+6\xi)}{2(1+10\xi)}\left(m^2 - \beta^2\right) - \frac{8\xi\alpha}{2(1+10\xi)}\right]. \tag{23}$$

To remain consistent, $m/\beta \geq 1$ for the chaotic inflation scenario, which requires

$$m_{\rm ch} \geq \sqrt{\beta^2 + \frac{8\xi\alpha}{(1+6\xi)}}, \tag{24}$$

where the subscript "ch" is to remind us that this inequality is to be satisfied under chaotic inflation conditions only.

For the new inflation scenario, it is helpful to rewrite Eq. (22) as

$$\frac{4\xi\alpha}{1+6\xi} = -\frac{1+10\xi}{1+6\xi}\ln\frac{m}{\beta} + \frac{1}{2}m^2 - \frac{1}{2}\beta^2. \tag{25}$$

As in [12], this equation may now be solved approximately for m under two limiting conditions: (a) $4\xi\alpha \ll (1+6\xi)$ and (b) $4\xi\alpha \gg (1+6\xi)$. However, as discussed below in Sec. III B, ξ is strongly constrained in the new inflation scenario of this model, based on sufficient inflation requirements: $\xi \leq 2.5 \times 10^{-3}$. Thus we need only consider the case (a). In this limit, m becomes

$$m_{\rm new} \simeq 1 - \sqrt{\frac{4\xi\alpha}{(1+6\xi)} + (\beta - 1)^2}, \tag{26}$$

where the subscript "new" is to remind us that this approximate solution for m applies only under the new in-

[2]Note that this value of $\phi_{\rm end}$ should be very close to the value of ϕ at the time inflation ends in the Jordan frame, since as $\phi \to v$, $\Omega(x) \to 1$, and the two frames coincide.

flation conditions. Note that given the constraint on ξ, the $(\beta-1)^2$ term will always remain over an order of magnitude smaller than the $4\xi\alpha/(1+6\xi)$ term.

Using Eqs. (20), (24), and (26), the PSRA parameters may now be written as functions of the nonminimal coupling strength ξ alone:

$$\epsilon = \frac{8\xi}{1+6\xi}\frac{1}{(m^2-1)^2},$$
$$\eta = \frac{8\xi}{1+6\xi}\frac{2-m^2}{(m^2-1)^2},$$
$$\zeta = \frac{4\sqrt{2}\xi}{1+6\xi}\sqrt{\left|\frac{m^2-4}{(m^2-1)^3}\right|}, \tag{27}$$

where the appropriate $m(\xi)$ is determined by the initial conditions.

Approximate first-order results for n_s may be written using Eq. (10), and taking the limits $m \gg 1$ for chaotic inflation initial conditions and $m \ll 1$ for new inflation initial conditions. In these limits, to first order, the spectral index may be written

$$n_{s,\,\mathrm{ch}} \simeq 1 - \frac{16\xi}{8\xi\alpha+1},$$
$$n_{s,\,\mathrm{new}} \simeq 1 - 16\xi, \tag{28}$$

where we have used $\alpha \simeq 60 \gg 1$ when evaluating $n_{s,\,\mathrm{ch}}$ and $\xi \ll 1$ when evaluating $n_{s,\,\mathrm{new}}$. It is important to remember that these expressions for n_s are limiting cases, corresponding loosely to $\xi \gtrsim 1$ and $\xi \lesssim 10^{-3}$, respectively; the second-order result for $n_{s,\,\mathrm{ch}}$, for example, has a positive slope for increasing ξ, unlike this approximate solution.

The full second-order results for n_s are plotted in Fig. 1. For the chaotic inflation case, $0.90 \le n_s \le 0.97$ for $\xi \ge 1.5\times10^{-3}$, which is obviously close to the observed $n_s \sim 1.00$ spectrum [7]. For the new inflation case, $0.93 \le n_s \le 0.97$ for $\xi \le 4\times10^{-3}$. Thus, induced-gravity inflation predicts a spectral index in close agreement with the experimental determinations.

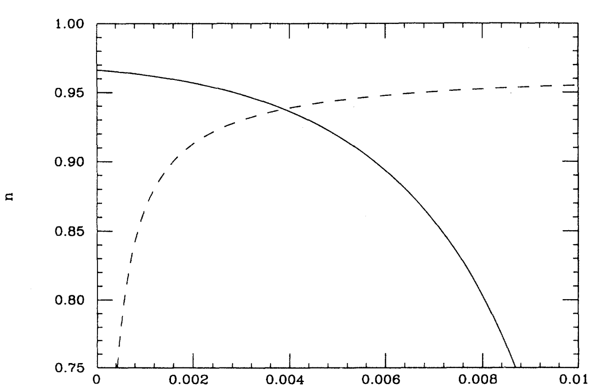

FIG. 1. Second-order results for the spectral index n_s for induced-gravity inflation, based on Eqs. (10), (24), (26), and (27), with $\alpha = 60$. The solid line is for new inflation initial conditions, and the dashed line is for chaotic inflation initial conditions. Note that for the new inflation scenario, $\xi > 2.5\times10^{-3}$ is forbidden due to sufficient inflation requirements.

The sensitivity of n_s to our assumption $e^{\alpha} \simeq e^{60}$ may be checked by calculating $\partial n_s/\partial\alpha$. From Eq. (10), it is clear that this requires calculating $\partial\epsilon/\partial\alpha$, $\partial\eta/\partial\alpha$, and $\partial\zeta/\partial\alpha$. For the case of chaotic inflation conditions, Eqs. (24) and (27) yield

$$\left.\frac{\partial\epsilon}{\partial\alpha}\right|_{\mathrm{ch}} = \frac{-128\xi^2}{(1+6\xi)^2}\frac{1}{(m^2-1)^3},$$
$$\left.\frac{\partial\eta}{\partial\alpha}\right|_{\mathrm{ch}} = \frac{64\xi^2}{(1+6\xi)^2}\frac{m^2-3}{(m^2-1)^3}. \tag{29}$$

When $\xi \gg 1$, $m_{\mathrm{ch}}^2 \to 4\alpha/3 \gg 1$, so

$$\left.\frac{\partial\epsilon}{\partial\alpha}\right|_{\mathrm{ch}} \propto \alpha^{-3} \sim 10^{-6},$$
$$\left.\frac{\partial\eta}{\partial\alpha}\right|_{\mathrm{ch}} \propto \alpha^{-2} \sim 10^{-4}. \tag{30}$$

When $\xi \ll 1$, $(m_{\mathrm{ch}}^2 - 1) \to 8\xi\alpha/(1+6\xi)$, so

$$\left.\frac{\partial\epsilon}{\partial\alpha}\right|_{\mathrm{ch}} \propto \xi^{-1}\alpha^{-3},$$
$$\left.\frac{\partial\eta}{\partial\alpha}\right|_{\mathrm{ch}} \propto \xi^{-1}\alpha^{-3}\,(8\xi\alpha-2)\,. \tag{31}$$

Because ζ only enters in n_s at second order in the PSRA expansion, $\partial\zeta/\partial\alpha$ has not been explicitly included, although it can easily be shown to behave similarly to $\partial\epsilon/\partial\alpha$ and $\partial\eta/\partial\alpha$. Likewise, for the new inflation case, Eqs. (26) and (27) yield $\partial\epsilon/\partial\alpha \propto \partial\eta/\partial\epsilon \propto \xi^4\alpha^2$, so that both of these deviations remain $\lesssim 10^{-8}$, given the independent constraint for new inflation conditions, $\xi \le 2.5\times10^{-3}$. If we expand n_s in a Taylor series as

$$n_s(\alpha) \simeq n_s(60) + \left(\frac{\partial n_s}{\partial\alpha}\right)_{\alpha=60}(\alpha-60) + \cdots\,, \tag{32}$$

then Eq. (31) can be used to place limits on the regions of ξ space, under chaotic inflation conditions, for which the assumption $n_s(60)$ remains accurate: requiring $n_s(\alpha) - n_s(60) \le 10^{-2}$, for $\alpha - 60 \sim 10$, limits ξ to $\xi \ge 10^3\alpha^{-3} \simeq 10^{-3}$. Note that under new inflation conditions, $n_s(\alpha) - n_s(60)$ will always remain $\le 10^{-7}$. The assumption that $\alpha = 60$, put in by hand to facilitate computation of ϕ_{HC}, therefore has negligible effects on the calculation of n_s, so long as $\xi \gtrsim 10^{-3}$ under chaotic inflation conditions. Thus, for the remainder of this paper, all numerical results for n_s will be calculated assuming $\alpha = 60$.

B. Comparison with Jordan frame results

In [12], n_s was calculated directly in the Jordan frame for a new inflation scenario of induced-gravity inflation. There it was assumed that the scales of interest crossed outside of the horizon, while the expansion was still predominantly quasi-power-law behavior, $a(t) \propto t^p$, where $p = (1+6\xi)/4\xi$. The result was

$$n_{s,\,J} = 1 - \frac{8\xi}{(1+2\xi)}. \tag{33}$$

This should be compared with the $m \ll 1$ limit of the Einstein-frame results in Eq. (28): $n_{s,\,E} \simeq 1 - 16\xi$. Obviously, the results differ between the two frames.

The difference may be traced to ambiguities between the quantum fluctuations of the scalar field in the two frames [17]. The usual procedure for calculating the density perturbation spectrum is to study the intrinsic curvature perturbation, given (during inflation) by [8, 14]:

$$\mathcal{R} = \frac{H}{\dot{\phi}}\,\delta\phi. \tag{34}$$

The spectrum of the density perturbation is then given by

$$\mathcal{P}_{\mathcal{R}}{}^{1/2} = \frac{H}{\dot{\phi}}\left(|\Delta\phi|^2\right)^{1/2}, \tag{35}$$

where $|\Delta\phi|^2$ is the two-point correlation function for the scalar field's quantum fluctuations, defined as [23, 24]

$$\left|\Delta\phi(\vec{k},\tau)\right|^2 \equiv k^3 \int \frac{d^3x}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \langle\delta\phi(\vec{x},\tau)\delta\phi(\vec{0},\tau)\rangle. \tag{36}$$

This is the basis for the PSRA formalism. The trouble is that although $\mathcal{R}$ is gauge invariant with respect to the choice of comoving or synchronous gauge [8], it is *not* invariant with respect to the conformal transformation of Eqs. (3) and (4). Labeling $\hat{\mathcal{R}}$ the curvature perturbation as evaluated in the Einstein frame, it is straightforward to show that

$$\hat{\mathcal{R}} \equiv \frac{\hat{H}}{d\varphi/d\hat{t}}\,\delta\varphi = \frac{\left(H + \dot{\Omega}/\Omega\right)}{\sqrt{\dot{\phi}^2 + 6\dot{\Omega}^2/\kappa^2}}\,\Omega\,\delta\varphi \neq \frac{H}{\dot{\phi}}\,\delta\phi, \tag{37}$$

or $\hat{\mathcal{R}} \neq \mathcal{R}$. For induced-gravity inflation, $\Omega = \sqrt{\kappa^2\xi}\,\phi$, so that during inflation, $\dot{\Omega}/\Omega = \dot{\phi}/\phi \ll H$ [25]; similarly, under the new inflation conditions, with $\xi \lesssim 10^{-3}$, then $6\dot{\Omega}^2/\kappa^2 = 6\xi\dot{\phi}^2 \ll \dot{\phi}^2$, giving $\hat{\mathcal{R}} \simeq (H/\dot{\phi})\,\Omega\,\delta\varphi$. For calculating n_s, however, it remains to compare the k dependence of $\delta\phi$ with that of $\delta\varphi$.

As shown in [12], near $t_{\rm HC}$, the linearized equation of motion for the fluctuations $\delta\phi$ is that of a nearly massless scalar field in an expanding background spacetime:

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{1}{a^2}\nabla^2\delta\phi \simeq 0, \tag{38}$$

where $\ddot{\delta\phi} \equiv \partial^2(\delta\phi)/\partial t^2$. Written in terms of conformal time $d\tau \equiv dt/a$ and a conformal field defined by $\psi \equiv a\delta\phi$, the equation of motion for each mode becomes

$$\tilde{\psi}_k'' - \frac{a''}{a}\tilde{\psi}_k + k^2\tilde{\psi}_k \simeq 0, \tag{39}$$

where primes (in this section only) denote $d/d\tau$, and we have performed a spatial Fourier transform. For a metric expanding as $a(t) \propto t^p$, the scale factor as a function of τ becomes $a(\tau) \propto [(1-p)\tau]^{p/(1-p)}$, and the equation of motion takes the form

$$\tilde{\psi}_k'' + \left[k^2 + \frac{p(1-2p)}{(1-p)^2}\frac{1}{\tau^2}\right]\tilde{\psi}_k \simeq 0. \tag{40}$$

Note that this approaches the equation of motion for a massless scalar field in a de Sitter background [17, 23, 26] as $p \to \infty$, as it should, given the form $a(t) \propto t^p$. If we next define the field χ as $\chi \equiv \tau^{-1/2}\psi$ and work in terms of the variable $x \equiv k\tau$, then the equation of motion takes the form of Bessel's equation:

$$\frac{d^2\tilde{\chi}_k}{dx^2} + \frac{1}{x}\frac{d\tilde{\chi}_k}{dx} + \left(1 - \frac{1}{x^2}\left[\frac{(3p-1)^2}{4(p-1)^2}\right]\right)\tilde{\chi}_k \simeq 0. \tag{41}$$

Mode solutions for the original field $\delta\phi$ may then be written in terms of Hankel functions:

$$\delta\tilde{\phi}_k(\tau) \simeq C_1\tau^\nu\left[A_k H_\nu^{(1)}(k\tau) + B_k H_\nu^{(2)}(k\tau)\right], \tag{42}$$

where C_1 is a constant, and

$$\nu = \nu_J = \frac{3p-1}{2(p-1)}. \tag{43}$$

The subscript "J" is to remind us that this solution is for the fluctuations in the Jordan frame. Again we see asymptotic agreement with the de Sitter case, which has $\nu_{\rm deS} = 3/2$.

The Bunch-Davies vacuum, as defined for the case of de Sitter expansion, corresponds to $A_k = 0$ [17, 27]; note that such a choice of vacuum is warranted for the case of power-law expansion as well, since, in the limit $p \to 0$, this vacuum choice yields mode solutions, which approach the ordinary Minkowski space solutions for massless scalar particles, $\propto k^{-1/2}\exp(i\vec{k}\cdot\vec{x} - ikt)$ [26, 28]. Taking the limit $k\tau \to 0$ (for long-wavelength modes [23]), the fluctuations $\delta\phi$ then behave, for this choice of vacuum, as

$$\delta\tilde{\phi}_k(\tau) \propto k^{-\nu_J}. \tag{44}$$

The two-point correlation function for these fluctuations then becomes

$$\left|\Delta\phi(\vec{k},\tau)\right|_J^2 \propto k^{3-2\nu_J}. \tag{45}$$

This gives

$$\mathcal{P}_{\mathcal{R}} \propto \delta_H^2 \propto |\Delta\phi|_J^2 \propto k^{3-2\nu_J}. \tag{46}$$

Using Eqs. (8), (43), and $p = (1+6\xi)/4\xi$ yields the result

$$n_{s,\,J} \simeq 1 - \frac{8\xi}{1+2\xi}. \tag{47}$$

This is the origin of Eq. (33).

The situation in the Einstein frame may now be compared: the fluctuations $\delta\varphi$ obey the equation of motion

$$\frac{d^2\delta\varphi}{d\hat{t}^2} + 3\hat{H}\frac{d\delta\varphi}{d\hat{t}} - \frac{1}{\hat{a}^2}\hat{\nabla}^2\delta\varphi + \frac{d^2U(\varphi)}{d\varphi^2}\delta\varphi = 0. \tag{48}$$

At $\hat{t}_{\rm HC}$, however,

$$\frac{\hat{k}^2}{\hat{a}^2(\hat{t}_{\rm HC})} = \hat{H}^2_{\rm HC} \simeq \frac{\kappa^2}{3} U(\varphi_{\rm HC}), \tag{49}$$

giving

$$\left(\frac{\hat{k}^2}{\hat{a}^2(\hat{t}_{\rm HC})} \right)^{-1} \frac{d^2 U(\varphi)}{d\varphi^2} \propto \xi \tag{50}$$

and under new inflation conditions $\xi \ll 1$, so in the Einstein frame the fluctuations $\delta\varphi$ also behave as a nearly massless scalar field.

The conformal transformation gives $\Omega \propto \phi \propto t$, and thus $\hat{t} \propto t^2$. Furthermore, $\hat{a}(\hat{t}) = \Omega\, a(t)$, so a Jordan-frame scale factor $a(t) \propto t^p$ corresponds to an Einstein-frame scale factor $\hat{a}(\hat{t}) \propto \hat{t}^{(p+1)/2}$. The conformal transformation does not affect $\vec{x}$, so $\hat{k} = k$. Proceeding as above, the equation of motion for the Einstein-frame fluctuations may be cast in the form of Bessel's equation, and mode solutions written as

$$\delta\tilde{\varphi}_k(\tau) = C_2 \tau^\nu \left[\hat{A}_k H^{(1)}_\nu(k\tau) + \hat{B}_k H^{(2)}_\nu(k\tau) \right], \tag{51}$$

with $C_2 \neq C_1$ another constant, and

$$\nu = \nu_E = \frac{3p+1}{2(p-1)} \neq \nu_J. \tag{52}$$

Note that this result also yields the de Sitter solution, $\nu_{\rm deS} = 3/2$, as $p \to \infty$.

If we attempt to define the Einstein-frame vacuum as $\hat{A}_k = 0$, then the two-point correlation function for the fluctuations $\delta\varphi$ becomes, as $k\tau \to 0$,

$$\left| \Delta\varphi(\vec{k}, \tau) \right|^2_E \propto k^{3-2\nu_E}, \tag{53}$$

and thus

$$n_{s,\,E} = 1 - \frac{16\xi}{1+2\xi}. \tag{54}$$

This reproduces the approximate first-order result for $n_{s,\,E}$, Eq. (28), given $\xi \lesssim 10^{-3}$.

It is now easy to see why n_s is unaffected by these ambiguities for the case of quasi–de Sitter expansion: when $a(t) \propto \exp(Ht)$, the Jordan-frame two-point correlation function for the quantum fluctuations takes the asymptotic form [17, 24]:

$$|\Delta\phi(k\tau)|^2 \to C_3 \left[1 + (k\tau)^2 \right]. \tag{55}$$

We saw above that $k = \hat{k}$; similarly, the conformal time, $d\tau \equiv dt/a$, remains invariant under the conformal transformation. Thus, for quasi–de Sitter inflation, the k dependence does not change between the two frames (although the magnitude of the correlation function does change between the two frames [17]), and the results for n_s obtained using the Einstein-frame formalism of section 2 should be accurate for the Jordan frame as well.

Put another way, we may understand the discrepancy between the two frames as follows: quasi–de Sitter expansion in the Jordan frame yields quasi–de Sitter expansion in the Einstein frame as well, so that $\nu_J = \nu_E = 3/2$. Quasi-power-law expansion in the Jordan frame likewise gives quasi-power-law expansion in the Einstein frame, but with a different power, so that $\nu_J \neq \nu_E$. This difference in ν (if the vacua in the two frames are defined to be $A_k = \hat{A}_k = 0$) is responsible for the different k dependencies of n_s. This discrepancy may be remedied by finding a gauge-invariant measure of the intrinsic curvature perturbation, which *also* remains invariant with respect to the conformal transformation. (Note that the combination presented in [25] does not circumvent the discrepancy between $\delta\phi$ and $\delta\varphi$ for models with quasi-power-law expansion.) Such a frame-independent formalism has been developed in [18], with which the Jordan-frame value for n_s does indeed match the Einstein-frame PSRA result.

Finally, having calculated the discrepancy between $n_{s,\,J}$ and $n_{s,\,E}$ for the new inflation scenario of induced-gravity inflation, it is important to consider how large a numerical difference this ambiguity amounts to. This can be done by finding the allowed region of ξ space, which yields sufficient inflation. In [13], Liddle, Parsons, and Barrow demonstrated that for inflationary models to solve the horizon and flatness problems, the model must provide at least 70 e folds of expansion of the comoving Hubble length, $(\hat{a}\hat{H})^{-1}$ (this is slightly different from the requirement ordinarily assumed in the literature, that the scale factor $\hat{a}$ grow by 70 e folds). To first order in the PSRA parameters, this requires

$$\bar{N}(\phi_0, \phi_{\rm end}) \equiv \ln \frac{\left(\hat{a}\hat{H}\right)_{\rm end}}{\left(\hat{a}\hat{H}\right)_0} = -\sqrt{\frac{\kappa^2}{2}} \int_{\phi_0}^{\phi_{\rm end}} \frac{1}{\sqrt{\epsilon(\phi)}} \left(1 - \frac{1}{3}\epsilon(\phi) - \frac{1}{3}\eta(\phi) \right) d\phi \geq 70. \tag{56}$$

Note that although $\bar{N}$ is written here in terms of the Jordan-frame field ϕ, it pertains, like the PSRA parameters ϵ and η, to the Einstein frame; that is, we require that the comoving Hubble length in the Einstein frame inflate by at least 70 e folds during inflation. For the new inflation scenario of induced-gravity inflation, with $\phi_o \ll v$ and $\phi_{\rm end} = \beta v$, this may be integrated to yield the closed-form expression

$$\bar{N} = \frac{2}{3\sqrt{1+6\xi}} \left[\beta - \ln\left(\frac{1-\beta}{1+\beta} \right) \right] + \frac{\sqrt{1+6\xi}}{4\xi} \beta \left(1 - \frac{1}{3}\beta^2 \right) \geq 70, \tag{57}$$

with $\beta(\xi)$ given in Eq. (21). This expression may be evaluated numerically, revealing[3] that $\bar{N} \geq 70$ for $\xi \leq 2.5 \times 10^{-3}$ (or $\bar{N} \geq 70$ for the Brans-Dicke parameter $\omega \geq 100$). Considering the quasi-power-law expansion for induced-gravity inflation under the new inflation conditions, this result makes sense: for small values of ξ, $a(t) \propto t^{1/4\xi}$, and thus $\xi \ll 1$ yields rapid expansion. Furthermore, as discussed in [12], there is no lower bound on ξ, as there is for extended inflation (stemming from bubble percolation requirements), because of the second-order phase transition in induced-gravity inflation. This means that the first-order result in the Jordan frame is $(1 - n_{s,\,J}) \leq 0.02$, while the first-order result in the Einstein frame is $(1 - n_{s,\,E}) \leq 0.04$. Thus, in either frame, induced-gravity inflation with the new inflation conditions predicts $n_s \simeq 1$.

In the following two sections, the Einstein frame formalism of Sec. II is applied to two other closely related GET models of inflation. We only present the results for n_s as determined by the Einstein-frame formalism of Sec. II; again, these should remain invariant between the Einstein frame and the Jordan frame for the cases of quasi–de Sitter inflation, and it is expected that the numerical discrepancies between frames is small for the quasi-power-law case.

IV. NONMINIMALLY COUPLED SCALAR WITH ϕ^4 SELF-INTERACTION

The action, Eq. (1), can be used to study a nonminimal coupling similar to, but distinct from, that of induced-gravity inflation. In this section we consider a model given by the action (see, e.g., [25]):

$$S = \int d^4x\sqrt{-g}\left[\left(\frac{1+\kappa^2\xi\phi^2}{2\kappa^2}\right)R - \frac{1}{2}\phi_{;\,\mu}\phi^{;\,\mu} - V(\phi)\right],$$

$$V(\phi) = \frac{\lambda}{4}\phi^4, \tag{58}$$

from which the Jordan-frame field equations in a flat Friedmann universe become

$$H^2 = \frac{\kappa^2}{3(1+\kappa^2\xi\phi^2)}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2 - 6\xi H\phi\dot{\phi}\right],$$

$$\ddot{\phi} + 3H\dot{\phi} + \left(\frac{\kappa^2\xi\phi^2(1+6\xi)}{1+\kappa^2\xi\phi^2(1+6\xi)}\right)\frac{\dot{\phi}^2}{\phi}$$

$$= \frac{1}{1+\kappa^2\xi\phi^2(1+6\xi)} \times\left[4\kappa^2\xi\phi V(\phi) - (1+\kappa^2\xi\phi^2)V'(\phi)\right], \tag{59}$$

where overdots denote time derivatives and primes denote $d/d\phi$. For these field equations, the Einstein frame inflationary attractor assumption becomes

$$H^2 \simeq \frac{\kappa^2}{3(1+\kappa^2\xi\phi^2)}V(\phi),$$

$$3H\dot{\phi} \simeq \frac{1}{(1+\kappa^2\xi\phi^2(1+6\xi))} \times[4\kappa^2\xi\phi V(\phi) - (1+\kappa^2\xi\phi^2)V'(\phi)]. \tag{60}$$

As for induced-gravity inflation, these slow-roll field equations may be integrated to yield $a(t)$:

$$\frac{a(t)}{a_B} = \left(\frac{1+\kappa^2\xi\phi^2(t)}{1+\kappa^2\xi\phi_o^2}\right)^{3/4} \times \exp\left[\frac{1+6\xi}{8}\kappa^2[\phi_o^2 - \phi^2(t)]\right]. \tag{61}$$

Note from the form of the potential, $V(\phi)$, that this model only admits chaotic inflation initial conditions, with $\phi_o \gg 0$, and, hence, during inflation the expansion is predominantly quasi–de Sitter.

A few words are in order concerning the sign of ξ in this model. In induced-gravity inflation, the sign of ξ is fixed by present conditions: $\xi > 0$ is required to yield the proper value of the Planck mass. Yet in this model, the sign of ξ is undetermined by present conditions: after inflation, $\phi \simeq 0$, and the present value of the Planck mass is independent of the model parameters. However, we will only consider values of $\xi \geq 0$ here; as Futamase and Maeda concluded in [16], a negative value of ξ (according to the sign conventions used here) would require $|\xi| \leq 10^{-3}$ in order to yield sufficient inflation. Such constraints do not apply for the sign choice $\xi \geq 0$.

Making the conformal transformation of Eqs. (3) and (4) yields

$$\Omega(x) = \sqrt{1+\kappa^2\xi\phi^2(t)},$$

$$U(\phi) = \left(1+\kappa^2\xi\phi^2\right)^{-2}V(\phi),$$

$$\frac{d\varphi}{d\phi} = \frac{[1+\kappa^2\xi\phi^2(1+6\xi)]^{1/2}}{1+\kappa^2\xi\phi^2}. \tag{62}$$

The first of the PSRA parameters thus becomes

$$\epsilon = \frac{8}{\kappa^2}\frac{1}{\phi^2[1+\kappa^2\xi\phi^2(1+6\xi)]}. \tag{63}$$

Setting $\epsilon = 1$, and writing $\kappa^2\xi\phi_{\rm end}^2 = \beta^2(\xi)$, we may solve for $\beta(\xi)$,

$$\beta = \sqrt{\frac{1}{2(1+6\xi)}\left(\sqrt{192\xi^2+32\xi+1} - 1\right)}. \tag{64}$$

Note that $\beta_{\rm max} = 1.07$, and $\beta \to 0$ as $\xi \to 0$.

If we similarly write $\kappa^2\xi\phi_{\rm HC}^2 = m^2(\xi)$, then the three PSRA parameters may be written

---

[3]Note that for chaotic inflation conditions, additional assumptions must be made about the value of ϕ_o before $\bar{N}$ can be used to place limits on ξ.

$$\epsilon = 8\xi \, \frac{1}{m^2[1+m^2(1+6\xi)]},$$
$$\eta = 4\xi \, \frac{3+m^2(1+12\xi)-2m^4(1+6\xi)}{m^2\left[1+m^2(1+6\xi)\right]^2},$$
$$\zeta = 4\sqrt{2}\xi \, \frac{\left|3+2m^2(-2+3\xi)-15m^4(1+6\xi)-6m^6(1+6\xi)^2+2m^8(1+6\xi)^2\right|^{1/2}}{m^2[1+m^2(1+6\xi)]^2}. \tag{65}$$

As for induced-gravity inflation, $m(\xi)$ can now be approximated by using Eq. (12). In this model, Eq. (12) may be rewritten

$$\frac{m^2}{\beta^2} = \frac{1+\beta^2}{\beta^2}\exp\left[\frac{1+6\xi}{10\xi}\left(m^2-\beta^2\right)-\frac{8\alpha}{10}\right]-\frac{1}{\beta^2}, \tag{66}$$

where, again, the consistency of the chaotic inflation conditions requires $m/\beta \geq 1$. Remarkably, the requirements for this model then take the same form as for induced-gravity inflation:

$$m_{\rm ch} \geq \sqrt{\beta^2 + \frac{8\xi\alpha}{1+6\xi}}, \tag{67}$$

with $\beta(\xi)$ for this model given by Eq. (64). The spectral index may now be calculated using Eq. (10), where, again, the three PSRA parameters of Eq. (65) are functions of ξ alone.

The first-order result, $n_s = 1 - 6\epsilon + 2\eta$, may be approximated in the limit $m \gg 1$, yielding

$$n_s \simeq 1 - \frac{32\xi}{16\xi\alpha - 1}. \tag{68}$$

Figure 2 shows the full second-order results. The spectral index satisfies $0.96 \leq n_s \leq 0.97$ for $\xi \geq 4 \times 10^{-3}$, again in very close agreement with the empirical results. The final model to be considered here is a close cousin to this model, with the ϕ^4 potential replaced by a Ginzburg-Landau potential.

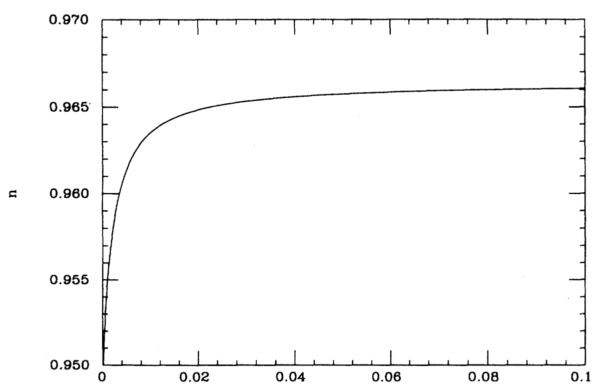

FIG. 2. Second-order results for the spectral index n_s for the model of Sec. IV, based on Eqs. (10), (65), and (67), with $\alpha = 60$. This model only admits chaotic inflation initial conditions.

V. NONMINIMALLY COUPLED SCALAR WITH GINZBURG-LANDAU POTENTIAL

As for the preceding model, the action for this last model may be written

$$S = \int d^4x\sqrt{-g}\left[\left(\frac{1+\kappa^2\xi\phi^2}{2\kappa^2}\right)R - \frac{1}{2}\phi_{;\,\mu}\phi^{;\,\mu} - V(\phi)\right],$$

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2. \tag{69}$$

There is a subtle difference between this model and the last, concerning the present value of the Planck mass: in this model, $\phi \simeq v$ at the end of inflation, and not $\simeq 0$, which means that the present value of the Planck mass is determined by

$$M_{\rm Pl}^2 = \frac{8\pi\left(1+\kappa^2\xi v^2\right)}{\kappa^2}. \tag{70}$$

In other words, $\kappa^2 \neq 8\pi M_{\rm Pl}^{-2}$; instead, κ^2 is a free parameter of the model, with dimensions $mass^{-2}$. For this reason, in this section only, the present value of Newton's gravitational constant will be written as $\kappa_N^2 \equiv 8\pi M_{\rm Pl}^{-2} \neq \kappa^2$. Equation (70) may then be rewritten

$$\frac{\kappa^2}{\kappa_N^2} = 1 + \kappa^2\xi v^2 \equiv 1 + \delta^2, \tag{71}$$

where the parameter $\delta^2 \equiv \kappa^2\xi v^2$ has been defined. The spectral index in this model is thus a function of the two free parameters, ξ and δ^2. Note also that, as above, we will only consider values of $\xi \geq 0$.

The Jordan-frame slow-roll field equations, Eq. (60), with the new potential and $\kappa^2 \neq \kappa_N^2$, can then be integrated for $a(t)$:

$$\frac{a(t)}{a_B} = \left(\frac{1+\kappa^2\xi\phi^2(t)}{1+\kappa^2\xi\phi_o^2}\right)^{3/4}\left(\frac{\phi(t)}{\phi_o}\right)^{\kappa_N^2 v^2/4} \times \exp\left[\frac{1+6\xi}{8}\kappa_N^2[\phi_o^2 - \phi^2(t)]\right]. \tag{72}$$

Although $\phi(t)$ cannot be solved exactly as a function of t, the opposite can be done, to study how ϕ evolves with t. That is, if we write the Jordan-frame field equations (60) as

$$H^2 \simeq \frac{\kappa^2}{3}A(\phi)\,V(\phi),$$

$$3H\frac{d\phi}{dt} \simeq B(\phi), \tag{73}$$

then during slow-roll, we may invert this expression and integrate to find t as a function of ϕ:

$$dt = \sqrt{3\kappa^2}\,\frac{\sqrt{A(\phi)\,V(\phi)}}{B(\phi)}\,d\phi, \tag{74}$$

or

$$t = \frac{\pm 1}{1+\delta^2}\sqrt{\frac{3\kappa^2}{4\lambda}}\ln\left(\frac{\phi(t)}{\phi_0}\,\frac{1+\sqrt{1+\kappa^2\xi\phi_0^2}}{1+\sqrt{1+\kappa^2\xi\phi^2(t)}}\right) \pm\frac{1+6\xi}{1+\delta^2}\sqrt{\frac{3\kappa^2}{4\lambda}}\left[\sqrt{1+\kappa^2\xi\phi^2(t)}-\sqrt{1+\kappa^2\xi\phi_0^2}\right], \tag{75}$$

where the + is for new inflation initial conditions and the − for chaotic inflation initial conditions. It can be verified numerically that the nonlinear terms in this expression are dominant only for very small values of $\kappa^2\xi\phi^2(t)$, and that for $\kappa^2\xi\phi^2(t)\sim 1$, it is a good approximation to assume $\phi(t)\propto t$. Thus, as for induced-gravity inflation, the expansion under chaotic inflation initial conditions is a quasi–de Sitter expansion, while the expansion under new inflation initial conditions is a quasi-power-law expansion.

The first PSRA parameter for this model is

$$\epsilon = \frac{8(1+\delta^2)}{\kappa_N^2}\,\frac{\phi^2}{(\phi^2-v^2)^2}\,\frac{1}{1+\kappa^2\xi\phi^2(1+6\xi)}. \tag{76}$$

Writing $\kappa^2\xi\phi_{\rm end}^2 = \beta^2(\xi)$, the equation for β becomes rather difficult to solve exactly:

$$0 = \beta^6(1+6\xi)+\beta^4-2\beta^3\delta^2(1+6\xi) +\beta^2\delta^2(\delta^2(1+6\xi)-\xi(1+\delta^2)^2)-2\beta\delta^2+\delta^4. \tag{77}$$

Instead, for the chaotic inflation conditions, two limiting approximations may be made: (a) $\phi_{\rm end}^2 \gg v^2$ (i.e., $\beta^2 \gg \delta^2$), so that the β calculated in the preceding section, Eq. (64), can serve as the approximate value here, and (b) $\phi_{\rm end}^2 \simeq v^2$ ($\beta^2\simeq\delta^2$). We will also assume $\beta^2\simeq\delta^2$ for the new inflation initial conditions. For case (a), we will define $\kappa^2\xi\phi_{\rm HC}^2 = m^2(\xi)$, and thus the PSRA parameters for case (a) may be written

$$\epsilon_a = 8\xi(1+\delta^2)^2\frac{m^2}{(m^2-\delta^2)^2[1+m^2(1+6\xi)]},$$
$$\eta_a = 4\xi(1+\delta^2)\frac{F_a+G_a\delta^2}{(m^2-\delta^2)^2[1+m^2(1+6\xi)]^2},$$
$$\zeta_a = 4\sqrt{2}\xi(1+\delta^2)\frac{\left|K_a+L_a\delta^2\right|^{1/2}}{\left|m^2-\delta^2\right|^{3/2}[1+m^2(1+6\xi)]^2}, \tag{78}$$

with $m^2 \gg \delta^2$, and

$$F_a = 3m^2+m^4(1+12\xi)-2m^6(1+6\xi),$$
$$G_a = -1+3m^2+4m^4(1+6\xi),$$
$$K_a = 3m^2+2m^4(-2+3\xi)-15m^6(1+6\xi) -6m^8(1+6\xi)^2+2m^{10}(1+6\xi)^2,$$
$$L_a = m^2(7+12\xi)+6m^4(1+9\xi) -9m^6(1+6\xi)-8m^8(1+6\xi)^2. \tag{79}$$

For both chaotic and new inflation conditions under case (b) $\beta^2\simeq\delta^2$, we will define $\kappa^2\xi\phi_{\rm HC}^2 = m^2(\xi)\,\delta^2$. Then the PSRA parameters become

$$\epsilon_b = \frac{8\xi[1+\delta^2]^2}{\delta^2}\,\frac{m^2}{(m^2-1)^2[1+m^2\delta^2(1+6\xi)]},$$
$$\eta_b = \frac{4\xi\,(1+\delta^2)}{\delta^2}\,\frac{F_b}{(m^2-1)^2\,[1+m^2\delta^2(1+6\xi)]^2},$$
$$\zeta_b = \frac{4\sqrt{2}\xi\,(1+\delta^2)}{\delta^2}\,\frac{|K_b|^{1/2}}{|m^2-1|^{3/2}\,[1+m^2\delta^2(1+6\xi)]^2}, \tag{80}$$

with

$$F_b = -1+3m^2(1+\delta^2)+m^4\delta^2((1+12\xi)+4\delta^2(1+6\xi)) -2m^6\delta^4(1+6\xi),$$
$$K_b = m^2(3+\delta^2(7+12\xi)) +2m^4\delta^2((-2+3\xi)+3\delta^2(1+9\xi)) -3m^6\delta^4(1+6\xi)\,(5+3\delta^2) -2m^8\delta^6(1+6\xi)^2\,(3+4\delta^2)+2m^{10}\delta^8(1+6\xi)^2. \tag{81}$$

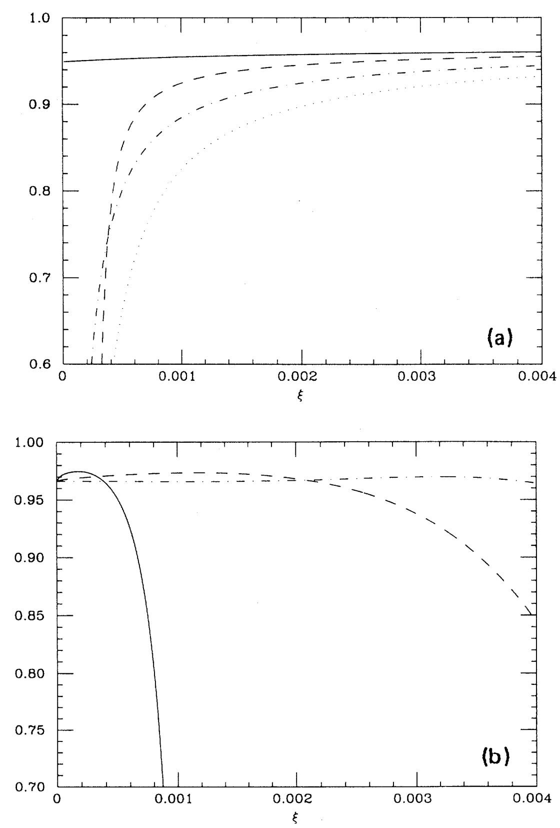

FIG. 3. Second-order results for the spectral index n_s for the model of Sec. V, based on Eqs. (10) and (78)–(82), with $\alpha = 60$. (a) Chaotic inflation initial conditions, with the free parameter $\delta^2 = 10^{-6}$ (solid line), 10^{-1} (dashed line), 1 (dot-dashed line), and 10 (dotted line). (b) New inflation initial conditions, with the free parameter $\delta^2 = 10^{-1}$ (solid line), 1 (dashed line), and 10 (dot-dashed line).

All that remains now is to calculate appropriate values for $m(\xi)$ for each of these cases.

Proceeding as above, $m(\xi)$ becomes, under each of these conditions,

$$m_{\mathrm{ch},\,a} \geq \sqrt{\beta^2 + \frac{8\xi\alpha}{1+6\xi}(1+\delta^2)},$$
$$m_{\mathrm{ch},\,b} \geq \sqrt{1 + \frac{8\xi\alpha}{1+6\xi}\frac{1+\delta^2}{\delta^2}},$$
$$m_{\mathrm{new},\,b} \simeq 1 - \sqrt{\frac{4\xi\alpha}{1+6\xi}\frac{1+\delta^2}{\delta^2}}, \tag{82}$$

where β for $m_{\mathrm{ch},\,a}$ is given by Eq. (64). Because of sufficient inflation constraints in the new inflation case, we have only considered $\xi \ll 1$. The three first-order limiting cases for the spectral index may thus be written

$$n_{s,\,\mathrm{ch},\,a} \simeq 1 - \frac{32\xi}{16\xi\alpha - 1},$$
$$n_{s,\,\mathrm{ch},\,b} \simeq 1 - \frac{16\xi(1+\delta^2)}{8\xi\alpha(1+\delta^2)+\delta^2},$$
$$n_{s,\,\mathrm{new},\,b} \simeq 1 - 8\xi\frac{1+\delta^2}{\delta^2}. \tag{83}$$

The full second-order results for various values of δ^2 are shown in Figs. 3(a) and 3(b). As for the other models, $0.90 \leq n_s \leq 0.97$ for many regions of allowed parameter space, yielding a spectral index in close agreement with observed values.

VI. CONCLUSIONS

The three closely related GET models of inflation considered above all predict values of n_s close to the observed, nearly scale-invariant spectrum of perturbations. For the quasi-power-law cases (new inflation initial conditions), the spectral index varies roughly linearly with the nonminimal coupling constant ξ, with negative slope. For large values of ξ, then, this negative slope dependence of n_s on ξ could drag the predictions for n_s below the experimentally observed values. Yet sufficient inflation requirements place stringent restrictions on $\xi \ll 1$; if such sufficient inflation requirements can be met, then the resulting spectral index deviates only little from $n_s = 1.00$. In the quasi–de Sitter expansion cases (chaotic inflation initial conditions), n_s again varies roughly linearly with ξ, but with positive slope; n_s thus remains close to $n_s = 1.00$ for most values of ξ. Note that these small deviations of n_s from the Harrison-Zel'dovich spectrum mean that each of the models considered here predicts very small values for the tensor-mode perturbation index n_T and the ratio of tensor to scalar mode amplitudes R: both n_T and R are proportional to ϵ to first order [13], and in each of the cases above, $0 < \epsilon < |\eta| \ll 1$.

Under new inflation initial conditions, the Einstein frame formalism employed here yields different forms of $n_s(\xi)$ from calculations conducted exclusively in the Jordan frame. The physical basis for these discrepancies is discussed in Sec. III B, and is further treated in [18]. However, again owing to the requirements from sufficient inflation, in the allowed regions of ξ space the numerical values for n_s differ negligibly between the two frames. Under chaotic inflation initial conditions, there are no discrepancies between the forms of $n_s(\xi)$ in the two frames.

Each of these models is able to produce acceptable spectra, even though their cousin-model extended inflation cannot, because they avoid *both* of the so-called ω problems which plagued extended inflation [10–12]. First, each of the models considered here exits inflation by slowly rolling towards the vacuum expectation value of its potential, thereby avoiding the strict requirements from bubble nucleation and percolation associated with a first-order phase transition. This means that there is no lower bound on ξ for these models. Second, by exiting inflation, all three of these models *also* exit the GET phase: after inflation, as ϕ settles in to v (or 0, for the $\lambda\phi^4$ model), the coefficient of the Ricci scalar in the action, Eq. (1), becomes the constant $1/(2\kappa_N^2)$. Thus, the second-order phase transition responsible for ending inflation simultaneously delivers the universe into the canonical Einstein-Hilbert gravitational form. Unlike extended inflation, then, present-day tests of Brans-Dicke gravitation versus Einsteinian general relativity place no restrictions on allowed values of ξ during the early Universe.

The approach used in this paper can be generalized further, by choosing a more general form for the GET action, Eq. (1). For example, specifically "stringy" effective actions, which often have the "wrong" sign for the kinetic term in Eq. (1) and different effective scalar potentials [3], can be studied, as can models with more than one scalar field coupled to the Ricci scalar (e.g., [29]). By studying these GET models of inflation with the methods employed here, we may further take advantage of the window on Planck-scale physics offered by the primordial spectrum of density perturbations.

ACKNOWLEDGMENTS

This research was conducted at Dartmouth College. It is a pleasure to thank Joseph Harris and Marcelo Gleiser for their hospitality. I would also like to thank Eric Carlson for many helpful discussions, Andrew Liddle for his comments on an earlier draft of this paper, and Redouane Fakir for supplying a copy of Ref. [17]. This research was supported by the NSF.

[1] See, e.g., S. L. Adler, Rev. Mod. Phys. **54**, 729 (1982).
[2] C. H. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
[3] See, e.g., E. J. Copeland, A. Lahiri, and D. Wands, Phys. Rev. D **50**, 4868 (1994); M. Gasperini, in *Second Paris Cosmology Colloquium*, edited by H. J. deVega and N. Sanchez (World Scientific, Singapore, 1995), and references therein; C. Angelantonj, L. Amendola, M. Litterio, and F. Occhionero, Phys. Rev. D **51**, 1607 (1995); R. Hol-

man, E. W. Kolb, S. L. Vadas, and Y. Wang, *ibid.* **43**, 995 (1991); and M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987), Vol. 2, Sec. 13.2.

[4] See, e.g., P. Candelas and S. Weinberg, in *Modern Kaluza-Klein Theories*, edited by T. Appelquist, A. Chodos, and P. G. O. Freund (Addison-Wesley, Reading, MA, 1987).

[5] T. Hirai and K. Maeda, Astrophys. J. **431**, 6 (1994).

[6] K. Maeda, J. A. Stein-Schabes, and T. Futamase, Phys. Rev. D **39**, 2848 (1989).

[7] See, e.g., K. M. Górski, G. Hinshaw, A. J. Banday, C. L. Bennett, E. L. Wright, A. Kogut, G. F. Smoot, and P. Lubin, Astrophys. J. **430**, L86 (1994); see also E. L. Wright, in *Proceedings of the 1994 CWRU Workshop on CMB Anisotropies Two Years After COBE: Observations, Theory and the Future*, edited by L. M. Krauss (World Scientific, Singapore, 1994); C. L. Bennett, Nucl. Phys. B (Proc. Suppl.) **38**, 415 (1995), and references therein.

[8] A. R. Liddle and D. H. Lyth, Phys. Rep. **231**, 1 (1993).

[9] D. La and P. J. Steinhardt, Phys. Rev. Lett. **62**, 376 (1989); R. Holman, E. W. Kolb, and Y. Wang, Phys. Rev. Lett. **65**, 17 (1990).

[10] A. R. Liddle and D. H. Lyth, in *Texas/PASCOS '92: Relativistic Astrophysics and Particle Cosmology*, Proceedings of the Workshop, Berkeley, California, 1992, edited by C. W. Akerlof and M. A. Srednicki [N.Y. Acad. Sci. **688**, 653 (1993)].

[11] P. J. Steinhardt and F. S. Accetta, Phys. Rev. Lett. **64**, 2740 (1990); J. García-Bellido and M. Quirós, Phys. Lett. B **243**, 45 (1990); R. Crittenden and P. J. Steinhardt, *ibid.* **293**, 32 (1992); P. J. Steinhardt, Class. Quantum Grav. **10**, S33 (1993); E. W. Kolb, Phys. Scr. T **36**, 199 (1991).

[12] D. I. Kaiser, Phys. Lett. B **340**, 23 (1994).

[13] A. R. Liddle, P. Parsons, and J. D. Barrow, Phys. Rev. D **50**, 7222 (1994).

[14] E. D. Stewart and D. H. Lyth, Phys. Lett. B **302**, 171 (1993).

[15] E. W. Kolb and S. L. Vadas, Phys. Rev. D **50**, 2479 (1994); see also M. S. Turner, *ibid.* **48**, 3502 (1993); **48**, 5539 (1993).

[16] See, e.g., A. Berkin and K. Maeda, Phys. Rev. D **44**, 1691 (1991); K. Maeda, *ibid.* **39**, 3159 (1989); T. Futamase and K. Maeda, *ibid.* **39**, 399 (1989).

[17] R. Fakir and S. Habib, Mod. Phys. Lett. A **8**, 2827 (1993).

[18] D. I. Kaiser, "Frame-Independent Calculation of Spectral Indices from Inflation," Report No. HUTP-95/A024, DART-HEP-95/02, astro-ph/9507048 (unpublished).

[19] D. I. Kaiser, Phys. Rev. D **49**, 6347 (1994).

[20] A. H. Guth and B. Jain, Phys. Rev. D **45**, 426 (1992).

[21] D. S. Salopek and J. R. Bond, Phys. Rev. D **42**, 3936 (1990).

[22] B. L. Spokoiny, Phys. Lett. **147B**, 39 (1984); F. S. Accetta, D. J. Zoller, and M. S. Turner, Phys. Rev. D **31**, 3046 (1985); F. Lucchin, S. Matarrese, and M. D. Pollock, Phys. Lett. **167B**, 163 (1986); R. Fakir and W. G. Unruh, Phys. Rev. D **41**, 1792 (1990); J. L. Cervantes-Cota and H. Dehnen, *ibid.* **51**, 395 (1995).

[23] See, e.g., L. F. Abbott and M. B. Wise, Nucl. Phys. B **244**, 541 (1984).

[24] See, e.g., A. H. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982).

[25] R. Fakir, S. Habib, and W. Unruh, Astrophys. J. **394**, 396 (1992); R. Fakir and W. G. Unruh, Phys. Rev. D **41**, 1783 (1990); N. Makino and M. Sasaki, Prog. Theor. Phys. **86**, 103 (1991).

[26] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[27] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London, Ser. A **360**, 117 (1978).

[28] T. S. Bunch and P. C. W. Davies, J. Phys. A **11**, 1315 (1978).

[29] A. Linde, Phys. Rev. D **49**, 748 (1994); A. M. Laycock and A. R. Liddle, *ibid.* **49**, 1827 (1994).