ABSTRACT DYNAMIC PROGRAMMING

3rd Edition

Dimitri P. Bertsekas

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Abstract Dynamic Programming

THIRD EDITION

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WWW: http://www.athenasc.com

Cover design: Dimitri Bertsekas

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Publisher's Cataloging-in-Publication Data

Bertsekas, Dimitri P.
Abstract Dynamic Programming: Third Edition
Includes bibliographical references and index
1. Mathematical Optimization. 2. Dynamic Programming. I. Title.
QA402.5 .B465 2022 519.703 01-75941

ISBN-10: 1-886529-47-7, ISBN-13: 978-1-886529-47-2

ABOUT THE AUTHOR

Dimitri Bertsekas studied Mechanical and Electrical Engineering at the National Technical University of Athens, Greece, and obtained his Ph.D. in system science from the Massachusetts Institute of Technology. He has held faculty positions with the Engineering-Economic Systems Department, Stanford University, and the Electrical Engineering Department of the University of Illinois, Urbana. From 1979 to 2019 he was in the faculty of the Electrical Engineering and Computer Science Department of the Massachusetts Institute of Technology (M.I.T.). In 2019, he joined the School of Computing and Augmented Intelligence at the Arizona State University, Tempe, AZ, as Fulton Professor of Computational Decision Making.

Professor Bertsekas' teaching and research have spanned several fields, including deterministic optimization, dynamic programming and stochastic control, large-scale and distributed computation, artificial intelligence, and data communication networks. He has authored or coauthored numerous research papers and nineteen books, several of which are currently used as textbooks in ASU and MIT classes, including "Dynamic Programming and Optimal Control," "Data Networks," "Introduction to Probability," "Nonlinear Programming," and "Reinforcement Learning and Optimal Control."

Professor Bertsekas was awarded the INFORMS 1997 Prize for Research Excellence in the Interface Between Operations Research and Computer Science for his book "Neuro-Dynamic Programming" (co-authored with John Tsitsiklis), the 2001 AACC John R. Ragazzini Education Award, the 2009 INFORMS Expository Writing Award, the 2014 AACC Richard Bellman Heritage Award, the 2014 INFORMS Khachiyan Prize for Life-Time Accomplishments in Optimization, the 2015 MOS/SIAM George B. Dantzig Prize, and the 2022 IEEE Control Systems Award. In 2018 he shared with his coauthor, John Tsitsiklis, the 2018 INFORMS John von Neumann Theory Prize for the contributions of the research monographs "Parallel and Distributed Computation" and "Neuro-Dynamic Programming." Professor Bertsekas was elected in 2001 to the United States National Academy of Engineering for "pioneering contributions to fundamental research, practice and education of optimization/control theory, and especially its application to data communication networks."

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Contents

Preface of the First Edition

This book aims at a unified and economical development of the core theory and algorithms of total cost sequential decision problems, based on the strong connections of the subject with fixed point theory. The analysis focuses on the abstract mapping that underlies dynamic programming (DP for short) and defines the mathematical character of the associated problem. Our discussion centers on two fundamental properties that this mapping may have: *monotonicity* and (weighted sup-norm) *contraction*. It turns out that the nature of the analytical and algorithmic DP theory is determined primarily by the presence or absence of these two properties, and the rest of the problem's structure is largely inconsequential.

In this book, with some minor exceptions, we will assume that monotonicity holds. Consequently, we organize our treatment around the contraction property, and we focus on four main classes of models:

(a) **Contractive models**, discussed in Chapter 2, which have the richest and strongest theory, and are the benchmark against which the theory of other models is compared. Prominent among these models are discounted stochastic optimal control problems. The development of these models is quite thorough and includes the analysis of recent approximation algorithms for large-scale problems (neuro-dynamic programming, reinforcement learning).

(b) **Semicontractive models**, discussed in Chapter 3 and parts of Chapter 4. The term "semicontractive" is used qualitatively here, to refer to a variety of models where some policies have a regularity/contraction-like property but others do not. A prominent example is stochastic shortest path problems, where one aims to drive the state of a Markov chain to a termination state at minimum expected cost. These models also have a strong theory under certain conditions, often nearly as strong as those of the contractive models.

(c) **Noncontractive models**, discussed in Chapter 4, which rely on just monotonicity. These models are more complex than the preceding ones and much of the theory of the contractive models generalizes in weaker form, if at all. For example, in general the associated Bellman equation need not have a unique solution, the value iteration method may work starting with some functions but not with others, and the policy iteration method may not work at all. Infinite horizon examples of these models are the classical positive and negative DP problems, first analyzed by Dubins and Savage, Blackwell, and

Strauch, which are discussed in various sources. Some new semicontractive models are also discussed in this chapter, further bridging the gap between contractive and noncontractive models.

(d) **Restricted policies and Borel space models**, which are discussed in Chapter 5. These models are motivated in part by the complex measurability questions that arise in mathematically rigorous theories of stochastic optimal control involving continuous probability spaces. Within this context, the admissible policies and DP mapping are restricted to have certain measurability properties, and the analysis of the preceding chapters requires modifications. Restricted policy models are also useful when there is a special class of policies with favorable structure, which is "closed" with respect to the standard DP operations, in the sense that analysis and algorithms can be confined within this class.

We do not consider average cost DP problems, whose character bears a much closer connection to stochastic processes than to total cost problems. We also do not address specific stochastic characteristics underlying the problem, such as for example a Markovian structure. Thus our results apply equally well to Markovian decision problems and to sequential minimax problems. While this makes our development general and a convenient starting point for the further analysis of a variety of different types of problems, it also ignores some of the interesting characteristics of special types of DP problems that require an intricate probabilistic analysis.

Let us describe the research content of the book in summary, deferring a more detailed discussion to the end-of-chapter notes. A large portion of our analysis has been known for a long time, but in a somewhat fragmentary form. In particular, the contractive theory, first developed by Denardo [Den67], has been known for the case of the unweighted sup-norm, but does not cover the important special case of stochastic shortest path problems where all policies are proper. Chapter 2 transcribes this theory to the weighted sup-norm contraction case. Moreover, Chapter 2 develops extensions of the theory to approximate DP, and includes material on asynchronous value iteration (based on the author's work [Ber82], [Ber83]), and asynchronous policy iteration algorithms (based on the author's joint work with Huizhen (Janey) Yu [BeY10a], [BeY10b], [YuB11a]). Most of this material is relatively new, having been presented in the author's recent book [Ber12a] and survey paper [Ber12b], with detailed references given there. The analysis of infinite horizon noncontractive models in Chapter 4 was first given in the author's paper [Ber77], and was also presented in the book by Bertsekas and Shreve [BeS78], which in addition contains much of the material on finite horizon problems, restricted policies models, and Borel space models. These were the starting point and main sources for our development.

The new research presented in this book is primarily on the semicontractive models of Chapter 3 and parts of Chapter 4. Traditionally, the theory of total cost infinite horizon DP has been bordered by two extremes: discounted models, which have a contractive nature, and positive and negative models, which do not have a contractive nature, but rely on an enhanced monotonicity structure (monotone increase and monotone decrease models, or in classical DP terms, positive and negative models). Between these two extremes lies a gray area of problems that are not contractive, and either do not fit into the categories of positive and negative models, or possess additional structure that is not exploited by the theory of these models. Included are stochastic shortest path problems, search problems, linear-quadratic problems, a host of queueing problems, multiplicative and exponential cost models, and others. Together these problems represent an important part of the infinite horizon total cost DP landscape. They possess important theoretical characteristics, not generally available for positive and negative models, such as the uniqueness of solution of Bellman's equation within a subset of interest, and the validity of useful forms of value and policy iteration algorithms.

Our semicontractive models aim to provide a unifying abstract DP structure for problems in this gray area between contractive and noncontractive models. The analysis is motivated in part by stochastic shortest path problems, where there are two types of policies: *proper*, which are the ones that lead to the termination state with probability one from all starting states, and *improper*, which are the ones that are not proper. Proper and improper policies can also be characterized through their Bellman equation mapping: for the former this mapping is a contraction, while for the latter it is not. In our more general semicontractive models, policies are also characterized in terms of their Bellman equation mapping, through a notion of *regularity*, which generalizes the notion of a proper policy and is related to classical notions of asymptotic stability from control theory.

In our development a policy is regular within a certain set if its cost function is the unique asymptotically stable equilibrium (fixed point) of the associated DP mapping within that set. *We assume that some policies are regular while others are not*, and impose various assumptions to ensure that attention can be focused on the regular policies. From an analytical point of view, this brings to bear the theory of fixed points of monotone mappings. From the practical point of view, this allows application to a diverse collection of interesting problems, ranging from stochastic shortest path problems of various kinds, where the regular policies include the proper policies, to linear-quadratic problems, where the regular policies include the stabilizing linear feedback controllers.

The definition of regularity is introduced in Chapter 3, and its theoretical ramifications are explored through extensions of the classical stochastic shortest path and search problems. In Chapter 4, semicontractive models are discussed in the presence of additional monotonicity structure, which brings to bear the properties of positive and negative DP models. With the aid of this structure, the theory of semicontractive models can be strengthened and can be applied to several additional problems, including risk-sensitive/exponential cost problems.

The book has a theoretical research monograph character, but requires a modest mathematical background for all chapters except the last one, essentially a first course in analysis. Of course, prior exposure to DP will definitely be very helpful to provide orientation and context. A few exercises have been included, either to illustrate the theory with examples and counterexamples, or to provide applications and extensions of the theory. Solutions of all the exercises can be found in Appendix D, at the book's internet site

http://www.athenasc.com/abstractdp.html

and at the author's web site

http://web.mit.edu/dimitrib/www/home.html

Additional exercises and other related material may be added to these sites over time.

I would like to express my appreciation to a few colleagues for interactions, recent and old, which have helped shape the form of the book. My collaboration with Steven Shreve on our 1978 book provided the motivation and the background for the material on models with restricted policies and associated measurability questions. My collaboration with John Tsitsiklis on stochastic shortest path problems provided inspiration for the work on semicontractive models. My collaboration with Janey (Huizhen) Yu played an important role in the book's development, and is reflected in our joint work on asynchronous policy iteration, on perturbation models, and on risk-sensitive models. Moreover Janey contributed significantly to the material on semicontractive models with many insightful suggestions. Finally, I am thankful to Mengdi Wang, who went through portions of the book with care, and gave several helpful comments.

Dimitri P. Bertsekas

Spring 2013

Preface to the Second Edition

The second edition aims primarily to amplify the presentation of the semicontractive models of Chapter 3 and Chapter 4, and to supplement it with a broad spectrum of research results that I obtained and published in journals and reports since the first edition was written. As a result, the size of this material more than doubled, and the size of the book increased by about 40%.

In particular, I have thoroughly rewritten Chapter 3, which deals with semicontractive models where stationary regular policies are sufficient. I expanded and streamlined the theoretical framework, and I provided new analyses of a number of shortest path-type applications (deterministic, stochastic, affine monotonic, exponential cost, and robust/minimax), as well as several types of optimal control problems with continuous state space (including linear-quadratic, regulation, and planning problems).

In Chapter 4, I have extended the notion of regularity to nonstationary policies (Section 4.4), aiming to explore the structure of the solution set of Bellman's equation, and the connection of optimality with other structural properties of optimal control problems. As an application, I have discussed in Section 4.5 the relation of optimality with classical notions of stability and controllability in continuous-spaces deterministic optimal control. In Section 4.6, I have similarly extended the notion of a proper policy to continuous-spaces stochastic shortest path problems.

I have also revised Chapter 1 a little (mainly with the addition of Section 1.2.5 on the relation between proximal algorithms and temporal difference methods), added to Chapter 2 some analysis relating to λ-policy iteration and randomized policy iteration algorithms (Section 2.5.3), and I have also added several new exercises (with complete solutions) to Chapters 1-4. Additional material relating to various applications can be found in some of my journal papers, reports, and video lectures on semicontractive models, which are posted at my web site.

In addition to the changes in Chapters 1-4, I have also eliminated from the second edition the analysis that deals with restricted policies (Chapter 5 and Appendix C of the first edition). This analysis is motivated in part by the complex measurability questions that arise in mathematically rigorous theories of stochastic optimal control with Borel state and control spaces. This material is covered in Chapter 6 of the monograph by Bertsekas and Shreve [BeS78], and followup research on the subject has been limited. Thus, I decided to just post Chapter 5 and Appendix C of the first edition at the book's web site (40 pages), and omit them from the second edition. As a result of this choice, the entire book now requires only a modest mathematical background, essentially a first course in analysis and in elementary probability.

The range of applications of dynamic programming has grown enormously in the last 25 years, thanks to the use of approximate simulation-based methods for large and challenging problems. Because approximations are often tied to special characteristics of specific models, their coverage in this book is limited to general discussions in Chapter 1 and to error bounds given in Chapter 2. However, much of the work on approximation methods so far has focused on finite-state discounted, and relatively simple deterministic and stochastic shortest path problems, for which there is solid and robust analytical and algorithmic theory (part of Chapters 2 and 3 in this monograph). As the range of applications becomes broader, I expect that the level of mathematical understanding projected in this book will become essential for the development of effective and reliable solution methods. In particular, much of the new material in this edition deals with infinite-state and/or complex shortest path type-problems, whose approximate solution will require new methodologies that transcend the current state of the art.

Dimitri P. Bertsekas

January 2018

Preface to the Third Edition

The third edition is based on the same theoretical framework as the second edition, but contains two major additions. The first is to highlight the central role of abstract DP methods in the conceptualization of reinforcement learning and approximate DP methods, as described in the author's recent book "Lessons from AlphaZero for Optimal, Model Predictive, and Adaptive Control," Athena Scientific, 2022. The main idea here is that approximation in value space with one-step lookahead amounts to a step of Newton's method for solving the abstract Bellman's equation. This material is included in summary form in view of its strong reliance on abstract DP visualization. Our presentation relies primarily on geometric illustrations rather than mathematical analysis, and is given in Section 1.3.

The second addition is a new Chapter 5 on abstract DP methods for minimax and zero sum game problems, which is based on the author's recent paper [Ber21c]. A primary motivation here is the resolution of some long-standing convergence difficulties of the "natural" policy iteration algorithm, which have been known since the Pollatschek and Avi-Itzhak method [PoA69] for finite-state Markov games. Mathematically, this "natural" algorithm is a form of Newton's method for solving the corresponding Bellman's equation, but Newton's method, contrary to the case of single-player DP problems, is not globally convergent in the case of a minimax problem, because the Bellman operator may have components that are neither convex nor concave. Our approach in Chapter 5 has been to introduce a special type of abstract Bellman operator for minimax problems, and modify the standard PI algorithm along the lines of the asynchronous optimistic PI algorithm of Section 2.6.3, which involves a parametric contraction mapping with a uniform fixed point.

The third edition also contains a number of small corrections and editorial changes. The author wishes to thank the contributions of several colleagues in this regard, and particularly Yuchao Li, who proofread with care large portions of the book.

Dimitri P. Bertsekas

February 2022

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Introduction

Contents

1.1 STRUCTURE OF DYNAMIC PROGRAMMING PROBLEMS

Dynamic programming (DP for short) is the principal method for analysis of a large and diverse class of sequential decision problems. Examples are deterministic and stochastic optimal control problems with a continuous state space, Markov and semi-Markov decision problems with a discrete state space, minimax problems, and sequential zero-sum games. While the nature of these problems may vary widely, their underlying structures turn out to be very similar. In all cases there is an underlying mapping that depends on an associated controlled dynamic system and corresponding cost per stage. This mapping, the DP (or Bellman) operator, provides a compact "mathematical signature" of the problem. It defines the cost function of policies and the optimal cost function, and it provides a convenient shorthand notation for algorithmic description and analysis.

More importantly, the structure of the DP operator defines the mathematical character of the associated problem. The purpose of this book is to provide an analysis of this structure, centering on two fundamental properties: *monotonicity* and (weighted sup-norm) *contraction*. It turns out that the nature of the analytical and algorithmic DP theory is determined primarily by the presence or absence of one or both of these two properties, and the rest of the problem's structure is largely inconsequential.

A Deterministic Optimal Control Example

To illustrate our viewpoint, let us consider a discrete-time deterministic optimal control problem described by a system equation

$$x_{k+1} = f(x_k, u_k), \qquad k = 0, 1, \ldots. \tag{1.1}$$

Here x_k is the state of the system taking values in a set X (the state space), and u_k is the control taking values in a set U (the control space).† At stage k, there is a cost

$$\alpha^k g(x_k, u_k)$$

incurred when u_k is applied at state x_k, where α is a scalar in $(0, 1]$ that has the interpretation of a discount factor when $\alpha < 1$. The controls are chosen as a function of the current state, subject to a constraint that depends on that state. In particular, at state x the control is constrained to take values in a given set $U(x) \subset U$. Thus we are interested in optimization over the set of (nonstationary) policies

$$\Pi = \big\{ \{\mu_0, \mu_1, \ldots\} \mid \mu_k \in \mathcal{M},\ k = 0, 1, \ldots \big\},$$

† Our discussion of this section is somewhat informal, without strict adherence to mathematical notation and rigor. We will introduce a rigorous mathematical framework later.

where $\mathcal{M}$ is the set of functions $\mu : X \mapsto U$ defined by

$$\mathcal{M} = \big\{\mu \mid \mu(x) \in U(x),\ \forall\ x \in X\big\}.$$

The total cost of a policy $\pi = \{\mu_0, \mu_1, \ldots\}$ over an infinite number of stages (an infinite horizon) and starting at an initial state x_0 is the limit superior of the N-step costs

$$J_\pi(x_0) = \limsup_{N\to\infty} \sum_{k=0}^{N-1} \alpha^k g\big(x_k, \mu_k(x_k)\big), \tag{1.2}$$

where the state sequence $\{x_k\}$ is generated by the deterministic system (1.1) under the policy π:

$$x_{k+1} = f\big(x_k, \mu_k(x_k)\big), \quad k = 0, 1, \ldots.$$

(We use limit superior rather than limit to cover the case where the limit does not exist.) The optimal cost function is

$$J^*(x) = \inf_{\pi\in\Pi} J_\pi(x), \qquad x \in X.$$

For any policy $\pi = \{\mu_0, \mu_1, \ldots\}$, consider the policy $\pi_1 = \{\mu_1, \mu_2, \ldots\}$ and write by using Eq. (1.2),

$$J_\pi(x) = g\big(x, \mu_0(x)\big) + \alpha J_{\pi_1}\big(f(x, \mu_0(x))\big).$$

We have for all $x \in X$

$$\begin{aligned} J^*(x) &= \inf_{\pi=\{\mu_0,\pi_1\}\in\Pi} \Big\{ g\big(x, \mu_0(x)\big) + \alpha J_{\pi_1}\big(f(x, \mu_0(x))\big)\Big\} \\ &= \inf_{\mu_0\in\mathcal{M}} \Big\{ g\big(x, \mu_0(x)\big) + \alpha \inf_{\pi_1\in\Pi} J_{\pi_1}\big(f(x, \mu_0(x))\big)\Big\} \\ &= \inf_{\mu_0\in\mathcal{M}} \Big\{ g\big(x, \mu_0(x)\big) + \alpha J^*\big(f(x, \mu_0(x))\big)\Big\}. \end{aligned}$$

The minimization over $\mu_0 \in \mathcal{M}$ can be written as minimization over all $u \in U(x)$, so we can write the preceding equation as

$$J^*(x) = \inf_{u\in U(x)} \Big\{ g(x, u) + \alpha J^*\big(f(x, u)\big)\Big\}, \qquad \forall\ x \in X. \tag{1.3}$$

This equation is an example of *Bellman's equation*, which plays a central role in DP analysis and algorithms. If it can be solved for J^*, an optimal stationary policy $\{\mu^*, \mu^*, \ldots\}$ may typically be obtained by minimization of the right-hand side for each x, i.e.,

$$\mu^*(x) \in \arg \min_{u\in U(x)} \Big\{ g(x, u) + \alpha J^*\big(f(x, u)\big)\Big\}, \qquad \forall\ x \in X. \tag{1.4}$$

We now note that both Eqs. (1.3) and (1.4) can be stated in terms of the expression

$$H(x,u,J) = g(x,u) + \alpha J\big(f(x,u)\big), \qquad x \in X,\ u \in U(x).$$

Defining

$$(T_\mu J)(x) = H\big(x,\mu(x),J\big), \qquad x \in X,$$

and

$$(TJ)(x) = \inf_{u \in U(x)} H(x,u,J), \qquad x \in X,$$

we see that Bellman's equation (1.3) can be written compactly as

$$J^* = TJ^*,$$

i.e., J^* is the fixed point of T, viewed as a mapping from the set of functions on X into itself. Moreover, it can be similarly seen that J_μ, the cost function of the stationary policy $\{\mu,\mu,\ldots\}$, is a fixed point of T_μ. In addition, the optimality condition (1.4) can be stated compactly as

$$T_{\mu^*} J^* = TJ^*.$$

We will see later that additional properties, as well as a variety of algorithms for finding J^* can be stated and analyzed using the mappings T and T_μ.

The mappings T_μ can also be used in the context of DP problems with a finite number of stages (a finite horizon). In particular, for a given policy $\pi = \{\mu_0,\mu_1,\ldots\}$ and a terminal cost $\alpha^N \bar J(x_N)$ for the state x_N at the end of N stages, consider the N-stage cost function

$$J_{\pi,N}(x_0) = \alpha^N \bar J(x_N) + \sum_{k=0}^{N-1} \alpha^k g\big(x_k,\mu_k(x_k)\big). \tag{1.5}$$

Then it can be verified by induction that for all initial states x_0, we have

$$J_{\pi,N}(x_0) = (T_{\mu_0} T_{\mu_1} \cdots T_{\mu_{N-1}} \bar J)(x_0). \tag{1.6}$$

Here $T_{\mu_0} T_{\mu_1} \cdots T_{\mu_{N-1}}$ is the composition of the mappings $T_{\mu_0}, T_{\mu_1}, \ldots T_{\mu_{N-1}}$, i.e., for all J,

$$(T_{\mu_0} T_{\mu_1} J)(x) = \big(T_{\mu_0}(T_{\mu_1} J)\big)(x), \qquad x \in X,$$

and more generally

$$(T_{\mu_0} T_{\mu_1} \cdots T_{\mu_{N-1}} J)(x) = \big(T_{\mu_0}(T_{\mu_1}(\cdots(T_{\mu_{N-1}} J)))\big)(x), \qquad x \in X,$$

(our notational conventions are summarized in Appendix A). Thus the finite horizon cost functions $J_{\pi,N}$ of π can be defined in terms of the mappings T_μ [cf. Eq. (1.6)], and so can the infinite horizon cost function J_π:

$$J_\pi(x) = \limsup_{N \to \infty} (T_{\mu_0} T_{\mu_1} \cdots T_{\mu_{N-1}} \bar J)(x), \qquad x \in X, \tag{1.7}$$

where $\bar J$ is the zero function, $\bar J(x) = 0$ for all $x \in X$.

Connection with Fixed Point Methodology

The Bellman equation (1.3) and the optimality condition (1.4), stated in terms of the mappings T_μ and T, highlight a central theme of this book, which is that DP theory is intimately connected with the theory of abstract mappings and their fixed points. Analogs of the Bellman equation, $J^* = TJ^*$, optimality conditions, and other results and computational methods hold for a great variety of DP models, and can be stated compactly as described above in terms of the corresponding mappings T_μ and T. The gain from this abstraction is greater generality and mathematical insight, as well as a more unified, economical, and streamlined analysis.

1.2 ABSTRACT DYNAMIC PROGRAMMING MODELS

In this section we formally introduce and illustrate with examples an abstract DP model, which embodies the ideas just discussed in Section 1.1.

1.2.1 Problem Formulation

Let X and U be two sets, which we loosely refer to as a set of "states" and a set of "controls," respectively. For each $x \in X$, let $U(x) \subset U$ be a nonempty subset of controls that are feasible at state x. We denote by $\mathcal{M}$ the set of all functions $\mu : X \mapsto U$ with $\mu(x) \in U(x)$, for all $x \in X$.

In analogy with DP, we refer to sequences $\pi = \{\mu_0, \mu_1, \ldots\}$, with $\mu_k \in \mathcal{M}$ for all k, as "nonstationary policies," and we refer to a sequence $\{\mu, \mu, \ldots\}$, with $\mu \in \mathcal{M}$, as a "stationary policy." In our development, stationary policies will play a dominant role, and with slight abuse of terminology, we will also refer to any $\mu \in \mathcal{M}$ as a "policy" when confusion cannot arise.

Let $\mathcal{R}(X)$ be the set of real-valued functions $J : X \mapsto \Re$, and let $H : X \times U \times \mathcal{R}(X) \mapsto \Re$ be a given mapping.† For each policy $\mu \in \mathcal{M}$, we consider the mapping $T_\mu : \mathcal{R}(X) \mapsto \mathcal{R}(X)$ defined by

$$(T_\mu J)(x) = H\big(x, \mu(x), J\big), \qquad \forall\, x \in X,\ J \in \mathcal{R}(X),$$

and we also consider the mapping T defined by ‡

$$(TJ)(x) = \inf_{u \in U(x)} H(x, u, J), \qquad \forall\, x \in X,\ J \in \mathcal{R}(X).$$

† Our notation and mathematical conventions are outlined in Appendix A. In particular, we denote by $\Re$ the set of real numbers, and by $\Re^n$ the space of n-dimensional vectors with real components.

‡ We assume that H, $T_\mu J$, and TJ are real-valued for $J \in \mathcal{R}(X)$ in the present chapter and in Chapter 2. In Chapters 3 and 4 we will allow $H(x, u, J)$, and hence also $(T_\mu J)(x)$ and $(TJ)(x)$, to take the values ∞ and $-\infty$.

We will generally refer to T and T_μ as the (abstract) *DP mappings* or *DP operators* or *Bellman operators* (the latter name is common in the artificial intelligence and reinforcement learning literature).

Similar to the deterministic optimal control problem of the preceding section, the mappings T_μ and T serve to define a multistage optimization problem and a DP-like methodology for its solution. In particular, for some function $\bar J \in \mathcal{R}(X)$, and nonstationary policy $\pi = \{\mu_0, \mu_1, \ldots\}$, we define for each integer $N \geq 1$ the functions

$$J_{\pi,N}(x) = (T_{\mu_0} T_{\mu_1} \cdots T_{\mu_{N-1}} \bar J)(x), \qquad x \in X,$$

where $T_{\mu_0} T_{\mu_1} \cdots T_{\mu_{N-1}}$ denotes the composition of the mappings T_{μ_0}, T_{μ_1}, $\ldots$, $T_{\mu_{N-1}}$, i.e.,

$$T_{\mu_0} T_{\mu_1} \cdots T_{\mu_{N-1}} J = \big(T_{\mu_0}(T_{\mu_1}(\cdots (T_{\mu_{N-2}}(T_{\mu_{N-1}} J))) \cdots)\big), \quad J \in \mathcal{R}(X).$$

We view $J_{\pi,N}$ as the "N-stage cost function" of π [cf. Eq. (1.5)]. Consider also the function

$$J_\pi(x) = \limsup_{N\to\infty} J_{\pi,N}(x) = \limsup_{N\to\infty} (T_{\mu_0} T_{\mu_1} \cdots T_{\mu_{N-1}} \bar J)(x), \qquad x \in X,$$

which we view as the "infinite horizon cost function" of π [cf. Eq. (1.7); we use lim sup for generality, since we are not assured that the limit exists]. We want to minimize J_π over π, i.e., to find

$$J^*(x) = \inf_\pi J_\pi(x), \qquad x \in X,$$

and a policy π^* that attains the infimum, if one exists.

The key connection with fixed point methodology is that J^* "typically" (under mild assumptions) can be shown to satisfy

$$J^*(x) = \inf_{u \in U(x)} H(x, u, J^*), \qquad \forall\, x \in X,$$

i.e., it is a fixed point of T. We refer to this as *Bellman's equation* [cf. Eq. (1.3)]. Another fact is that if an optimal policy π^* exists, it "typically" can be selected to be stationary, $\pi^* = \{\mu^*, \mu^*, \ldots\}$, with $\mu^* \in \mathcal{M}$ satisfying an optimality condition, such as for example

$$(T_{\mu^*} J^*)(x) = (T J^*)(x), \qquad x \in X,$$

[cf. Eq. (1.4)]. Several other results of an analytical or algorithmic nature also hold under appropriate conditions, which will be discussed in detail later.

However, Bellman's equation and other related results may not hold without T_μ and T having some special structural properties. Prominent among these are a monotonicity assumption that typically holds in DP problems, and a contraction assumption that holds for some important classes of problems. We describe these assumptions next.

1.2.2 Monotonicity and Contraction Properties

Let us now formalize the monotonicity and contraction assumptions. We will require that both of these assumptions hold for most of the next chapter, and we will gradually relax the contraction assumption in Chapters 3 and 4. Recall also our assumption that T_μ and T map $\mathcal{R}(X)$ (the space of real-valued functions over X) into $\mathcal{R}(X)$. In Chapters 3 and 4 we will relax this assumption as well.

Assumption 1.2.1: (Monotonicity) If $J, J' \in \mathcal{R}(X)$ and $J \leq J'$, then

$$H(x,u,J) \leq H(x,u,J'), \qquad \forall\, x \in X,\ u \in U(x).$$

Note that by taking infimum over $u \in U(x)$, we have

$$J(x) \leq J'(x),\ \ \forall\, x \in X \quad \Rightarrow \quad \inf_{u\in U(x)} H(x,u,J) \leq \inf_{u\in U(x)} H(x,u,J'),\ \ \forall\, x \in X,$$

or equivalently,†

$$J \leq J' \qquad \Rightarrow \qquad TJ \leq TJ'.$$

Another way to arrive at this relation, is to note that the monotonicity assumption is equivalent to

$$J \leq J' \quad \Rightarrow \quad T_\mu J \leq T_\mu J', \qquad \forall\, \mu \in \mathcal{M},$$

and to use the simple but important fact

$$\inf_{u\in U(x)} H(x,u,J) = \inf_{\mu\in\mathcal{M}} (T_\mu J)(x), \qquad \forall\, x \in X,\ J \in \mathcal{R}(X),$$

i.e., for a fixed $x \in X$, *infimum over u is equivalent to infimum over μ*. This is true because for any μ, there is no coupling constraint between the controls $\mu(x)$ and $\mu(x')$ that correspond to two different states x and x', i.e., the set $\mathcal{M} = \big\{\mu \mid \mu(x) \in U(x),\, \forall\, x \in X\big\}$ can be viewed as the Cartesian product $\Pi_{x\in X} U(x)$. We will be writing this relation as $TJ = \inf_{\mu\in\mathcal{M}} T_\mu J$.

For the contraction assumption, we introduce a function $v : X \mapsto \Re$ with

$$v(x) > 0, \qquad \forall\, x \in X.$$

† Unless otherwise stated, in this book, inequalities involving functions, minima and infima of a collection of functions, and limits of function sequences are meant to be pointwise; see Appendix A for our notational conventions.

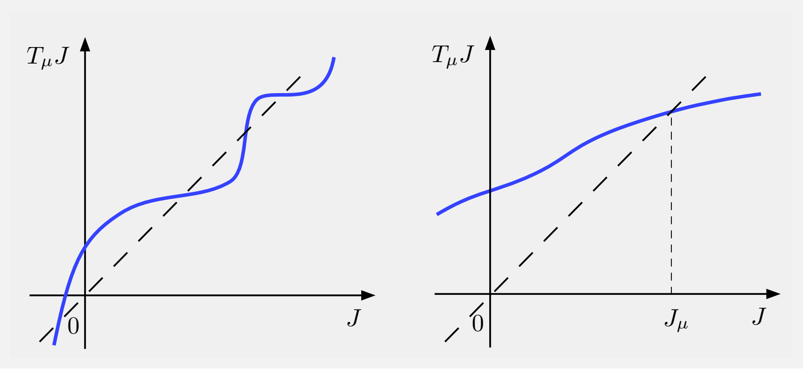


Figure 1.2.1. Illustration of the monotonicity and the contraction assumptions in one dimension. The mapping T_μ on the left is monotone but is not a contraction. The mapping T_μ on the right is both monotone and a contraction. It has a unique fixed point at J_μ.

Let us denote by $\mathcal{B}(X)$ the space of real-valued functions J on X such that $J(x)/v(x)$ is bounded as x ranges over X, and consider the weighted sup-norm

$$\|J\| = \sup_{x\in X} \frac{\big|J(x)\big|}{v(x)}$$

on $\mathcal{B}(X)$. The properties of $\mathcal{B}(X)$ and some of the associated fixed point theory are discussed in Appendix B. In particular, as shown there, $\mathcal{B}(X)$ is a complete normed space, so any mapping from $\mathcal{B}(X)$ to $\mathcal{B}(X)$ that is a contraction or an m-stage contraction for some integer $m > 1$, with respect to $\|\cdot\|$, has a unique fixed point (cf. Props. B.1 and B.2).

Assumption 1.2.2: (Contraction) For all $J \in \mathcal{B}(X)$ and $\mu \in \mathcal{M}$, the functions $T_\mu J$ and TJ belong to $\mathcal{B}(X)$. Furthermore, for some $\alpha \in (0,1)$, we have

$$\|T_\mu J - T_\mu J'\| \le \alpha\|J - J'\|, \qquad \forall\ J, J' \in \mathcal{B}(X),\ \mu \in \mathcal{M}. \qquad (1.8)$$

Figure 1.2.1 illustrates the monotonicity and the contraction assumptions. It can be shown that the contraction condition (1.8) implies that

$$\|TJ - TJ'\| \le \alpha\|J - J'\|, \qquad \forall\ J, J' \in \mathcal{B}(X), \qquad (1.9)$$

so that T is also a contraction with modulus α. To see this we use Eq. (1.8) to write

$$(T_\mu J)(x) \le (T_\mu J')(x) + \alpha\|J - J'\|\, v(x), \qquad \forall\ x \in X,$$

from which, by taking infimum of both sides over $\mu \in \mathcal{M}$, we have

$$\frac{(TJ)(x) - (TJ')(x)}{v(x)} \le \alpha \|J - J'\|, \qquad \forall\, x \in X.$$

Reversing the roles of J and J', we also have

$$\frac{(TJ')(x) - (TJ)(x)}{v(x)} \le \alpha \|J - J'\|, \qquad \forall\, x \in X,$$

and combining the preceding two relations, and taking the supremum of the left side over $x \in X$, we obtain Eq. (1.9).

Nearly all mappings related to DP satisfy the monotonicity assumption, and many important ones satisfy the weighted sup-norm contraction assumption as well. When both assumptions hold, the most powerful analytical and computational results can be obtained, as we will show in Chapter 2. These are:

(a) Bellman's equation has a unique solution, i.e., T and T_μ have unique fixed points, which are the optimal cost function J^* and the cost functions J_μ of the stationary policies $\{\mu, \mu, \ldots\}$, respectively [cf. Eq. (1.3)].

(b) A stationary policy $\{\mu^*, \mu^*, \ldots\}$ is optimal if and only if

$$T_{\mu^*} J^* = TJ^*,$$

[cf. Eq. (1.4)].

(c) J^* and J_μ can be computed by the *value iteration* method,

$$J^* = \lim_{k\to\infty} T^k J, \qquad J_\mu = \lim_{k\to\infty} T_\mu^k J,$$

starting with any $J \in \mathcal{B}(X)$.

(d) J^* can be computed by the *policy iteration* method, whereby we generate a sequence of stationary policies via

$$T_{\mu^{k+1}} J_{\mu^k} = TJ_{\mu^k},$$

starting from some initial policy μ^0 [here J_{μ^k} is obtained as the fixed point of T_{μ^k} by several possible methods, including value iteration as in (c) above].

These are the most favorable types of results one can hope for in the DP context, and they are supplemented by a host of other results, involving approximate and/or asynchronous implementations of the value and policy iteration methods, and other related methods that combine features of both. As the contraction property is relaxed and is replaced by various weaker assumptions, some of the preceding results may hold in weaker form. For example J^* turns out to be a solution of Bellman's equation in most of the models to be discussed, but it may not be the unique solution. The interplay between the monotonicity and contraction-like properties, and the associated results of the form (a)-(d) described above is a recurring analytical theme in this book.

1.2.3 Some Examples

In what follows in this section, we describe a few special cases, which indicate the connections of appropriate forms of the mapping H with the most popular total cost DP models. In all these models the monotonicity Assumption 1.2.1 (or some closely related version) holds, but the contraction Assumption 1.2.2 may not hold, as we will indicate later. Our descriptions are by necessity brief, and the reader is referred to the relevant textbook literature for more detailed discussion.

Example 1.2.1 (Stochastic Optimal Control - Markovian Decision Problems)

Consider the stationary discrete-time dynamic system

$$x_{k+1} = f(x_k, u_k, w_k), \qquad k = 0, 1, \ldots, \tag{1.10}$$

where for all k, the state x_k is an element of a space X, the control u_k is an element of a space U, and w_k is a random "disturbance," an element of a space W. We consider problems with infinite state and control spaces, as well as problems with discrete (finite or countable) state space (in which case the underlying system is a Markov chain). However, for technical reasons that relate to measure-theoretic issues, *we assume that W is a countable set.*

The control u_k is constrained to take values in a given nonempty subset $U(x_k)$ of U, which depends on the current state x_k [$u_k \in U(x_k)$, for all $x_k \in X$]. The random disturbances w_k, $k = 0, 1, \ldots$, are characterized by probability distributions $P(\cdot \mid x_k, u_k)$ that are identical for all k, where $P(w_k \mid x_k, u_k)$ is the probability of occurrence of w_k, when the current state and control are x_k and u_k, respectively. Thus the probability of w_k may depend explicitly on x_k and u_k, but not on values of prior disturbances $w_{k-1}, \ldots, w_0$.

Given an initial state x_0, we want to find a policy $\pi = \{\mu_0, \mu_1, \ldots\}$, where $\mu_k : X \mapsto U$, $\mu_k(x_k) \in U(x_k)$, for all $x_k \in X$, $k = 0, 1, \ldots$, that minimizes the cost function

$$J_\pi(x_0) = \limsup_{N\to\infty} \mathop{E}_{\substack{w_k \\ k=0,1,\ldots}} \left\{ \sum_{k=0}^{N-1} \alpha^k g\big(x_k, \mu_k(x_k), w_k\big) \right\}, \tag{1.11}$$

where $\alpha \in (0, 1]$ is a discount factor, subject to the system equation constraint

$$x_{k+1} = f\big(x_k, \mu_k(x_k), w_k\big), \qquad k = 0, 1, \ldots.$$

This is a classical problem, which is discussed extensively in various sources, including the author's text [Ber12a]. It is usually referred to as the *stochastic optimal control problem* or the *Markovian Decision Problem* (MDP for short).

Note that the expected value of the N-stage cost of π,

$$\mathop{E}_{\substack{w_k \\ k=0,1,\ldots}} \left\{ \sum_{k=0}^{N-1} \alpha^k g\big(x_k, \mu_k(x_k), w_k\big) \right\},$$

is defined as a (possibly countably infinite) sum, since the disturbances w_k, $k = 0, 1, \ldots$, take values in a countable set. Indeed, the reader may verify that all the subsequent mathematical expressions that involve an expected value can be written as summations over a finite or a countable set, so they make sense without resort to measure-theoretic integration concepts.†

In what follows we will often impose appropriate assumptions on the cost per stage g and the scalar α, which guarantee that the infinite horizon cost $J_\pi(x_0)$ is defined as a limit (rather than as a lim sup):

$$J_\pi(x_0) = \lim_{N\to\infty} \mathop{E}_{\substack{w_k\\ k=0,1,\ldots}} \left\{ \sum_{k=0}^{N-1} \alpha^k g\big(x_k, \mu_k(x_k), w_k\big) \right\}.$$

In particular, it can be shown that the limit exists if $\alpha < 1$ and the expected value of $|g|$ is uniformly bounded, i.e., for some $B > 0$,

$$E\big\{\big|g(x,u,w)\big|\big\} \le B, \qquad \forall\ x \in X,\ u \in U(x). \tag{1.12}$$

In this case, we obtain the classical discounted infinite horizon DP problem, which generally has the most favorable structure of all infinite horizon stochastic DP models (see [Ber12a], Chapters 1 and 2).

To make the connection with abstract DP, let us define

$$H(x,u,J) = E\big\{g(x,u,w) + \alpha J\big(f(x,u,w)\big)\big\},$$

so that

$$(T_\mu J)(x) = E\Big\{g\big(x,\mu(x),w\big) + \alpha J\big(f(x,\mu(x),w)\big)\Big\},$$

and

$$(TJ)(x) = \inf_{u\in U(x)} E\big\{g(x,u,w) + \alpha J\big(f(x,u,w)\big)\big\}.$$

Similar to the deterministic optimal control problem of Section 1.1, the N-stage cost of π, can be expressed in terms of T_μ:

$$(T_{\mu_0}\cdots T_{\mu_{N-1}}\bar J)(x_0) = \mathop{E}_{\substack{w_k\\ k=0,1,\ldots}} \left\{ \sum_{k=0}^{N-1} \alpha^k g\big(x_k, \mu_k(x_k), w_k\big) \right\},$$

† As noted in Appendix A, the formula for the expected value of a random variable w defined over a space Ω is

$$E\{w\} = E\{w^+\} + E\{w^-\},$$

where w^+ and w^- are the positive and negative parts of w,

$$w^+(\omega) = \max\big\{0, w(\omega)\big\}, \qquad w^-(\omega) = \min\big\{0, w(\omega)\big\}, \qquad \forall\ \omega \in \Omega.$$

In this way, taking also into account the rule $\infty - \infty = \infty$ (see Appendix A), $E\{w\}$ is well-defined as an extended real number if Ω is finite or countably infinite.

where $\bar{J}$ is the zero function, $\bar{J}(x) = 0$ for all $x \in X$. The same is true for the infinite-stage cost [cf. Eq. (1.11)]:

$$J_\pi(x_0) = \limsup_{N\to\infty} (T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J})(x_0).$$

It can be seen that the mappings T_μ and T are monotone, and it is well-known that if $\alpha < 1$ and the boundedness condition (1.12) holds, they are contractive as well (under the unweighted sup-norm); see e.g., [Ber12a], Chapter 1. In this case, the model has the powerful analytical and algorithmic properties (a)-(d) mentioned at the end of the preceding subsection. In particular, the optimal cost function J^* [i.e., $J^*(x) = \inf_\pi J_\pi(x)$ for all $x \in X$] can be shown to be the unique solution of the fixed point equation $J^* = TJ^*$, also known as Bellman's equation, which has the form

$$J^*(x) = \inf_{u\in U(x)} E\Big\{g(x,u,w) + \alpha J^*\big(f(x,u,w)\big)\Big\}, \qquad x \in X,$$

and parallels the one given for deterministic optimal control problems [cf. Eq. (1.3)].

These properties can be expressed and analyzed in an abstract setting by using just the mappings T_μ and T, both when T_μ and T are contractive (see Chapter 2), and when they are only monotone and not contractive while either $g \geq 0$ or $g \leq 0$ (see Chapter 4). Moreover, under some conditions, it is possible to analyze these properties in cases where T_μ is contractive for some but not all μ (see Chapter 3, and Section 4.4).

Example 1.2.2 (Finite-State Discounted Markovian Decision Problems)

In the special case of the preceding example where the number of states is finite, the system equation (1.10) may be defined in terms of the transition probabilities

$$p_{xy}(u) = \text{Prob}\big(y = f(x,u,w) \mid x\big), \qquad x, y \in X,\ u \in U(x),$$

so H takes the form

$$H(x,u,J) = \sum_{y\in X} p_{xy}(u)\big(g(x,u,y) + \alpha J(y)\big).$$

When $\alpha < 1$ and the boundedness condition

$$\big|g(x,u,y)\big| \leq B, \qquad \forall\ x, y \in X,\ u \in U(x),$$

[cf. Eq. (1.12)] holds (or more simply, when U is a finite set), the mappings T_μ and T are contraction mappings with respect to the standard (unweighted) sup-norm. This is a classical model, referred to as *discounted finite-state MDP*, which has a favorable theory and has found extensive applications (cf. [Ber12a], Chapters 1 and 2). The model is additionally important, because it is often used for computational solution of continuous state space problems via discretization.

Example 1.2.3 (Discounted Semi-Markov Problems)

With x, y, and u as in Example 1.2.2, consider a mapping of the form

$$H(x,u,J) = G(x,u) + \sum_{y \in X} m_{xy}(u) J(y),$$

where G is some function representing expected cost per stage, and $m_{xy}(u)$ are nonnegative scalars with

$$\sum_{y \in X} m_{xy}(u) < 1, \qquad \forall\ x \in X,\ u \in U(x).$$

The equation $J^* = TJ^*$ is Bellman's equation for a finite-state continuous-time semi-Markov decision problem, after it is converted into an equivalent discrete-time problem (cf. [Ber12a], Section 1.4). Again, the mappings T_μ and T are monotone and can be shown to be contraction mappings with respect to the unweighted sup-norm.

Example 1.2.4 (Discounted Zero-Sum Dynamic Games)

Let us consider a zero-sum game analog of the finite-state MDP Example 1.2.2. Here there are two players that choose actions at each stage: the first (called the *minimizer*) may choose a move i out of n moves and the second (called the *maximizer*) may choose a move j out of m moves. Then the minimizer gives a specified amount a_{ij} to the maximizer, called a *payoff*. The minimizer wishes to minimize a_{ij}, and the maximizer wishes to maximize a_{ij}.

The players use mixed strategies, whereby the minimizer selects a probability distribution $u = (u_1, \ldots, u_n)$ over his n possible moves and the maximizer selects a probability distribution $v = (v_1, \ldots, v_m)$ over his m possible moves. Thus the probability of selecting i and j is $u_i v_j$, and the expected payoff for this stage is $\sum_{i,j} a_{ij} u_i v_j$ or $u'Av$, where A is the $n \times m$ matrix with components a_{ij}.

In a single-stage version of the game, the minimizer must minimize $\max_{v \in V} u'Av$ and the maximizer must maximize $\min_{u \in U} u'Av$, where U and V are the sets of probability distributions over $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively. A fundamental result (which will not be proved here) is that these two values are equal:

$$\min_{u \in U} \max_{v \in V} u'Av = \max_{v \in V} \min_{u \in U} u'Av. \tag{1.13}$$

Let us consider the situation where a separate game of the type just described is played at each stage. The game played at a given stage is represented by a "state" x that takes values in a finite set X. The state evolves according to transition probabilities $q_{xy}(i,j)$ where i and j are the moves selected by the minimizer and the maximizer, respectively (here y represents the next game to be played after moves i and j are chosen at the game represented by x). When the state is x, under $u \in U$ and $v \in V$, the one-stage expected payoff is $u'A(x)v$, where $A(x)$ is the $n \times m$ payoff matrix, and the state transition probabilities are

$$p_{xy}(u,v) = \sum_{i=1}^{n} \sum_{j=1}^{m} u_i v_j q_{xy}(i,j) = u' Q_{xy} v,$$

where Q_{xy} is the $n \times m$ matrix that has components $q_{xy}(i,j)$. Payoffs are discounted by $\alpha \in (0,1)$, and the objectives of the minimizer and maximizer, roughly speaking, are to minimize and to maximize the total discounted expected payoff. This requires selections of u and v to strike a balance between obtaining favorable current stage payoffs and playing favorable games in future stages.

We now introduce an abstract DP framework related to the sequential move selection process just described. We consider the mapping G given by

$$\begin{aligned} G(x,u,v,J) &= u'A(x)v + \alpha \sum_{y \in X} p_{xy}(u,v) J(y) \\ &= u' \left(A(x) + \alpha \sum_{y \in X} Q_{xy} J(y) \right) v, \end{aligned} \tag{1.14}$$

where $\alpha \in (0,1)$ is discount factor, and the mapping H given by

$$H(x,u,J) = \max_{v \in V} G(x,u,v,J).$$

The corresponding mappings T_μ and T are

$$(T_\mu J)(x) = \max_{v \in V} G\big(x,\mu(x),v,J\big), \qquad x \in X,$$

and

$$(TJ)(x) = \min_{u \in U} \max_{v \in V} G(x,u,v,J).$$

It can be shown that T_μ and T are monotone and (unweighted) sup-norm contractions. Moreover, the unique fixed point J^* of T satisfies

$$J^*(x) = \min_{u \in U} \max_{v \in V} G(x,u,v,J^*), \qquad \forall\ x \in X,$$

(see [Ber12a], Section 1.6.2).

We now note that since

$$A(x) + \alpha \sum_{y \in X} Q_{xy} J(y)$$

[cf. Eq. (1.14)] is a matrix that is independent of u and v, we may view $J^*(x)$ as the value of a static game (which depends on the state x). In particular, from the fundamental minimax equality (1.13), we have

$$\min_{u \in U} \max_{v \in V} G(x,u,v,J^*) = \max_{v \in V} \min_{u \in U} G(x,u,v,J^*), \qquad \forall\ x \in X.$$

This implies that J^* is also the unique fixed point of the mapping

$$(\overline{T}J)(x) = \max_{v \in V} \overline{H}(x, v, J),$$

where

$$\overline{H}(x, v, J) = \min_{u \in U} G(x, u, v, J),$$

i.e., J^* is the fixed point regardless of the order in which minimizer and maximizer select mixed strategies at each stage.

In the preceding development, we have introduced J^* as the unique fixed point of the mappings T and $\overline{T}$. However, J^* also has an interpretation in game theoretic terms. In particular, it can be shown that $J^*(x)$ is the value of a dynamic game, whereby at state x the two opponents choose multistage (possibly nonstationary) policies that consist of functions of the current state, and continue to select moves using these policies over an infinite horizon. For further discussion of this interpretation, we refer to [Ber12a] and to books on dynamic games such as [FiV96]; see also [PaB99] and [Yu14] for an analysis of the undiscounted case ($\alpha = 1$) where there is a termination state, as in the stochastic shortest path problems of the subsequent Example 1.2.6. An alternative and more general formulation of sequential zero-sum games, which allows for an infinite state space, will be given in Chapter 5.

Example 1.2.5 (Minimax Problems)

Consider a minimax version of Example 1.2.1, where w is not random but is rather chosen from within a set $W(x, u)$ by an antagonistic opponent. Let

$$H(x, u, J) = \sup_{w \in W(x,u)} \Big[g(x, u, w) + \alpha J\big(f(x, u, w)\big) \Big].$$

Then the equation $J^* = TJ^*$ is Bellman's equation for an infinite horizon minimax DP problem. A special case of this mapping arises in zero-sum dynamic games (cf. Example 1.2.4). We will also discuss alternative and more general abstract DP formulations of minimax problems in Chapter 5.

Example 1.2.6 (Stochastic Shortest Path Problems)

The stochastic shortest path (SSP for short) problem is the special case of the stochastic optimal control Example 1.2.1 where:

(a) There is no discounting ($\alpha = 1$).

(b) The state space is $X = \{t, 1, \ldots, n\}$ and we are given transition probabilities, denoted by

$$p_{xy}(u) = P(x_{k+1} = y \mid x_k = x, u_k = u), \qquad x, y \in X,\ u \in U(x).$$

(c) The control constraint set $U(x)$ is finite for all $x \in X$.

(d) A cost $g(x,u)$ is incurred when control $u \in U(x)$ is selected at state x.

(e) State t is a special termination state, which is cost-free and absorbing, i.e., for all $u \in U(t)$,

$$g(t,u) = 0, \qquad p_{tt}(u) = 1.$$

To simplify the notation, we have assumed that the cost per stage does not depend on the successor state, which amounts to using expected cost per stage in all calculations.

Since the termination state t is cost-free, the cost starting from t is zero for every policy. Accordingly, for all cost functions, we ignore the component that corresponds to t, and define

$$H(x,u,J) = g(x,u) + \sum_{y=1}^{n} p_{xy}(u)J(y), \quad x = 1,\ldots,n,\ u \in U(x),\ J \in \Re^n.$$

The mappings T_μ and T are defined by

$$(T_\mu J)(x) = g\big(x,\mu(x)\big) + \sum_{y=1}^{n} p_{xy}\big(\mu(x)\big)J(y), \qquad x = 1,\ldots,n,$$

$$(TJ)(x) = \min_{u \in U(x)} \left[g(x,u) + \sum_{y=1}^{n} p_{xy}(u)J(y) \right], \qquad x = 1,\ldots,n.$$

Note that the matrix that has components $p_{xy}(u)$, $x,y = 1,\ldots,n$, is substochastic (some of its row sums may be less than 1) because there may be a positive transition probability from a state x to the termination state t. Consequently T_μ may be a contraction for some μ, but not necessarily for all $\mu \in \mathcal{M}$.

The SSP problem has been discussed in many sources, including the books [Pal67], [Der70], [Whi82], [Ber87], [BeT89], [HeL99], [Ber12a], and [Ber17a], where it is sometimes referred to by earlier names such as "first passage problem" and "transient programming problem." In the framework that is most relevant to our purposes, given in the paper by Bertsekas and Tsitsiklis [BeT91], there is a classification of stationary policies for SSP into *proper* and *improper*. We say that $\mu \in \mathcal{M}$ is proper if, when using μ, there is positive probability that termination will be reached after at most n stages, regardless of the initial state; i.e., if

$$\rho_\mu = \max_{x=1,\ldots,n} P\{x_n \neq 0 \mid x_0 = x, \mu\} < 1.$$

Otherwise, we say that μ is improper. It can be seen that μ is proper if and only if in the Markov chain corresponding to μ, each state x is connected to the termination state with a path of positive probability transitions.

For a proper policy μ, it can be shown that T_μ is a weighted sup-norm contraction, as well as an n-stage contraction with respect to the unweighted sup-norm. For an improper policy μ, T_μ is not a contraction with respect to any norm. Moreover, T also need not be a contraction with respect to any norm (think of the case where there is only one policy, which is improper). However, T is a weighted sup-norm contraction in the important special case where all policies are proper (see [BeT96], Prop. 2.2, or [Ber12a], Chapter 3).

Nonetheless, even in the case where there are improper policies and T is not a contraction, results comparable to the case of discounted finite-state MDP are available for SSP problems assuming that:

(a) There exists at least one proper policy.

(b) For every improper policy there is an initial state that has infinite cost under this policy.

Under the preceding two assumptions, referred to as the *strong SSP conditions* in Section 3.5.1, it was shown in [BeT91] that T has a unique fixed point J^*, the optimal cost function of the SSP problem. Moreover, a policy $\{\mu^*, \mu^*, \ldots\}$ is optimal if and only if

$$T_{\mu^*} J^* = T J^*.$$

In addition, J^* and J_μ can be computed by value iteration,

$$J^* = \lim_{k\to\infty} T^k J, \qquad J_\mu = \lim_{k\to\infty} T_\mu^k J,$$

starting with any $J \in \Re^n$ (see [Ber12a], Chapter 3, for a textbook account). These properties are in analogy with the desirable properties (a)-(c), given at the end of the preceding subsection in connection with contractive models.

Regarding policy iteration, it works in its strongest form when there are no improper policies, in which case the mappings T_μ and T are weighted sup-norm contractions. When there are improper policies, modifications to the policy iteration method are needed; see [Ber12a], [YuB13a], and also Section 3.6.2, where these modifications will be discussed in an abstract setting.

In Section 3.5.1 we will also consider SSP problems where the strong SSP conditions (a) and (b) above are not satisfied. Then we will see that unusual phenomena can occur, including that J^* may not be a solution of Bellman's equation. Still our line of analysis of Chapter 3 will apply to such problems.

Example 1.2.7 (Deterministic Shortest Path Problems)

The special case of the SSP problem where the state transitions are deterministic is the classical shortest path problem. Here, we have a graph of n nodes $x = 1, \ldots, n$, plus a destination t, and an arc length a_{xy} for each directed arc (x, y). At state/node x, a policy μ chooses an outgoing arc from x. Thus the controls available at x can be identified with the outgoing neighbors of x [the nodes u such that (x, u) is an arc]. The corresponding mapping H is

$$H(x,u,J) = \begin{cases} a_{xu} + J(u) & \text{if } u \neq t, \\ a_{xt} & \text{if } u = t, \end{cases} \qquad x = 1, \ldots, n.$$

A stationary policy μ defines a graph whose arcs are $\big(x, \mu(x)\big)$, $x = 1, \ldots, n$. The policy μ is proper if and only if this graph is acyclic (it consists of a tree of directed paths leading from each node to the destination). Thus there exists a proper policy if and only if each node is connected to the destination with a directed path. Furthermore, an improper policy has finite cost starting from every initial state if and only if all the cycles of the corresponding graph have nonnegative cycle cost. It follows that the favorable analytical and algorithmic results described for SSP in the preceding example hold if the given graph is connected and the costs of all its cycles are positive. We will see later that significant complications result if the cycle costs are allowed to be zero, even though the shortest path problem is still well posed in the sense that shortest paths exist if the given graph is connected (see Section 3.1).

Example 1.2.8 (Multiplicative and Risk-Sensitive Models)

With x, y, u, and transition probabilities $p_{xy}(u)$, as in the finite-state MDP of Example 1.2.2, consider the mapping

$$H(x,u,J) = \sum_{y\in X} p_{xy}(u)g(x,u,y)J(y) = E\big\{g(x,u,y)J(y) \mid x,u\big\}, \tag{1.15}$$

where g is a scalar function satisfying $g(x,u,y) \geq 0$ for all x, y, u (this is necessary for H to be monotone). This mapping corresponds to the multiplicative model of minimizing over all $\pi = \{\mu_0,\mu_1,\ldots\}$ the cost

$$\begin{aligned} J_\pi(x_0) = \limsup_{N\to\infty} E\Big\{ & g\big(x_0,\mu_0(x_0),x_1\big)g\big(x_1,\mu_1(x_1),x_2\big)\cdots \\ & g\big(x_{N-1},\mu_{N-1}(x_{N-1}),x_N\big) \;\big|\; x_0\Big\}, \end{aligned} \tag{1.16}$$

where the state sequence $\{x_0,x_1,\ldots\}$ is generated using the transition probabilities $p_{x_kx_{k+1}}\big(\mu_k(x_k)\big)$.

To see that the mapping H of Eq. (1.15) corresponds to the cost function (1.16), let us consider the unit function

$$\bar J(x) \equiv 1, \qquad x \in X,$$

and verify that for all $x_0 \in X$, we have

$$\begin{aligned} (T_{\mu_0}T_{\mu_1}\cdots T_{\mu_{N-1}}\bar J)(x_0) = E\Big\{ & g\big(x_0,\mu_0(x_0),x_1\big)g\big(x_1,\mu_1(x_1),x_2\big)\cdots \\ & g\big(x_{N-1},\mu_{N-1}(x_{N-1}),x_N\big) \;\big|\; x_0\Big\}, \end{aligned} \tag{1.17}$$

so that

$$J_\pi(x) = \limsup_{N\to\infty} (T_{\mu_0}T_{\mu_1}\cdots T_{\mu_{N-1}}\bar J)(x), \qquad x \in X.$$

Indeed, taking into account that $\bar J(x) \equiv 1$, we have

$$\begin{aligned} (T_{\mu_{N-1}}\bar J)(x_{N-1}) &= E\big\{g\big(x_{N-1},\mu_{N-1}(x_{N-1}),x_N\big)\bar J(x_N) \mid x_{N-1}\big\} \\ &= E\big\{g\big(x_{N-1},\mu_{N-1}(x_{N-1}),x_N\big) \mid x_{N-1}\big\}, \end{aligned}$$

$$
\begin{aligned}
(T_{\mu_{N-2}}T_{\mu_{N-1}}\bar{J})(x_{N-2}) &= \big((T_{\mu_{N-2}}(T_{\mu_{N-1}}\bar{J})\big)(x_{N-2})\\
&= E\Big\{g\big(x_{N-2},\mu_{N-2}(x_{N-2}),x_{N-1}\big)\\
&\qquad \cdot E\Big\{g\big(x_{N-1},\mu_{N-1}(x_{N-1}),x_N\big) \mid x_{N-1}\Big\} \mid x_{N-2}\Big\},
\end{aligned}
$$

and continuing similarly,

$$
\begin{aligned}
(T_{\mu_0}T_{\mu_1}\cdots T_{\mu_{N-1}}\bar{J})(x_0) = E\bigg\{g\big(x_0,\mu_0(x_0),x_1\big)E\Big\{g\big(x_1,\mu_1(x_1),x_2\big)\cdots\\
E\Big\{g\big(x_{N-1},\mu_{N-1}(x_{N-1}),x_N\big) \mid x_{N-1}\Big\} \mid x_{N-2}\Big\}\cdots\Big\} \mid x_0\bigg\},
\end{aligned}
$$

which by using the iterated expectations formula (see e.g., [BeT08]) proves the expression (1.17).

An important special case of a multiplicative model is when g has the form

$$g(x,u,y) = e^{h(x,u,y)}$$

for some one-stage cost function h. We then obtain a finite-state MDP with an exponential cost function,

$$J_\pi(x_0) = \limsup_{N\to\infty} E\bigg\{e^{\big(h(x_0,\mu_0(x_0),x_1)+\cdots+h(x_{N-1},\mu_{N-1}(x_{N-1}),x_N)\big)}\bigg\},$$

which is often used to introduce risk aversion in the choice of policy through the convexity of the exponential.

There is also a multiplicative version of the infinite state space stochastic optimal control problem of Example 1.2.1. The mapping H takes the form

$$H(x,u,J) = E\Big\{g(x,u,w)J\big(f(x,u,w)\big)\Big\},$$

where $x_{k+1} = f(x_k,u_k,w_k)$ is the underlying discrete-time dynamic system; cf. Eq. (1.10).

Multiplicative models and related risk-sensitive models are discussed extensively in the literature, mostly for the exponential cost case and under different assumptions than ours; see e.g., [HoM72], [Jac73], [Rot84], [ChS87], [Whi90], [JBE94], [FlM95], [HeM96], [FeM97], [BoM99], [CoM99], [BoM02], [BBB08], [Ber16a]. The works of references [DeR79], [Pat01], and [Pat07] relate to the stochastic shortest path problems of Example 1.2.6, and are the closest to the semicontractive models discussed in Chapters 3 and 4, based on the author's paper [Ber16a]; see the next example and Section 3.5.2.

Example 1.2.9 (Affine Monotonic Models)

Consider a finite state space $X = \{1,\ldots,n\}$ and a (possibly infinite) control constraint set $U(x)$ for each state x. For each policy μ, let the mapping T_μ be given by

$$T_\mu J = b_\mu + A_\mu J, \tag{1.18}$$

where b_μ is a vector of $\Re^n$ with components $b\big(x,\mu(x)\big)$, $x = 1,\ldots,n$, and A_μ is an $n\times n$ matrix with components $A_{xy}\big(\mu(x)\big)$, $x,y = 1,\ldots,n$. We assume that $b(x,u)$ and $A_{xy}(u)$ are nonnegative,

$$b(x,u) \geq 0, \qquad A_{xy}(u) \geq 0, \qquad \forall\ x,y = 1,\ldots,n,\ u \in U(x).$$

Thus T_μ and T map nonnegative functions to nonnegative functions $J : X \mapsto [0, \infty]$.

This model was introduced in the first edition of this book, and was elaborated on in the author's paper [Ber16a]. Special cases of the model include the finite-state Markov and semi-Markov problems of Examples 1.2.1-1.2.3, and the stochastic shortest path problem of Example 1.2.6, with A_μ being the transition probability matrix of μ (perhaps appropriately discounted), and b_μ being the cost per stage vector of μ, which is assumed nonnegative. An interesting affine monotonic model of a different type is the multiplicative cost model of the preceding example, where the initial function is $\bar{J}(x) \equiv 1$ and the cost accumulates multiplicatively up to reaching a termination state t. In the exponential case of this model, the cost of a generated path starting from some initial state accumulates additively as in the SSP case, up to reaching t. However, the cost of the model is the expected value of the *exponentiated* cost of the path up to reaching t. It can be shown then that the mapping T_μ has the form

$$\begin{aligned}(T_\mu J)(x) = {} & p_{xt}\big(\mu(x)\big)\exp\big(g(x, \mu(x), t)\big) \\ & + \sum_{y=1}^{n} p_{xy}(\mu(x))\exp\big(g(x, \mu(x), y)\big)J(y), \qquad x \in X,\end{aligned}$$

where $p_{xy}(u)$ is the probability of transition from x to y under u, and $g(x, u, y)$ is the cost of the transition; see Section 3.5.2 for a detailed derivation. Clearly T_μ has the affine monotonic form (1.18).

Example 1.2.10 (Aggregation)

Aggregation is an approximation approach that replaces a large DP problem with a simpler problem obtained by "combining" many of its states together into *aggregate states*. This results in an "aggregate" problem with fewer states, which may be solvable by exact DP methods. The optimal cost-to-go function of this problem is then used to approximate the optimal cost function of the original problem.

Consider an n-state Markovian decision problem with transition probabilities $p_{ij}(u)$. To construct an aggregation framework, we introduce a finite set $\mathcal{A}$ of aggregate states. We generically denote the aggregate states by letters such as x and y, and the original system states by letters such as i and j. The approximation framework is specified by combining in various ways the aggregate states and the original system states to form a larger system (see Fig. 1.2.2). To specify the probabilistic structure of this system, we introduce two (somewhat arbitrary) choices of probability distributions, which relate the original system states with the aggregate states:

(1) For each aggregate state x and original system state i, we specify the *disaggregation probability* d_{xi}. We assume that $d_{xi} \geq 0$ and

$$\sum_{i=1}^{n} d_{xi} = 1, \qquad \forall\, x \in \mathcal{A}.$$

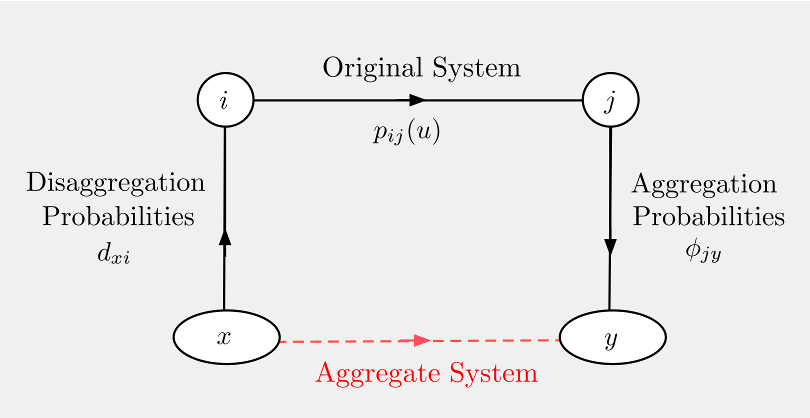


Figure 1.2.2 Illustration of the relation between aggregate and original system states.

Roughly, d_{xi} may be interpreted as the "degree to which x is represented by i."

(2) For each aggregate state y and original system state j, we specify the *aggregation probability* ϕ_{jy}. We assume that $\phi_{jy} \geq 0$ and

$$\sum_{y \in \mathcal{A}} \phi_{jy} = 1, \qquad \forall\, j = 1, \ldots, n.$$

Roughly, ϕ_{jy} may be interpreted as the "degree of membership of j in the aggregate state y."

The aggregation and disaggregation probabilities specify a dynamic system involving both aggregate and original system states (cf. Fig. 1.2.2). In this system:

(i) From aggregate state x, we generate original system state i according to d_{xi}.

(ii) We generate transitions from original system state i to original system state j according to $p_{ij}(u)$, with cost $g(i, u, j)$.

(iii) From original system state j, we generate aggregate state y according to ϕ_{jy}.

Illustrative examples of aggregation frameworks are given in the books [Ber12a] and [Ber17a]. One possibility is *hard aggregation*, where aggregate states are identified with the sets of a partition of the state space. For another type of common scheme, think of the case where the original system states form a fine grid in some space, which is "aggregated" into a much coarser grid. In particular let us choose a collection of "representative" original system states, and associate each one of them with an aggregate state. Thus, each aggregate state x is associated with a unique representative state i_x, and the

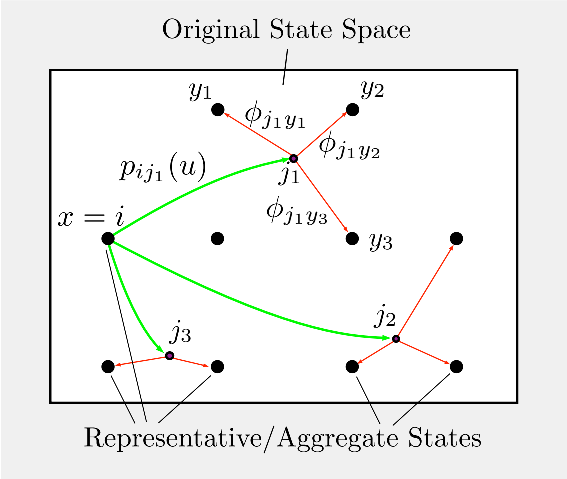


Figure 1.2.3 Aggregation based on a small subset of representative states (these are shown with larger dark circles, while the other (nonrepresentative) states are shown with smaller dark circles). In this figure, from representative state $x = i$, there are three possible transitions, to states j_1, j_2, and j_3, according to $p_{ij_1}(u), p_{ij_2}(u), p_{ij_3}(u)$, and each of these states is associated with a convex combination of representative states using the aggregation probabilities. For example, j_1 is associated with $\phi_{j_1y_1}y_1 + \phi_{j_1y_2}y_2 + \phi_{j_1y_3}y_3$.

disaggregation probabilities are

$$d_{xi} = \begin{cases} 1 & \text{if } i = i_x, \\ 0 & \text{if } i \neq i_x. \end{cases} \tag{1.19}$$

The aggregation probabilities are chosen to represent each original system state j with a convex combination of aggregate/representative states; see Fig. 1.2.3. It is also natural to assume that the aggregation probabilities map representative states to themselves, i.e.,

$$\phi_{jy} = \begin{cases} 1 & \text{if } j = j_y, \\ 0 & \text{if } j \neq j_y. \end{cases}$$

This scheme makes intuitive geometrical sense as an interpolation scheme in the special case where both the original and the aggregate states are associated with points in a Euclidean space. The scheme may also be extended to problems with a continuous state space. In this case, the state space is discretized with a finite grid, and the states of the grid are viewed as the aggregate states. The disaggregation probabilities are still given by Eq. (1.19), while the aggregation probabilities may be arbitrarily chosen to represent each original system state with a convex combination of representative states.

As an extension of the preceding schemes, suppose that through some special insight into the problem's structure or some preliminary calculation, we know some features of the system's state that can "predict well" its cost. Then it seems reasonable to form the aggregate states by grouping together states with "similar features," or to form aggregate states by using "representative features" instead of representative states. This is called "feature-based aggregation;" see the books [BeT96] (Section 3.1) and [Ber12a] (Section 6.5) for a description and analysis.

Given aggregation and disaggregation probabilities, one may define an *aggregate problem* whose states are the aggregate states. This problem involves an aggregate discrete-time system, which we will describe shortly. We require that the control is applied with knowledge of the current aggregate state only (rather than the original system state).† To this end, we assume that the control constraint set $U(i)$ is independent of the state i, and we denote it by U. Then, by adding the probabilities of all the relevant paths in Fig. 1.2.2, it can be seen that the transition probability from aggregate state x to aggregate state y under control $u \in U$ is

$$\hat{p}_{xy}(u) = \sum_{i=1}^{n} d_{xi} \sum_{j=1}^{n} p_{ij}(u)\phi_{jy}.$$

The corresponding expected transition cost is given by

$$\hat{g}(x,u) = \sum_{i=1}^{n} d_{xi} \sum_{j=1}^{n} p_{ij}(u)g(i,u,j).$$

These transition probabilities and costs define the aggregate problem.

We may compute the optimal costs-to-go $\hat{J}(x)$, $x \in \mathcal{A}$, of this problem by using some exact DP method. Then, the costs-to-go of each state j of the original problem are usually approximated by

$$\tilde{J}(j) = \sum_{y \in \mathcal{A}} \phi_{jy}\hat{J}(y).$$

Example 1.2.11 (Distributed Aggregation)

The abstract DP framework is useful not only in modeling DP problems, but also in modeling algorithms arising in DP and even other contexts. We illustrate this with an example from Bertsekas and Yu [BeY10] that relates to the distributed solution of large-scale discounted finite-state MDP using cost function approximation based on aggregation. ‡ It involves a partition of the n states into m subsets for the purposes of distributed computation, and yields a corresponding approximation $(V_1,\ldots,V_m)$ to the cost vector J^*.

In particular, we have a discounted n-state MDP (cf. Example 1.2.2), and we introduce aggregate states $S_1,\ldots,S_m$, which are disjoint subsets of

† An alternative form of aggregate problem, where the control may depend on the original system state is discussed in Section 6.5.2 of the book [Ber12a].

‡ See [Ber12a], Section 6.5.2, for a more detailed discussion. Other examples of algorithmic mappings that come under our framework arise in asynchronous policy iteration (see Sections 2.6.3, 3.6.2, and [BeY10], [BeY12], [YuB13a]), and in constrained forms of policy iteration (see [Ber11c], or [Ber12a], Exercise 2.7).

the original state space with $S_1 \cup \cdots \cup S_n = \{1, \ldots, n\}$. We envision a network of processors $\ell = 1, \ldots, m$, each assigned to the computation of a local cost function V_ℓ, defined on the corresponding aggregate state/subset S_ℓ:

$$V_\ell = \{V_{\ell y} \mid y \in S_\ell\}.$$

Processor ℓ also maintains a scalar aggregate cost R_ℓ for its aggregate state, which is a weighted average of the detailed cost values $V_{\ell x}$ within S_ℓ:

$$R_\ell = \sum_{x \in S_\ell} d_{\ell x} V_{\ell x},$$

where $d_{\ell x}$ are given probabilities with $d_{\ell x} \geq 0$ and $\sum_{x \in S_\ell} d_{\ell x} = 1$. The aggregate costs R_ℓ are communicated between processors and are used to perform the computation of the local cost functions V_ℓ (we will discuss computation models of this type in Section 2.6).

We denote $J = (V_1, \ldots, V_m, R_1, \ldots, R_m)$. We introduce the mapping $H(x, u, J)$ defined for each of the n states x by

$$H(x, u, J) = W_\ell(x, u, V_\ell, R_1, \ldots, R_m), \qquad \text{if } x \in S_\ell,$$

where for $x \in S_\ell$

$$W_\ell(x, u, V_\ell, R_1, \ldots, R_m) = \sum_{y=1}^{n} p_{xy}(u) g(x, u, y) + \alpha \sum_{y \in S_\ell} p_{xy}(u) V_{\ell y} + \alpha \sum_{y \notin S_\ell} p_{xy}(u) R_{s(y)},$$

and for each original system state y, we denote by $s(y)$ the index of the subset to which y belongs [i.e., $y \in S_{s(y)}$].

We may view H as an abstract mapping on the space of J, and aim to find its fixed point $J^* = (V_1^*, \ldots, V_m^*, R_1^*, \ldots, R_m^*)$. Then, for $\ell = 1, \ldots, m$, we may view V_ℓ^* as an approximation to the optimal cost vector of the original MDP starting at states $x \in S_\ell$, and we may view R_ℓ^* as a form of aggregate cost for S_ℓ. The advantage of this formulation is that it involves significant decomposition and parallelization of the computations among the processors, when performing various DP algorithms. In particular, the computation of $W_\ell(x, u, V_\ell, R_1, \ldots, R_m)$ depends on just the local vector V_ℓ, whose dimension may be potentially much smaller than n.

1.2.4 Approximation Models - Projected and Aggregation Bellman Equations

Given an abstract DP model described by a mapping H, we may be interested in fixed points of related mappings other than T and T_μ. Such mappings may arise in various contexts, such as for example distributed asynchronous aggregation in Example 1.2.11. An important context is *subspace approximation*, whereby T_μ and T are restricted onto a subspace of functions for the purpose of approximating their fixed points. Much of the theory of approximate DP, neuro-dynamic programming, and reinforcement learning relies on such approximations (there are quite a few books, which collectively contain extensive accounts these subjects, such as Bertsekas and Tsitsiklis [BeT96], Sutton and Barto [SuB98], Gosavi [Gos03], Cao [Cao07], Chang, Fu, Hu, and Marcus [CFH07], Meyn [Mey07], Powell [Pow07], Borkar [Bor08], Haykin [Hay08], Busoniu, Babuska, De Schutter, and Ernst [BBD10], Szepesvari [Sze10], Bertsekas [Ber12a], [Ber17a], and Vrabie, Vamvoudakis, and Lewis [VVL13]).

For an illustration, consider the approximate evaluation of the cost vector of a discrete-time Markov chain with states $i = 1, \ldots, n$. We assume that state transitions (i, j) occur at time k according to given transition probabilities p_{ij}, and generate a cost $\alpha^k g(i, j)$, where $\alpha \in (0, 1)$ is a discount factor. The cost function over an infinite number of stages can be shown to be the unique fixed point of the Bellman equation mapping $T : \Re^n \mapsto \Re^n$ whose components are given by

$$(TJ)(i) = \sum_{j=1}^{n} p_{ij}(u)\big(g(i, j) + \alpha J(j)\big), \qquad i = 1, \ldots, n,\ J \in \Re^n.$$

This is the same as the mapping T in the discounted finite-state MDP Example 1.2.2, except that we restrict attention to a single policy. Finding the cost function of a fixed policy is the important policy evaluation subproblem that arises prominently within the context of policy iteration. It also arises in the context of a simplified form of policy iteration, the *rollout algorithm*; see e.g., [BeT96], [Ber12a], [Ber17a]. In some artificial intelligence contexts, policy iteration is referred to as *self-learning*, and in these contexts the policy evaluation is almost always done approximately, sometimes with the use of neural networks.

A prominent approach for approximation of the fixed point of T is based on the solution of lower-dimensional equations defined on the subspace $\{\Phi r \mid r \in \Re^s\}$ that is spanned by the columns of a given $n \times s$ matrix Φ. Two such approximating equations have been studied extensively (see [Ber12a], Chapter 6, for a detailed account and references; also [BeY07], [BeY09], [YuB10], [Ber11a] for extensions to abstract contexts beyond approximate DP). These are:

(a) The *projected equation*

$$\Phi r = \Pi_\xi T(\Phi r), \tag{1.20}$$

where Π_ξ denotes projection onto S with respect to a weighted Euclidean norm

$$\|J\|_\xi = \sqrt{\sum_{i=1}^{n} \xi_i \big(J(i)\big)^2} \tag{1.21}$$

with $\xi = (\xi_1, \ldots, \xi_n)$ being a probability distribution with positive components (sometimes a seminorm projection is used, whereby some of the components ξ_i may be zero; see Yu and Bertsekas [YuB12]).

(b) The *aggregation equation*

$$\Phi r = \Phi DT(\Phi r), \tag{1.22}$$

with D being an $s \times n$ matrix whose rows are restricted to be probability distributions; these are the disaggregation probabilities of Example 1.2.10. Also, in this approach, the rows of Φ are restricted to be probability distributions; these are the aggregation probabilities of Example 1.2.10.

We now see that solving the projected equation (1.20) and the aggregation equation (1.22) amounts to finding a fixed point of the mappings $\Pi_\xi T$ and ΦDT, respectively. These mappings derive their structure from the DP operator T, so they have some DP-like properties, which can be exploited for analysis and computation.

An important fact is that the aggregation mapping ΦDT preserves the monotonicity and the sup-norm contraction property of T, while the projected equation mapping $\Pi_\xi T$ generally does not. The reason for preservation of monotonicity is the nonnegativity of the components of the matrices Φ and D (see the author's survey paper [Ber11c] for a discussion of the importance of preservation of monotonicity in various DP operations). The reason for preservation of sup-norm contraction is that the matrices Φ and D are sup-norm nonexpansive, because their rows are probability distributions. In fact, it can be verified that the solution r of Eq. (1.22) can be viewed as the *exact* DP solution of the "aggregate" DP problem that represents a lower-dimensional approximation of the original (see Example 1.2.10). The preceding observations are important for our purposes, as they indicate that much of the theory developed in this book applies to approximation-related mappings based on aggregation.

By contrast, the projected equation mapping $\Pi_\xi T$ need not be monotone, because the components of Π_ξ need not be nonnegative. Moreover while the projection Π_ξ is nonexpansive with respect to the projection norm $\|\cdot\|_\xi$, it need not be nonexpansive with respect to the sup-norm. As a result the projected equation mapping $\Pi_\xi T$ need not be a sup-norm contraction. These facts play a significant role in approximate DP methodology.

1.2.5 Multistep Models - Temporal Difference and Proximal Algorithms

An important possibility for finding a fixed point of T is to replace T with another mapping, say F, such that F and T have the same fixed points. For example, F may offer some advantages in terms of algorithmic convenience or quality of approximation when used in conjunction with projection or aggregation [cf. Eqs. (1.20) and (1.22)]. Alternatively, F may be the mapping of some iterative method $x_{k+1} = F(x_k)$ that is suitable for computing fixed points of T.

In this book we will not consider in much detail the possibility of using an alternative mapping F to find a fixed point of a mapping T. We will just mention here some multistep versions of T, which have been used widely for approximations, particularly in connection with the projected equation approach. An important example is the mapping $T^{(\lambda)} : \Re^n \mapsto \Re^n$, defined for a given $\lambda \in (0,1)$ as follows: $T^{(\lambda)}$ transforms a vector $J \in \Re^n$ to the vector $T^{(\lambda)}J \in \Re^n$, whose n components are given by

$$\big(T^{(\lambda)}J\big)(i) = (1-\lambda)\sum_{\ell=0}^{\infty} \lambda^\ell (T^{\ell+1}J)(i), \qquad i = 1,\dots,n,\ J \in \Re^n,$$

for $\lambda \in (0,1)$, where T^ℓ is the ℓ-fold composition of T with itself ℓ times. Here there should be conditions that guarantee the convergence of the infinite series in the preceding definition. The multistep analog of the projected Eq. (1.20) is

$$\Phi r = \Pi_\xi T^{(\lambda)}(\Phi r).$$

The popular temporal difference methods, such as TD(λ), LSTD(λ), and LSPE(λ), aim to solve this equation (see the book references on approximate DP, neuro-dynamic programming, and reinforcement learning cited earlier). The mapping $T^{(\lambda)}$ also forms the basis for the λ-policy iteration method to be discussed in Sections 2.5, 3.2.4, and 4.3.3.

The multistep analog of the aggregation Eq. (1.22) is

$$\Phi r = \Phi D T^{(\lambda)}(\Phi r),$$

and methods that are similar to the temporal difference methods can be used for its solution. In particular, a multistep method based on the mapping $T^{(\lambda)}$ is the, so-called, λ-aggregation method (see [Ber12a], Chapter 6), as well as other forms of aggregation (see [Ber12a], [YuB12]).

In the case where T is a linear mapping of the form

$$TJ = AJ + b,$$

where b is a vector in $\Re^n$, and A is an $n \times n$ matrix with eigenvalues strictly within the unit circle, there is an interesting connection between the multistep mapping $T^{(\lambda)}$ and another mapping of major importance in numerical convex optimization. This is the *proximal mapping*, associated with T and a scalar $c > 0$, and denoted by $P^{(c)}$. In particular, for a given $J \in \Re^n$, the vector $P^{(c)}J$ is defined as the unique vector $Y \in \Re^n$ that solves the equation

$$Y - AY - b = \frac{1}{c}(J - Y).$$

Equivalently,

$$P^{(c)}J = \left(\frac{c+1}{c}I - A\right)^{-1}\left(b + \frac{1}{c}J\right), \tag{1.23}$$

where I is the identity matrix. Then it can be shown (see Exercise 1.2 or the papers [Ber16b], [Ber18c]) that if

$$c = \frac{\lambda}{1-\lambda},$$

we have

$$T^{(\lambda)} = T \cdot P^{(c)} = P^{(c)} \cdot T.$$

Moreover, the vectors J, $P^{(c)}J$, and $T^{(\lambda)}J$ are colinear and satisfy

$$T^{(\lambda)}J = J + \frac{c+1}{c}\big(P^{(c)}J - J\big).$$

The preceding formulas show that $T^{(\lambda)}$ and $P^{(c)}$ are closely related, and that iterating with $T^{(\lambda)}$ is "faster" than iterating with $P^{(c)}$, since the eigenvalues of A are within the unit circle, so that T is a contraction. In addition, methods such as TD(λ), LSTD(λ), LSPE(λ), and their projected versions, which are based on $T^{(\lambda)}$, can be adapted to be used with $P^{(c)}$.

A more general form of multistep approach, introduced and studied in the paper [YuB12], replaces $T^{(\lambda)}$ with a mapping $T^{(w)} : \Re^n \mapsto \Re^n$ that has components

$$\big(T^{(w)}J\big)(i) = \sum_{\ell=1}^{\infty} w_{i\ell}(T^{\ell}J)(i), \qquad i = 1,\ldots,n,\ J \in \Re^n,$$

where w is a vector sequence whose ith component, $(w_{i1}, w_{i2}, \ldots)$, is a probability distribution over the positive integers. Then the multistep analog of the projected equation (1.20) is

$$\Phi r = \Pi_{\xi} T^{(w)}(\Phi r), \tag{1.24}$$

while the multistep analog of the aggregation equation (1.22) is

$$\Phi r = \Phi D T^{(w)}(\Phi r). \tag{1.25}$$

The mapping $T^{(\lambda)}$ is obtained for $w_{i\ell} = (1-\lambda)\lambda^{\ell-1}$, independently of the state i. A more general version, where λ depends on the state i, is obtained for $w_{i\ell} = (1-\lambda_i)\lambda_i^{\ell-1}$. The solution of Eqs. (1.24) and (1.25) by simulation-based methods is discussed in the paper [YuB12]; see also Exercise 1.3.

Let us also note that there is a connection between projected equations of the form (1.24) and aggregation equations of the form (1.25). This connection is based on the use of a seminorm [this is given by the same expression as the norm $\|\cdot\|_{\xi}$ of Eq. (1.21), with some of the components of ξ allowed to be 0]. In particular, the most prominent cases of aggregation equations can be viewed as seminorm projected equations because, for these cases, ΦD is a seminorm projection (see [Ber12a], p. 639, [YuB12], Section 4). Moreover, they can also be viewed as projected equations where the projection is oblique (see [Ber12a], Section 7.3.6).

1.3 ABSTRACT VISUALIZATIONS - NEWTON'S METHOD

In this section we will use geometric illustrations to obtain insight into Bellman's equation, and the algorithms of value iteration (VI) and policy iteration (PI). We will also discuss some reinforcement learning methods, such as approximation in value space together with some of the properties of the associated one-step or multistep lookahead policies. To this end, we will focus on the stochastic optimal control problem of Example 1.2.1, where

$$(TJ)(x) = \inf_{u\in U(x)} E\Big\{g(x,u,w) + \alpha J\big(f(x,u,w)\big)\Big\}, \qquad \text{for all } x, \quad (1.26)$$

and

$$(T_\mu J)(x) = E\Big\{g\big(x,\mu(x),w\big) + \alpha J\big(f(x,\mu(x),w)\big)\Big\}, \qquad \text{for all } x. \quad (1.27)$$

Our geometric illustrations will make use of some special properties of the operators T and T_μ. These are:

(a) T and T_μ are monotone, i.e., they satisfy Assumption 1.2.1.

(b) T_μ is linear, in the sense that it has the form $T_\mu J = G + A_\mu J$, where $G \in R(X)$ is some function and $A_\mu : R(X) \mapsto R(X)$ is an operator such that for any functions J_1, J_2, and scalars γ_1, γ_2, we have

$$A_\mu(\gamma_1 J_1 + \gamma_2 J_2) = \gamma_1 A_\mu J_1 + \gamma_2 A_\mu J_2.$$

This is true because of the linearity of the expected value operation in Eq. (1.27).

(c) We have

$$(TJ)(x) = \min_{\mu\in\mathcal{M}} (T_\mu J)(x), \qquad \text{for all } x, \qquad (1.28)$$

where $\mathcal{M}$ is the set of stationary policies. This is true because for any policy μ, there is no coupling constraint between the controls $\mu(x)$ and $\mu(x')$ that correspond to two different states x and x'.

(d) $(TJ)(x)$ is a concave function of J for every x, which follows from the linearity of T_μ and the alternative definition of T given by Eq. (1.28).

We illustrate these properties graphically with an example.

Example 1.3.1 (A Two-State and Two-Control Example)

Assume that there are two states 1 and 2, and two controls u and v. Consider the policy μ that applies control u at state 1 and control v at state 2. Then the operator T_μ takes the form

$$(T_\mu J)(1) = \sum_{y=1}^{2} p_{1j}(u)\big(g(1,u,y) + \alpha J(y)\big), \qquad (1.29)$$

$$(T_\mu J)(2) = \sum_{y=1}^{2} p_{2y}(v)\big(g(2,v,y) + \alpha J(y)\big), \tag{1.30}$$

where $p_{xy}(u)$ and $p_{xy}(v)$ are the probabilities that the next state will be y, when the current state is x, and the control is u or v, respectively. Clearly, $(T_\mu J)(1)$ and $(T_\mu J)(2)$ are linear functions of J. Also the operator T of the Bellman equation $J = TJ$ takes the form

$$\begin{aligned}(TJ)(1) = \min \Bigg[&\sum_{y=1}^{2} p_{1y}(u)\big(g(1,u,y) + \alpha J(y)\big), \\ &\qquad \sum_{y=1}^{2} p_{1y}(v)\big(g(1,v,y) + \alpha J(y)\big)\Bigg],\end{aligned} \tag{1.31}$$

$$\begin{aligned}(TJ)(2) = \min \Bigg[&\sum_{y=1}^{2} p_{2y}(u)\big(g(2,u,y) + \alpha J(y)\big), \\ &\qquad \sum_{y=1}^{2} p_{2y}(v)\big(g(2,v,y) + \alpha J(y)\big)\Bigg].\end{aligned} \tag{1.32}$$

Thus, $(TJ)(1)$ and $(TJ)(2)$ are concave and piecewise linear as functions of the two-dimensional vector J (with two pieces; more generally, as many linear pieces as the number of controls). This concavity property holds in general since $(TJ)(x)$ is the minimum of a collection of linear functions of J, one for each $u \in U(x)$. Figure 1.3.1 illustrates $(T_\mu J)(1)$ for the cases where $\mu(1) = u$ and $\mu(1) = v$, $(T_\mu J)(2)$ for the cases where $\mu(2) = u$ and $\mu(2) = v$, $(TJ)(1)$, and $(TJ)(2)$, as functions of $J = \big(J(1), J(2)\big)$.

Mathematically the concavity property of T manifests itself in that the set

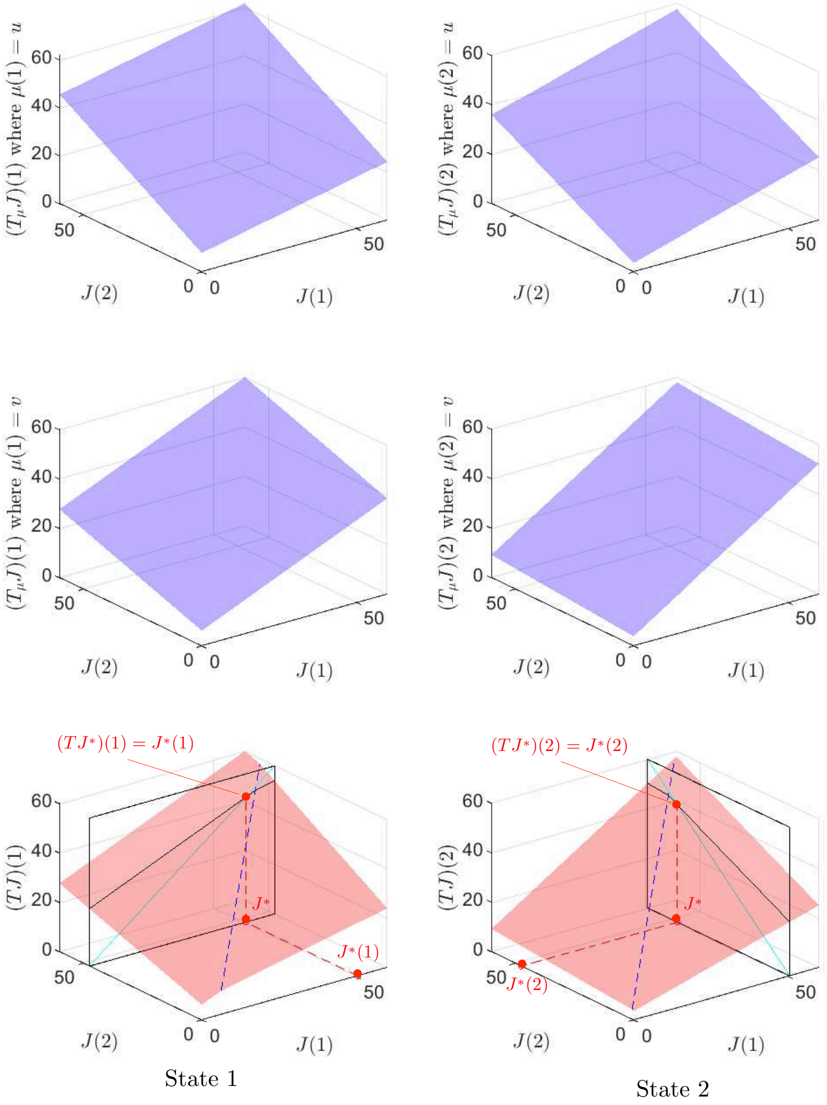
$$C = \big\{(J,\xi) \in R(X) \times R(X) \mid (TJ)(x) \geq \xi(x), \text{ for all } x \in X\big\} \tag{1.33}$$

is convex as a subset of $R(X) \times R(X)$, where $R(X)$ is the set of real-valued functions over the state space X. This convexity property is verified by showing that given (J_1,ξ_1) and (J_2,ξ_2) in C, and $\gamma \in [0,1]$, we have

$$\big(\gamma J_1 + (1-\gamma)J_2,\ \gamma\xi_1 + (1-\gamma)\xi_2\big) \in C.$$

The proof of this is straightforward by using the concavity of $(TJ)(x)$ for each x.

Critical properties from the DP point of view are whether T and T_μ have fixed points; equivalently, whether the Bellman equations $J = TJ$ and $J = T_\mu J$ have solutions within the class of real-valued functions, and whether the set of solutions includes J^* and J_μ, respectively. It may thus be important to verify that T or T_μ are contraction mappings. This is true



State 1

State 2

Figure 1.3.1 Geometric illustrations of the Bellman operators T_μ and T for states 1 and 2 in Example 1.3.1; cf. Eqs. (1.29)-(1.32). The problem's transition probabilities are: $p_{11}(u) = 0.3$, $p_{12}(u) = 0.7$, $p_{21}(u) = 0.4$, $p_{22}(u) = 0.6$, $p_{11}(v) = 0.6$, $p_{12}(v) = 0.4$, $p_{21}(v) = 0.9$, $p_{22}(v) = 0.1$. The stage costs are $g(1,u,1) = 3$, $g(1,u,2) = 10$, $g(2,u,1) = 0$, $g(2,u,2) = 6$, $g(1,v,1) = 7$, $g(1,v,2) = 5$, $g(2,v,1) = 3$, $g(2,v,2) = 12$. The discount factor is $\alpha = 0.9$, and the optimal costs are $J^*(1) = 50.59$ and $J^*(2) = 47.41$. The optimal policy is $\mu^*(1) = v$ and $\mu^*(2) = u$. The figure also shows the one-dimensional "slices" of T that pass through J^*.

for example in the benign case of discounted problems with bounded cost per stage. However, for undiscounted problems, asserting the contraction property of T or T_μ may be more complicated, and even impossible. In this book we will deal extensively with such questions and related issues regarding the solution set of the Bellman equations.

Geometrical Interpretations

We will now interpret the Bellman operators geometrically, starting with T_μ, which is linear as noted earlier. Figure 1.3.2 illustrates its form. Note here that the functions J and $T_\mu J$ are multidimensional. They have as many scalar components $J(x)$ and $(T_\mu J)(x)$, respectively, as there are states x, but they can only be shown projected onto one dimension. The cost function J_μ satisfies $J_\mu = T_\mu J_\mu$, so it is obtained from the intersection of the graph of $T_\mu J$ and the 45 degree line, when J_μ is real-valued. We interpret the situation where J_μ is not real-valued with lack of system stability under μ [so μ will be viewed as unstable if we have $J_\mu(x) = \infty$ for some initial states x]. For further discussion of stability issues, see the book [Ber22].

The form of the Bellman operator T is illustrated in Fig. 1.3.3. Again the functions J, J^*, TJ, $T_\mu J$, etc, are multidimensional, but they are shown projected onto one dimension. The Bellman equation $J = TJ$ may have one or many real-valued solutions. It may also have no real-valued solution in exceptional situations, as we will discuss later. The figure assumes that the Bellman equations $J = TJ$ and $J = T_\mu J$ have a unique real-valued solution, which is true if T and T_μ are contraction mappings, as is the case for discounted problems with bounded cost per stage. Otherwise, these equations may have no solution or multiple solutions within the class of real-valued functions. The equation $J = TJ$ typically has J^* as a solution, but may have more than one solution in cases where either $\alpha = 1$ or $\alpha < 1$, and the cost per stage is unbounded.

Example 1.3.2 (A Two-State and Infinite Controls Problem)

Let us consider the mapping T for a problem that involves two states, 1 and 2, but an infinite number of controls. In particular, the control space at both states is the unit interval, $U(1) = U(2) = [0,1]$. Here $(TJ)(1)$ and $(TJ)(2)$ are given by

$$(TJ)(1) = \min_{u\in[0,1]} \big\{g_1 + r_{11}u^2 + r_{12}(1-u)^2 + \alpha uJ(1) + \alpha(1-u)J(2)\big\},$$

$$(TJ)(2) = \min_{u\in[0,1]} \big\{g_2 + r_{21}u^2 + r_{22}(1-u)^2 + \alpha uJ(1) + \alpha(1-u)J(2)\big\}.$$

The control u at each state $x = 1, 2$ has the meaning of a probability that we must select at that state. In particular, we control the probabilities u and $(1-u)$ of moving to states $y = 1$ and $y = 2$, at a control cost that is quadratic

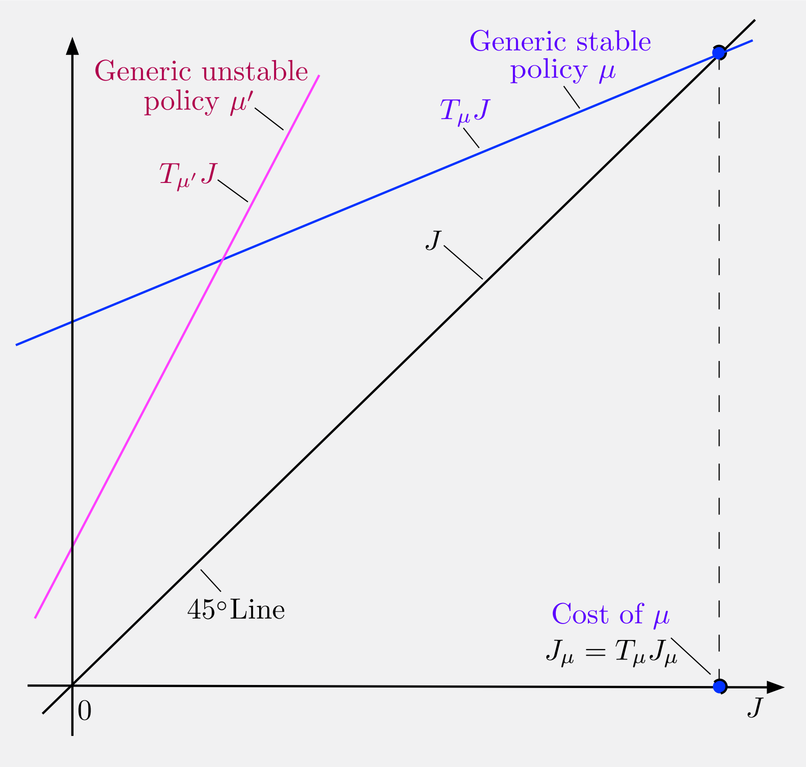


Figure 1.3.2 Geometric interpretation of the linear Bellman operator T_μ and the corresponding Bellman equation. The graph of T_μ is a plane in the space $R(X) \times R(X)$, and when projected on a one-dimensional plane that corresponds to a single state and passes through J_μ, it becomes a line. Then there are three cases:

(a) The line has slope less than 45 degrees, so it intersects the 45-degree line at a unique point, which is equal to J_μ, the solution of the Bellman equation $J = T_\mu J$. This is true if T_μ is a contraction mapping, as is the case for discounted problems with bounded cost per stage.

(b) The line has slope less than 45 degrees. Then it intersects the 45-degree line at a unique point, which is a solution of the Bellman equation $J = T_\mu J$, but is not equal to J_μ. Then J_μ is not real-valued; we consider such μ to be *unstable* under μ.

(c) The line has slope exactly equal to 45 degrees. This is an exceptional case where the Bellman equation $J = T_\mu J$ has an infinite number of real-valued solutions or no real-valued solution at all; we will provide examples where this occurs later.

in u and $(1-u)$, respectively. For this problem $(TJ)(1)$ and $(TJ)(2)$ can be calculated in closed form, so they are easy to plot and understand. They are piecewise quadratic, unlike the corresponding plots of Fig. 1.3.1, which are piecewise linear; see Fig. 1.3.4.

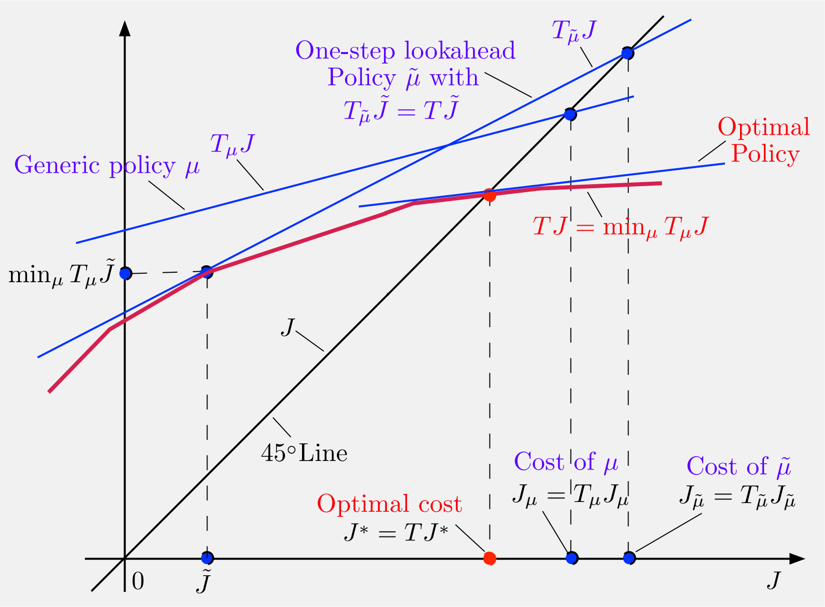


Figure 1.3.3 Geometric interpretation of the Bellman operator T, and the corresponding Bellman equation. For a fixed x, the function $(TJ)(x)$ can be written as $\min_\mu (T_\mu J)(x)$, so it is concave as a function of J. The optimal cost function J^* satisfies $J^* = TJ^*$, so it is obtained from the intersection of the graph of TJ and the 45 degree line shown, assuming J^* is real-valued.

Note that the graph of T lies below the graph of every operator T_μ, and is in fact obtained as the lower envelope of the graphs of T_μ as μ ranges over the set of policies $\mathcal{M}$. In particular, for any given function $\tilde J$, for every x, the value $(T\tilde J)(x)$ is obtained by finding a support hyperplane/subgradient of the graph of the concave function $(T\,J)(x)$ at $\tilde J$, as shown in the figure. This support hyperplane is defined by the control $\mu(x)$ of a policy $\tilde\mu$ that attains the minimum of $(T_\mu \tilde J)(x)$ over μ:

$$\tilde\mu(x) \in \arg\min_{\mu\in\mathcal{M}} (T_\mu \tilde J)(x)$$

(there may be multiple policies attaining this minimum, defining multiple support hyperplanes). This construction also shows how the minimization

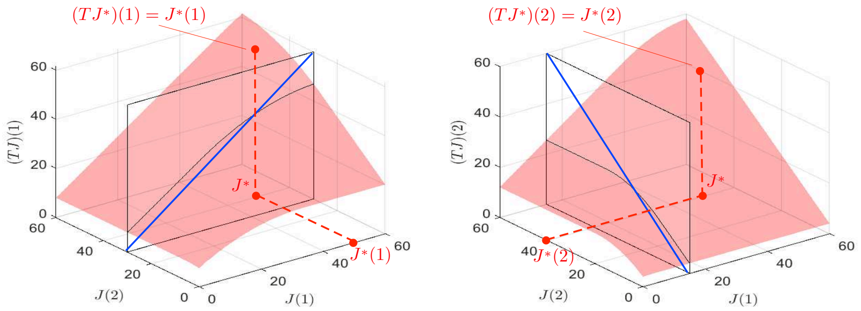
$$(T\tilde J)(x) = \min_{\mu\in\mathcal{M}} (T_\mu \tilde J)(x)$$

corresponds to a linearization of the mapping T at the point $\tilde J$.

Visualization of Value Iteration

The operator notation simplifies algorithmic descriptions, derivations, and proofs related to DP. For example, the value iteration (VI) algorithm can be written in the compact form

$$J_{k+1} = TJ_k, \qquad k = 0, 1, \ldots,$$



State 1

State 2

Figure 1.3.4 Illustration of the Bellman operator T for states 1 and 2 in Example 1.3.2. The parameter values are $g_1 = 5$, $g_2 = 3$, $r_{11} = 3$, $r_{12} = 15$, $r_{21} = 9$, $r_{22} = 1$, and the discount factor is $\alpha = 0.9$. The optimal costs are $J^*(1) = 49.7$ and $J^*(2) = 40.0$, and the optimal policy is $\mu^*(1) = 0.59$ and $\mu^*(2) = 0$. The figure also shows the one-dimensional slices of the operators at $J(1) = 15$ and $J(2) = 30$, together with the corresponding 45-degree lines.

as illustrated in Fig. 1.3.5. Moreover, the VI algorithm for a given policy μ can be written as

$$J_{k+1} = T_\mu J_k, \qquad k = 0, 1, \ldots,$$

and it can be similarly interpreted, except that the graph of the function $T_\mu J$ is linear. Also we will see shortly that there is a similarly compact description for the policy iteration algorithm.

1.3.1 Approximation in Value Space and Newton's Method

Let us now interpret a major approximate DP approach, known as *approximation in value space*, in terms of abstract geometric constructions. Here we approximate J^* with some function $\tilde{J}$, and we obtain by minimization a corresponding policy, called a *one-step lookahead policy*. In particular, for a given $\tilde{J}$, a one-step lookahead policy $\tilde{\mu}$ is characterized by the equation

$$T_{\tilde{\mu}} \tilde{J} = T \tilde{J},$$

as in Fig. 1.3.6. This equation implies that the graph of $T_{\tilde{\mu}} J$ just touches the graph of TJ at $\tilde{J}$, as shown in the figure. In mathematical terms, the set

$$C_{\tilde{\mu}} = \big\{ (J, \xi) \mid T_{\tilde{\mu}} J \geq \xi \big\},$$

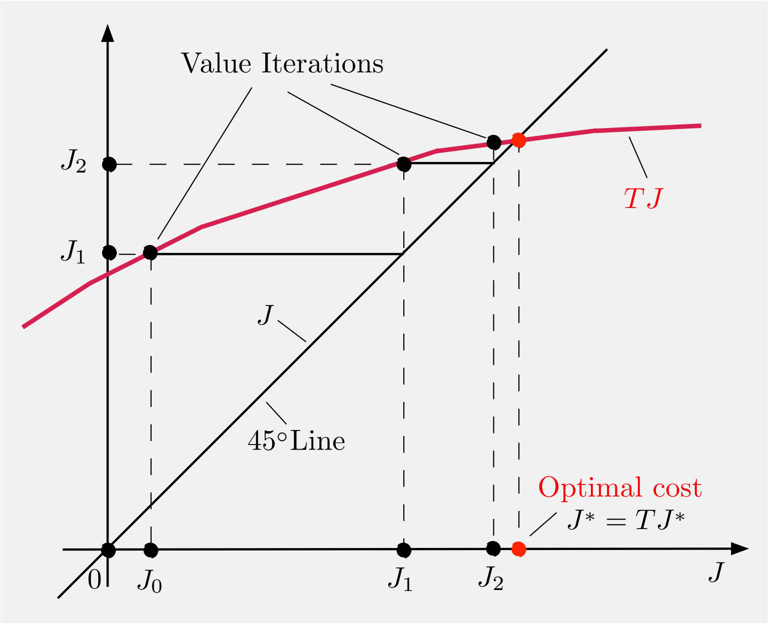


Figure 1.3.5 Geometric interpretation of the VI algorithm $J_{k+1} = TJ_k$, starting from some initial function J_0. Successive iterates are obtained through the staircase construction shown in the figure. The VI algorithm $J_{k+1} = T_\mu J_k$ for a given policy μ can be similarly interpreted, except that the graph of the function $T_\mu J$ is linear.

contains the convex set C of Eq. (1.33) (since $TJ \geq \xi$ implies that $T_{\tilde{\mu}} J \geq \xi$), and has a common point $(\tilde{J}, T_{\tilde{\mu}} \tilde{J})$ with C. Moreover, for each state $x \in X$ the hyperplane $H_{\tilde{\mu}}(x)$

$$H_{\tilde{\mu}}(x) = \Big\{ \big(J(x), \xi(x)\big) \mid (T_{\tilde{\mu}} J)(x) \geq \xi(x) \Big\},$$

supports from above the convex set

$$\Big\{ \big(J(x), \xi(x)\big) \mid (TJ)(x) \geq \xi(x) \Big\}$$

at the point $\big(\tilde{J}(x), (T\tilde{J})(x)\big)$ and defines a subgradient of $(TJ)(x)$ at $\tilde{J}$. Note that the one-step lookahead policy $\tilde{\mu}$ need not be unique, since T need not be differentiable.

Thus, the equation

$$J = T_{\tilde{\mu}} J$$

is a pointwise (for each x) linearization of the equation

$$J = TJ$$

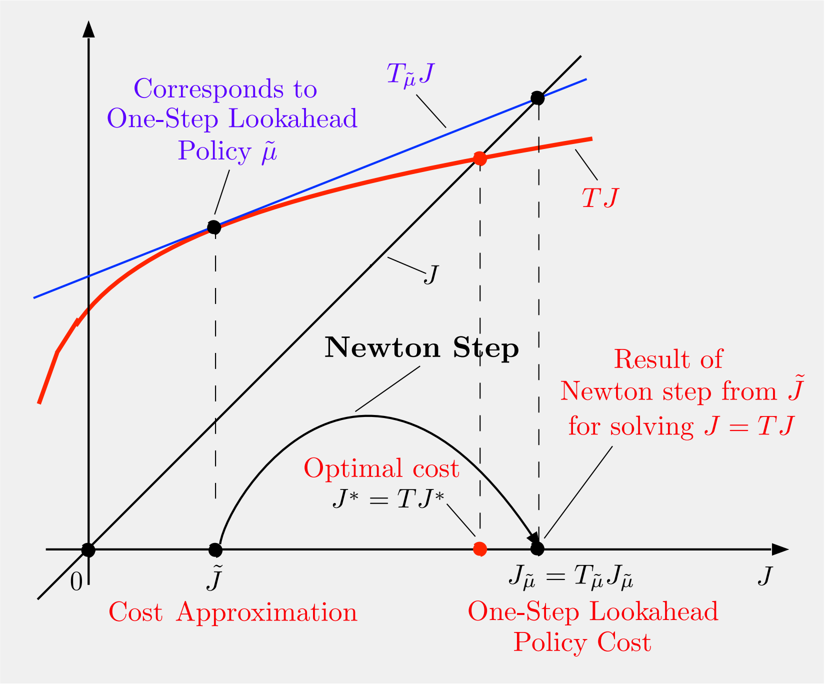


Figure 1.3.6 Geometric interpretation of approximation in value space and the one-step lookahead policy $\tilde{\mu}$ as a step of Newton's method. Given $\tilde{J}$, we find a policy $\tilde{\mu}$ that attains the minimum in the relation

$$T\tilde{J} = \min_{\mu} T_{\mu}\tilde{J}.$$

This policy satisfies $T\tilde{J} = T_{\tilde{\mu}}\tilde{J}$, so the graph of TJ and $T_{\tilde{\mu}}J$ touch at $\tilde{J}$, as shown. It may not be unique. Because TJ has concave components, the equation

$$J = T_{\tilde{\mu}}J$$

is the linearization of the equation $J = TJ$ at $\tilde{J}$. The linearized equation is solved at the typical step of Newton's method to provide the next iterate, which is just $J_{\tilde{\mu}}$.

at $\tilde{J}$, and its solution, $J_{\tilde{\mu}}$, can be viewed as the result of a Newton iteration at the point $\tilde{J}$. In summary, *the Newton iterate at $\tilde{J}$ is $J_{\tilde{\mu}}$*, the solution of the linearized equation $J = T_{\tilde{\mu}}J$.†

We may also consider approximation in value space with ℓ-step looka-

† The classical Newton's method for solving a fixed point problem of the form $y = T(y)$, where y is an n-dimensional vector, operates as follows: At the current iterate y_k, we linearize T and find the solution y_{k+1} of the corresponding linear fixed point problem. Assuming T is differentiable, the linearization is obtained

head using $\tilde{J}$. This is the same as approximation in value space with one-step lookahead using the $(\ell-1)$-fold operation of T on $\tilde{J}$, $T^{\ell-1}\tilde{J}$. Thus it can be interpreted as a Newton step starting from $T^{\ell-1}\tilde{J}$, the result of $\ell-1$ value iterations applied to $\tilde{J}$. This is illustrated in Fig. 1.3.7.†

1.3.2 Policy Iteration and Newton's Method

Another major class of infinite horizon algorithms is based on *policy iteration* (PI for short). We will discuss several abstract versions of PI in subsequent chapters, under a variety of assumptions. Generally, each iteration of the PI algorithm starts with a policy (which we call *current* or *base* policy), and generates another policy (which we call *new* or *rollout* policy, respectively). As an example, for the stochastic optimal control problem

by using a first order Taylor expansion:

$$y_{k+1} = T(y_k) + \frac{\partial T(y_k)}{\partial y}(y_{k+1} - y_k),$$

where $\partial T(y_k)/\partial y$ is the $n \times n$ Jacobian matrix of T evaluated at the vector y_k. The most commonly given convergence rate property of Newton's method is *quadratic convergence*. It states that near the solution y^*, we have

$$\|y_{k+1} - y^*\| = O\big(\|y_k - y^*\|^2\big),$$

where $\|\cdot\|$ is the Euclidean norm, and holds assuming the Jacobian matrix exists and is Lipschitz continuous (see [Ber16], Section 1.4). There are extensions of Newton's method that are based on solving a linearized system at the current iterate, but relax the differentiability requirement to piecewise differentiability, and/or component concavity, while maintaining the superlinear convergence property of the method.

The structure of the Bellman operators (1.26) and (1.27), with their monotonicity and concavity properties, tends to enhance the convergence and rate of convergence properties of Newton's method, even in the absence of differentiability, as evidenced by the convergence analysis of PI, and the extensive favorable experience with rollout, PI, and MPC. In this connection, it is worth noting that in the case of Markov games, where the concavity property does not hold, the PI method may oscillate, as shown by Pollatschek and Avi-Itzhak [PoA69], and needs to be modified to restore its global convergence; see the author's paper [Ber21c]. We will discuss abstract versions of game and minimax contexts n Chapter 5.

† Variants of Newton's method that involve combinations of first order iterative methods, such as the Gauss-Seidel and Jacobi algorithms, and Newton's method, and they belong to the general family of *Newton-SOR methods* (SOR stands for "successive over-relaxation"); see the classic book by Ortega and Rheinboldt [OrR70] (Section 13.4).

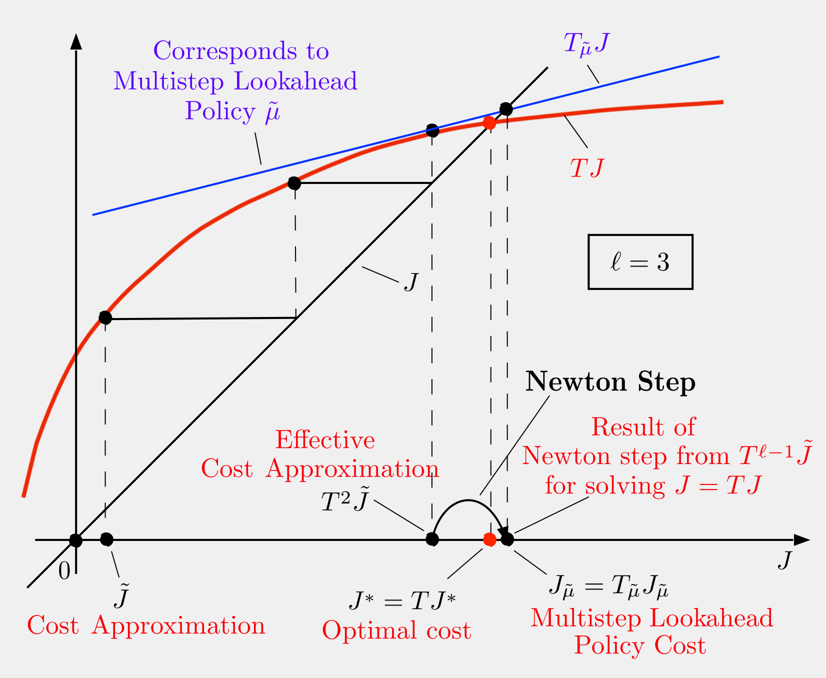


Figure 1.3.7 Geometric interpretation of approximation in value space with ℓ-step lookahead (in this figure $\ell = 3$). It is the same as approximation in value space with one-step lookahead using $T^{\ell-1}\tilde{J}$ as cost approximation. It can be viewed as a Newton step at the point $T^{\ell-1}\tilde{J}$, the result of $\ell - 1$ value iterations applied to $\tilde{J}$. Note that as ℓ increases the cost function $J_{\tilde{\mu}}$ of the ℓ-step lookahead policy $\tilde{\mu}$ approaches more closely the optimal J^*, and that $\lim_{\ell\to\infty} J_{\tilde{\mu}} = J^*$.

of Example 1.2.1, given the base policy μ, a policy iteration consists of two phases:

(a) *Policy evaluation*, which computes the cost function J_μ. One possibility is to solve the corresponding Bellman equation

$$J_\mu(x) = E\Big\{g\big(x,\mu(x),w\big) + \alpha J_\mu\big(f(x,\mu(x),w)\big)\Big\}, \qquad \text{for all } x. \tag{1.34}$$

However, the value $J_\mu(x)$ for any x can also be computed by Monte Carlo simulation, by averaging over many randomly generated trajectories the cost of the policy starting from x. Other possibilities include the use of specialized simulation-based methods, based on the projected and aggregation Bellman equations discussed in Section 1.2.4, for which there is extensive literature (see e.g., the books [BeT96], [SuB98], [Ber12a], [Ber19b]).

(b) *Policy improvement*, which computes the rollout policy $\tilde{\mu}$ using the one-step lookahead minimization

$$\tilde{\mu}(x) \in \arg\min_{u \in U(x)} E\Big\{g(x,u,w) + \alpha J_\mu\big(f(x,u,w)\big)\Big\}, \qquad \text{for all } x. \tag{1.35}$$

It is generally expected (and can be proved under mild conditions) that the rollout policy is improved in the sense that $J_{\tilde{\mu}}(x) \leq J_\mu(x)$ for all x.

Thus the PI process generates a sequence of policies $\{\mu^k\}$, by obtaining μ^{k+1} through a policy improvement operation using J_{μ^k} in place of J_μ in Eq. (1.35), which is obtained through policy evaluation of the preceding policy μ^k using Eq. (1.34). In subsequent chapters, we will show under appropriate assumptions that general forms of PI have interesting and often solid convergence properties, which may hold even when the method is implemented (with appropriate modifications) in unconventional computing environments, involving asynchronous distributed computation.

In terms of our abstract notation, the PI algorithm can be written in a compact form. For the generated policy sequence $\{\mu^k\}$, the policy evaluation phase obtains J_{μ^k} from the equation

$$J_{\mu^k} = T_{\mu^k} J_{\mu^k}, \tag{1.36}$$

while the policy improvement phase obtains μ^{k+1} through the equation

$$T_{\mu^{k+1}} J_{\mu^k} = T J_{\mu^k}. \tag{1.37}$$

As Fig. 1.3.8 illustrates, PI can be viewed as Newton's method for solving the Bellman equation in the function space of cost functions J. In particular, *the policy improvement Eq. (1.37) is the Newton step starting from J_{μ^k}, and yields μ^{k+1} as the corresponding one-step lookahead/rollout policy.*

The interpretation of PI as a form of Newton's method has a long history, for which we refer to the original works for linear quadratic problems by Kleinman [Klei68],† and for finite-state infinite horizon discounted and Markov game problems by Pollatschek and Avi-Itzhak [PoA69] (who also showed that the method may oscillate in the game case; see the discussion in Chapter 5).

† This was part of Kleinman's Ph.D. thesis [Kle67] at M.I.T., supervised by M. Athans. Kleinman gives credit for the one-dimensional version of his results to Bellman and Kalaba [BeK65]. Note also that the first proposal of the PI method was given by Bellman in his classic book [Bel57], under the name "approximation in policy space."

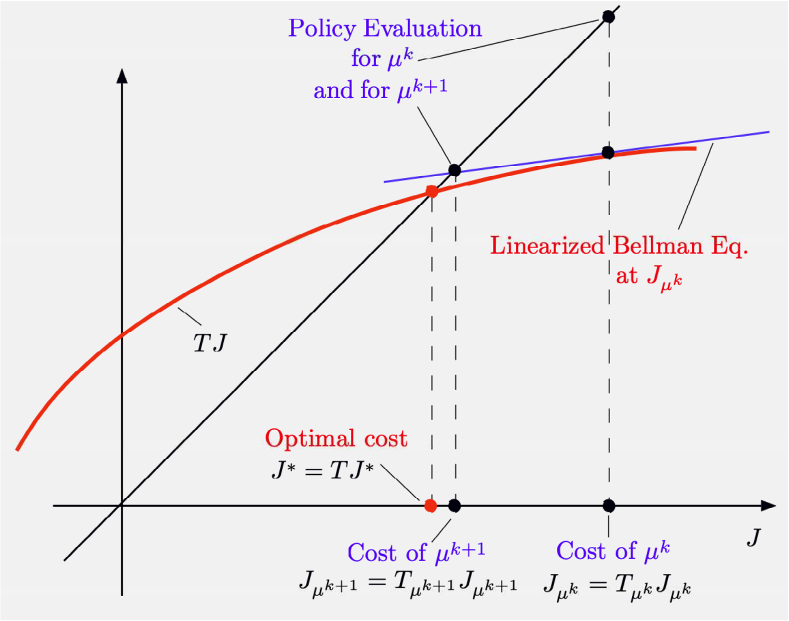


Figure 1.3.8 Geometric interpretation of a policy iteration. Starting from the stable current policy μ^k, it evaluates the corresponding cost function J_{μ^k}, and computes the next policy μ^{k+1} according to $T_{\mu^{k+1}} J_{\mu^k} = TJ_{\mu^k}$. The corresponding cost function $J_{\mu^{k+1}}$ is obtained as the solution of the linearized equation $J = T_{\mu^{k+1}} J$, so it is the result of a Newton step for solving the Bellman equation $J = TJ$, starting from J_{μ^k}. Note than in policy iteration, the Newton step always starts at a function J_μ, which satisfies $J_\mu \geq J^*$.

1.4 ORGANIZATION OF THE BOOK

The examples of the preceding sections have illustrated how the monotonicity assumption is satisfied for most DP models, while the contraction assumption may or may not be satisfied. In particular, the contraction assumption is satisfied for the mapping H in Examples 1.2.1-1.2.5, assuming that there is discounting and that the cost per stage is bounded, but it need not hold in the SSP Example 1.2.6, the multiplicative Example 1.2.8, and the affine monotonic Example 1.2.9.

The main theme of this book is that the presence or absence of monotonicity and contraction is the primary determinant of the analytical and algorithmic theory of a typical total cost DP model. In our development, with few exceptions, we will assume that monotonicity holds. Consequently, the rest of the book is organized around the presence or absence of the contraction property. In the next three chapters we will discuss the following three types of models.

(a) **Contractive models:** These models, discussed in Chapter 2, have the richest and strongest algorithmic theory, and are the benchmark against which the theory of other models is compared. Prominent among them are discounted stochastic optimal control problems (cf. Example 1.2.1), finite-state discounted MDP (cf. Example 1.2.2), and some special types of SSP problems (cf. Example 1.2.6).

(b) **Semicontractive models:** In these models T_μ is monotone but it need not be a contraction for all $\mu \in \mathcal{M}$. Most deterministic, stochastic, and minimax-type shortest path problems of practical interest are of this type. One of the difficulties here is that under certain circumstances, some of the cost functions of the problem may take the values $+\infty$ or $-\infty$, and the mappings T_μ and T must accordingly be allowed to deal with such functions.

The salient characteristic of semicontractive models is that policies are separated into those that "behave well" with respect to our optimization framework and those that do not. It turns out that the notion of contraction is not sufficiently general for our purposes. We will thus introduce a related notion of "regularity," which is based on the idea that a policy μ should be considered "well-behaved" if the dynamic system defined by T_μ has J_μ as an asymptotically stable equilibrium within some domain. Our models and analysis are patterned to a large extent after the SSP problems of Example 1.2.6 (the regular μ correspond to the proper policies). We show that the (restricted) optimal cost function over just the regular policies may have favorable value and policy iteration properties. By contrast, the optimal cost function over all policies J^* may not be obtainable by these algorithms, and indeed J^* may not be a solution of Bellman's equation, as we will show with a simple example in Section 3.1.2.

The key idea is that under certain conditions, the restricted optimization (under the regular policies only) is well behaved, both analytically and algorithmically. Under still stronger conditions, which directly or indirectly guarantee that there exists an optimal regular policy, we prove that semicontractive models have strong properties, sometimes almost as strong as those of the contractive models.

In Chapter 3, we develop the basic theory of semicontractive models for the case where the regular policies are stationary, while in Chapter 4 (Section 4.4), we extend the notion of regularity to nonstationary policies. Moreover, we illustrate the theory with a variety of interesting shortest path-type problems (stochastic, minimax, affine monotonic, and risk sensitive/exponential cost), linear-quadratic optimal control problems, and deterministic and stochastic optimal control problems.

(c) **Noncontractive models:** These models rely on just the monotonicity property of T_μ, and are more complex than the preceding ones. As in semicontractive models, the various cost functions of the problem may take the values $+\infty$ or $-\infty$, and in fact the optimal cost function may take the values ∞ and $-\infty$ as a matter of course (rather than on an exceptional basis, as in semicontractive models). The complications are considerable, and much of the theory of the contractive models generalizes in weaker form, if at all. For example, in general the fixed point equation $J = TJ$ need not have a unique solution, the value iteration method may work starting with some functions but not with others, and the policy iteration method may not work at all. Of course some of these weaknesses may not appear in the presence of additional structure, and we will discuss in Sections 4.4-4.6 noncontractive models that also have some semicontractive structure, and corresponding favorable properties.

Examples of DP problems from each of the above model categories, mostly special cases of the specific DP models discussed in Section 1.2, are scattered throughout the book, both to illustrate the theory and its exceptions, and to illustrate the beneficial role of additional special structure. In some other types of models there are restrictions to the set of policies, so that $\mathcal{M}$ may be a strict subset of the set of functions $\mu : X \mapsto U$ with $\mu(x) \in U(x)$ for all $x \in X$. Such restrictions may include measurability (needed to establish a mathematically rigorous probabilistic framework) or special structure that enhances the characterization of optimal policies and facilitates their computation. These models were treated in Chapter 5 of the first edition of this book, and also in Chapter 6 of [BeS78].†

Algorithms

Our discussion of algorithms centers on abstract forms of value and policy iteration, and is organized along three characteristics: *exact, approximate, and asynchronous*. The exact algorithms represent idealized versions, the approximate represent implementations that use approximations of various kinds, and the asynchronous involve irregular computation orders, where the costs and controls at different states are updated at different iterations (for example the cost of a single state being iterated at a time, as in Gauss-Seidel and other methods; see [Ber12a] for several examples of distributed asynchronous DP algorithms).

Approximate and asynchronous implementations have been the subject of intensive investigations since the 1980s, in the context of the solution of large-scale problems. Some of this methodology relies on the use of simulation, which is asynchronous by nature and is prominent in approximate DP. Generally, the monotonicity and sup-norm contraction structures of

† Chapter 5 of the first edition is accessible from the author's web site and the book's web page, and uses terminology and notation that are consistent with the present edition of the book.

many of the prominent DP models favors the use of asynchronous algorithms in DP, as first shown in the author's paper [Ber82], and discussed at various points, starting with Section 2.6.

1.5 NOTES, SOURCES, AND EXERCISES

This monograph is written in a mathematical style that emphasizes simplicity and abstraction. According to the relevant Wikipedia article:

"Abstraction in mathematics is the process of extracting the underlying essence of a mathematical concept, removing any dependence on real world objects with which it might originally have been connected, and generalizing it so that it has wider applications or matching among other abstract descriptions of equivalent phenomena ... The advantages of abstraction are:

(1) It reveals deep connections between different areas of mathematics.

(2) Known results in one area can suggest conjectures in a related area.

(3) Techniques and methods from one area can be applied to prove results in a related area.

One disadvantage of abstraction is that highly abstract concepts can be difficult to learn. A degree of mathematical maturity and experience may be needed for conceptual assimilation of abstractions."

Consistent with the preceding view of abstraction, our aim has been to construct a minimalist framework, where the important mathematical structures stand out, while the application context is deliberately blurred. Of course, our development has to pass the test of relevance to applications. In this connection, we note that our presentation has integrated the relation of our abstract DP models with the applications of Section 1.2, and particularly discounted stochastic optimal control models (Chapter 2), shortest path-type models (Chapters 3 and 4), undiscounted deterministic and stochastic optimal control models (Chapter 4), and minimax and zero-sum game problems (Chapter 5). We have given illustrations of the abstract mathematical theory using these models and others throughout the text. A much broader and accessible account of applications is given in the author's two-volume DP textbook.

Section 1.2: The abstract style of mathematical development has a long history in DP. In particular, the connection between DP and fixed point theory may be traced to Shapley [Sha53], who exploited contraction mapping properties in analysis of the two-player dynamic game model of Example 1.2.4. Since that time the underlying contraction properties of discounted DP problems with bounded cost per stage have been explicitly or implicitly used by most authors that have dealt with the subject. Moreover, the value of the abstract viewpoint as the basis for economical and insightful analysis has been widely recognized.

An abstract DP model, based on unweighted sup-norm contraction assumptions, was introduced in the paper by Denardo [Den67]. This model pointed to the fundamental connections between DP and fixed point theory, and provided generality and insight into the principal analytical and algorithmic ideas underlying the discounted DP research up to that time. Abstract DP ideas were also researched earlier, notably in the paper by Mitten (Denardo's Ph.D. thesis advisor) [Mit64]; see also Denardo and Mitten [DeM67]. The properties of monotone contractions were also used in the analysis of sequential games by Zachrisson [Zac64].

Two abstract DP models that rely only on monotonicity properties were given by the author in the papers [Ber75], [Ber77]. They were patterned after the negative cost DP problem of Blackwell [Bla65] and the positive cost DP problem of Strauch [Str66] (see the monotone decreasing and monotone increasing models of Section 4.3). These two abstract DP models, together with the finite horizon models of Section 4.2, were used extensively in the book by Bertsekas and Shreve [BeS78] for the analysis of both discounted and undiscounted DP problems, ranging over MDP, minimax, multiplicative, and Borel space models.

Extensions of the analysis of the author's [Ber77] were given by Verdu and Poor [VeP87], which considered additional structure that allows the development of backward and forward value iterations, and in the thesis by Szepesvari [Sze98a], [Sze98b], which introduced non-Markovian policies into the abstract DP framework. The model of [Ber77] was also used by Bertsekas [Ber82], and Bertsekas and Yu [BeY10], to develop asynchronous value and policy iteration methods for abstract contractive and noncontractive DP models. Another line of related research involving abstract DP mappings that are not necessarily scalar-valued was initiated by Mitten [Mit74], and was followed up by a number of authors, including Sobel [Sob75], Morin [Mor82], and Carraway and Morin [CaM88].

Section 1.3: The central role of the abstract DP framework and Newton's method in the conceptualization of reinforcement learning and approximate DP methods, was highlighted in the author's recent book [Ber22]. It was described in more mathematical detail in the book [Ber20].

Section 1.4: Generally, noncontractive total cost DP models with some special structure beyond monotonicity, fall in three major categories: monotone increasing models principally represented by positive cost DP, monotone decreasing models principally represented by negative cost DP, and transient models, exemplified by the SSP model of Example 1.2.6, where the decision process terminates after a period that is random and subject to control. Abstract DP models patterned after the first two categories have been known since the author's papers [Ber75], [Ber77], and are further discussed in Section 4.3. The semicontractive models of Chapter 3 and

Sections 4.4-4.6 (introduced and analyzed in the first edition of this book, as well as the subsequent series of papers and reports, [Ber15], [Ber16a], [BeY16], [Ber17b], [Ber17c], [Ber17d], [Ber19c]), are patterned after the third category. Their analysis is based on the idea of separating policies into those that are well-behaved (these are called *regular*, and have contraction-like properties) and those that are not (these are called *irregular*). The objective of the analysis is then to explain the detrimental effects of the irregular policies, and to delineate the kind of model structure that can limit these effects. As far as the author knows, this idea is new in the context of abstract DP. One of the aims of the present monograph is to develop this idea and to show that it leads to an important and insightful paradigm for conceptualization and solution of major classes of practical DP problems.

EXERCISES

1.1 (Multistep Contraction Mappings)

This exercise shows how starting with an abstract mapping, we can obtain multistep mappings with the same fixed points and a stronger contraction modulus. Consider a set of mappings $T_\mu : \mathcal{B}(X) \mapsto \mathcal{B}(X)$, $\mu \in \mathcal{M}$, satisfying the contraction Assumption 1.2.2, let m be a positive integer, and let $\mathcal{M}_m$ be the set of m-tuples $\nu = (\mu_0, \ldots, \mu_{m-1})$, where $\mu_k \in \mathcal{M}$, $k = 1, \ldots, m-1$. For each $\nu = (\mu_0, \ldots, \mu_{m-1}) \in \mathcal{M}_m$, define the mapping $\overline{T}_\nu$, by

$$\overline{T}_\nu J = T_{\mu_0} \cdots T_{\mu_{m-1}} J, \qquad \forall\ J \in \mathcal{B}(X).$$

Show the contraction properties

$$\|\overline{T}_\nu J - \overline{T}_\nu J'\| \le \alpha^m \|J - J'\|, \qquad \forall\ J, J' \in \mathcal{B}(X), \tag{1.38}$$

and

$$\|\overline{T} J - \overline{T} J'\| \le \alpha^m \|J - J'\|, \qquad \forall\ J, J' \in \mathcal{B}(X), \tag{1.39}$$

where $\overline{T}$ is defined by

$$(\overline{T} J)(x) = \inf_{(\mu_0, \ldots, \mu_{m-1}) \in \mathcal{M}_m} (T_{\mu_0} \cdots T_{\mu_{m-1}} J)(x), \qquad \forall\ J \in \mathcal{B}(X),\ x \in X.$$

Solution: By the contraction property of $T_{\mu_0}, \ldots, T_{\mu_{m-1}}$, we have for all $J, J' \in B(X)$,

$$\begin{aligned} \|\overline{T}_\nu J - \overline{T}_\nu J'\| &= \|T_{\mu_0} \cdots T_{\mu_{m-1}} J - T_{\mu_0} \cdots T_{\mu_{m-1}} J'\| \\ &\le \alpha \|T_{\mu_1} \cdots T_{\mu_{m-1}} J - T_{\mu_1} \cdots T_{\mu_{m-1}} J'\| \\ &\le \alpha^2 \|T_{\mu_2} \cdots T_{\mu_{m-1}} J - T_{\mu_2} \cdots T_{\mu_{m-1}} J'\| \\ &\ \ \vdots \\ &\le \alpha^m \|J - J'\|, \end{aligned}$$

thus showing Eq. (1.38).

We have from Eq. (1.38)

$$(T_{\mu_0}\cdots T_{\mu_{m-1}}J)(x) \le (T_{\mu_0}\cdots T_{\mu_{m-1}}J')(x) + \alpha^m \|J-J'\|\, v(x), \qquad \forall\ x \in X,$$

and by taking infimum of both sides over $(T_{\mu_0}\cdots T_{\mu_{m-1}}) \in \mathcal{M}_m$ and dividing by $v(x)$, we obtain

$$\frac{(\overline{T}J)(x) - (\overline{T}J')(x)}{v(x)} \le \alpha^m \|J-J'\|, \qquad \forall\ x \in X.$$

Similarly

$$\frac{(\overline{T}J')(x) - (\overline{T}J)(x)}{v(x)} \le \alpha^m \|J-J'\|, \qquad \forall\ x \in X,$$

and by combining the last two relations and taking supremum over $x \in X$, Eq. (1.39) follows.

1.2 (Relation of Multistep and Proximal Mappings [Ber16b], [Ber18c])

Consider a linear mapping of the form

$$TJ = AJ + b,$$

where b is a vector in $\Re^n$, and A is an $n \times n$ matrix with eigenvalues strictly within the unit circle. Let $\lambda \in (0,1)$ and $c = \frac{\lambda}{1-\lambda}$, and consider the multistep mapping

$$\big(T^{(\lambda)}J\big)(i) = (1-\lambda)\sum_{\ell=0}^{\infty} \lambda^{\ell}(T^{\ell+1}J)(i), \qquad i = 1,\ldots,n,\ J \in \Re^n,$$

and the proximal mapping

$$P^{(c)}J = \left(\frac{c+1}{c}I - A\right)^{-1}\left(b + \frac{1}{c}J\right);$$

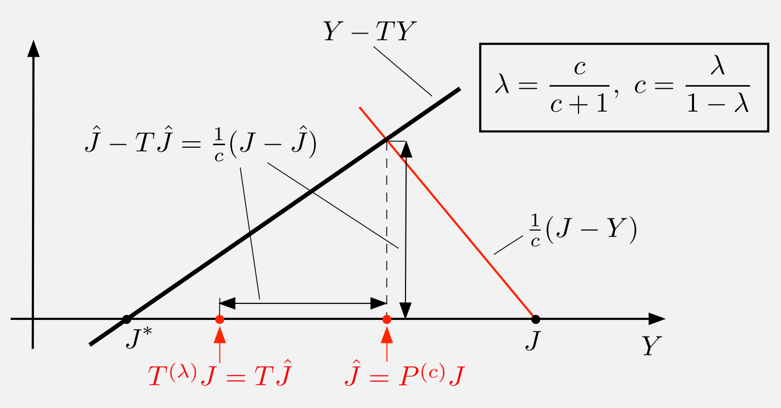
cf. Eq. (1.23) [equivalently, for a given J, $P^{(c)}J$ is the unique vector $Y \in \Re^n$ that solves the equation

$$Y - TY = \frac{1}{c}(J - Y),$$

(cf. Fig. 1.5.1)].

(a) Show that $P^{(c)}$ is given by

$$P^{(c)} = (1-\lambda)\sum_{\ell=0}^{\infty} \lambda^{\ell} T^{\ell},$$



$Y - TY$
$\lambda = \frac{c}{c+1}, \ c = \frac{\lambda}{1-\lambda}$
$\hat J - T\hat J = \frac{1}{c}(J - \hat J)$
$\frac{1}{c}(J - Y)$
J^*
J
Y
$T^{(\lambda)}J = T\hat J$
$\hat J = P^{(c)}J$

Figure 1.5.1. Illustration of the iterates $T^{(\lambda)}J$ and $P^{(c)}J$ for finding the fixed point J^* of a linear mapping T. Given J, we find the proximal iterate $\hat J = P^{(c)}J$ and then add the amount $\frac{1}{c}\big(\hat J - J\big)$ to obtain $T^{(\lambda)}J = TP^{(c)}J$. If T is a contraction mapping, $T^{(\lambda)}J$ is closer to J^* than $P^{(c)}J$.

and can be written as

$$P^{(c)}J = \overline{A}^{(\lambda)}J + \overline{b}^{(\lambda)},$$

where

$$\overline{A}^{(\lambda)} = (1-\lambda)\sum_{\ell=0}^{\infty}\lambda^\ell A^\ell, \qquad \overline{b}^{(\lambda)} = \sum_{\ell=0}^{\infty}\lambda^{\ell+1}A^\ell b.$$

(b) Verify that

$$T^{(\lambda)}J = A^{(\lambda)}J + b^{(\lambda)},$$

where

$$A^{(\lambda)} = (1-\lambda)\sum_{\ell=0}^{\infty}\lambda^\ell A^{\ell+1}, \qquad b^{(\lambda)} = \sum_{\ell=0}^{\infty}\lambda^{\ell}A^\ell b,$$

and show that

$$T^{(\lambda)} = TP^{(c)} = P^{(c)}T, \tag{1.40}$$

and that for all $J \in \Re^n$,

$$P^{(c)}J = J + \lambda\big(T^{(\lambda)}J - J\big), \qquad T^{(\lambda)}J = J + \frac{c+1}{c}\big(P^{(c)}J - J\big). \tag{1.41}$$

Thus $T^{(\lambda)}J$ is obtained by extrapolation along the line segment $P^{(c)}J - J$, as illustrated in Fig. 1.5.1. Note that since T is a contraction mapping, $T^{(\lambda)}J$ is closer to J^* than $P^{(c)}J$.

(c) Show that for a given $J \in \Re^n$, the multistep and proximal iterates $T^{(\lambda)}J$ and $P^{(c)}J$ are the unique fixed points of the contraction mappings W_J and $\overline{W}_J$ given by

$$W_JY = (1-\lambda)TJ + \lambda TY, \qquad \overline{W}_JY = (1-\lambda)J + \lambda TY, \qquad Y \in \Re^n,$$

respectively.

(d) Show that the fixed point property of part (c) yields the following formula for the multistep mapping $T^{(\lambda)}$:

$$T^{(\lambda)}J = (1 - \lambda A)^{-1}\big(b + (1 - \lambda)AJ\big). \tag{1.42}$$

(e) (*Multistep Contraction Property for Nonexpansive A [BeY09]*) Instead of assuming that A has eigenvalues strictly within the unit circle, assume that the matrix $I - A$ is invertible and A is nonexpansive [i.e., has all its eigenvalues within the unit circle (possibly on the unit circle)]. Show that $A^{(\lambda)}$ is contractive (i.e., has eigenvalues that lie strictly within the unit circle) and its eigenvalues have the form

$$\theta_i = (1 - \lambda)\sum_{\ell=0}^{\infty}\lambda^\ell\zeta_i^{\ell+1} = \frac{\zeta_i(1 - \lambda)}{1 - \zeta_i\lambda}, \qquad i = 1, \ldots, n, \tag{1.43}$$

where ζ_i, $i = 1, \ldots, n$, are the eigenvalues of A. *Note*: For an intuitive explanation of the result, note that the eigenvalues of $A^{(\lambda)}$ can be viewed as convex combinations of complex numbers from the unit circle at least two of which are different from each other, since $\zeta_i \neq 1$ by assumption (the nonzero corresponding eigenvalues of A and A^2 are different from each other). As a result the eigenvalues of $A^{(\lambda)}$ lie strictly within the unit circle.

(f) (*Contraction Property of Projected Multistep Mappings*) Under the assumptions of part (e), show that $\lim_{\lambda\to 1} A^{(\lambda)} = 0$. Furthermore, for any $n \times n$ matrix W, the matrix $WA^{(\lambda)}$ is contractive for λ sufficiently close to 1. In particular the projected mapping $\Pi A^{(\lambda)}$ and corresponding projected proximal mapping (cf. Section 1.2.5) become contractions as $\lambda \to 1$.

Solution: (a) The inverse in the definition of $P^{(c)}$ is written as

$$\left(\frac{c+1}{c}I - A\right)^{-1} = \left(\frac{1}{\lambda}I - A\right)^{-1} = \lambda(I - \lambda A)^{-1} = \lambda\sum_{\ell=0}^{\infty}(\lambda A)^\ell.$$

Thus, using the equation $\frac{1}{c} = \frac{1-\lambda}{\lambda}$,

$$\begin{aligned} P^{(c)}J &= \left(\frac{c+1}{c}I - A\right)^{-1}\left(b + \frac{1}{c}J\right) \\ &= \lambda\sum_{\ell=0}^{\infty}(\lambda A)^\ell\left(b + \frac{1-\lambda}{\lambda}J\right) \\ &= (1 - \lambda)\sum_{\ell=0}^{\infty}(\lambda A)^\ell J + \lambda\sum_{\ell=0}^{\infty}(\lambda A)^\ell b, \end{aligned}$$

which is equal to $\overline{A}^{(\lambda)}J + \overline{b}^{(\lambda)}$. The formula $P^{(c)} = (1 - \lambda)\sum_{\ell=0}^{\infty}\lambda^\ell T^\ell$ follows from this expression.

(b) The formula $T^{(\lambda)}J = A^{(\lambda)}J + b^{(\lambda)}$ is verified by straightforward calculation. We have,

$$\begin{aligned} TP^{(c)}J &= A\big(\overline{A}^{(\lambda)}J + \overline{b}^{(\lambda)}\big) + b \\ &= (1-\lambda)\sum_{\ell=0}^{\infty}\lambda^{\ell}A^{\ell+1}J + \sum_{\ell=0}^{\infty}\lambda^{\ell+1}A^{\ell+1}b + b = A^{(\lambda)}J + b^{(\lambda)} \\ &= T^{(\lambda)}J, \end{aligned}$$

thus proving the left side of Eq. (1.40). The right side is proved similarly. The interpolation/extrapolation formula (1.41) follows by a straightforward calculation from the definition of $T^{(\lambda)}$. As an example, to show the left side of Eq. (1.41), we write

$$\begin{aligned} J + \lambda\big(T^{(\lambda)}J - J\big) &= (1-\lambda)J + \lambda T^{(\lambda)}J \\ &= (1-\lambda)J + \lambda\left((1-\lambda)\sum_{\ell=0}^{\infty}\lambda^{\ell}A^{\ell+1}J + \sum_{\ell=0}^{\infty}\lambda^{\ell}A^{\ell}b\right) \\ &= (1-\lambda)\left(J + \sum_{\ell=1}^{\infty}\lambda^{\ell}A^{\ell}J\right) + \sum_{\ell=0}^{\infty}\lambda^{\ell+1}A^{\ell}b \\ &= \overline{A}^{(\lambda)}J + \overline{b}^{(\lambda)} \\ &= P^{(c)}J. \end{aligned}$$

(c) To show that $T^{(\lambda)}J$ is the fixed point of W_J, we must verify that

$$T^{(\lambda)}J = W_J\big(T^{(\lambda)}J\big),$$

or equivalently that

$$T^{(\lambda)}J = (1-\lambda)TJ + \lambda T\big(T^{(\lambda)}J\big) = (1-\lambda)TJ + \lambda T^{(\lambda)}(TJ).$$

The right-hand side, in view of the interpolation formula

$$(1-\lambda)J + \lambda T^{(\lambda)}J = P^{(c)}J, \qquad \forall\ x \in \Re^n,$$

is equal to $P^{(c)}(TJ)$, which from the formula $T^{(\lambda)} = P^{(c)}T$ [cf. part (b)], is equal to $T^{(\lambda)}J$. The proof is similar for $\overline{W}_J$.

(d) The fixed point property of part (c) states that $T^{(\lambda)}J$ is the unique solution of the following equation in Y:

$$Y = (1-\lambda)TJ + \lambda TY = (1-\lambda)(AJ + b) + \lambda(AY + b),$$

from which the desired relation follows.

(e), (f) The formula (1.43) follows from the expression for $A^{(\lambda)}$ given in part (b). This formula can be used to show that the eigenvalues of $A^{(\lambda)}$ lie strictly within the unit circle, using also the fact that the matrices A^m, $m \geq 1$, and $A^{(\lambda)}$ have the same eigenvectors (see [BeY09] for details). Moreover, the eigenvalue formula shows that all eigenvalues of $A^{(\lambda)}$ converge to 0 as $\lambda \to 1$, so that $\lim_{\lambda\to 1} A^{(\lambda)} = 0$. This also implies that $WA^{(\lambda)}$ is contractive for λ sufficiently close to 1.

1.3 (State-Dependent Weighted Multistep Mappings [YuB12])

Consider a set of mappings $T_\mu : \mathcal{B}(X) \mapsto \mathcal{B}(X)$, $\mu \in \mathcal{M}$, satisfying the contraction Assumption 1.2.2. Consider also the mappings $T_\mu^{(w)} : \mathcal{B}(X) \mapsto \mathcal{B}(X)$ defined by

$$(T_\mu^{(w)} J)(x) = \sum_{\ell=1}^{\infty} w_\ell(x) \big(T_\mu^\ell J\big)(x), \qquad x \in X,\ J \in \mathcal{B}(X),$$

where $w_\ell(x)$ are nonnegative scalars such that for all $x \in X$,

$$\sum_{\ell=1}^{\infty} w_\ell(x) = 1.$$

Show that

$$\frac{\big|(T_\mu^{(w)} J)(x) - (T_\mu^{(w)} J')(x)\big|}{v(x)} \leq \sum_{\ell=1}^{\infty} w_\ell(x)\, \alpha^\ell \|J - J'\|, \qquad \forall\ x \in X,$$

where α is the contraction modulus of T_μ, so that $T_\mu^{(w)}$ is a contraction with modulus

$$\bar{\alpha} = \sup_{x \in X} \sum_{\ell=1}^{\infty} w_\ell(x)\, \alpha^\ell \leq \alpha.$$

Show also that for all $\mu \in \mathcal{M}$, the mappings T_μ and $T_\mu^{(w)}$ have the same fixed point.

Solution: By the contraction property of T_μ, we have for all $J, J' \in B(X)$ and $x \in X$,

$$\begin{aligned}\frac{\big|(T_\mu^{(w)} J)(x) - (T_\mu^{(w)} J')(x)\big|}{v(x)} &= \frac{\big|\sum_{\ell=1}^{\infty} w_\ell(x)(T_\mu^\ell J)(x) - \sum_{\ell=1}^{\infty} w_\ell(x)(T_\mu^\ell J')(x)\big|}{v(x)} \\ &\leq \sum_{\ell=1}^{\infty} w_\ell(x) \|T_\mu^\ell J - T_\mu^\ell J'\| \\ &\leq \left(\sum_{\ell=1}^{\infty} w_\ell(x) \alpha^\ell\right) \|J - J'\|,\end{aligned}$$

showing the contraction property of $T_\mu^{(w)}$.

Let J_μ be the fixed point of T_μ. By using the relation $(T_\mu^\ell J_\mu)(x) = J_\mu(x)$, we have for all $x \in X$,

$$(T_\mu^{(w)} J_\mu)(x) = \sum_{\ell=1}^{\infty} w_\ell(x) \big(T_\mu^\ell J_\mu\big)(x) = \left(\sum_{\ell=1}^{\infty} w_\ell(x)\right) J_\mu(x) = J_\mu(x),$$

so J_μ is the fixed point of $T_\mu^{(w)}$ [which is unique since $T_\mu^{(w)}$ is a contraction].

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