APPENDIX A:

Notation and Mathematical Conventions

In this appendix we collect our notation, and some related mathematical facts and conventions.

A.1 SET NOTATION AND CONVENTIONS

If $X$ is a set and $x$ is an element of $X$, we write $x \in X$. A set can be specified in the form $X = \{ x \mid x \text{ satisfies } P \}$, as the set of all elements satisfying property $P$. The union of two sets $X_1$ and $X_2$ is denoted by $X_1 \cup X_2$, and their intersection by $X_1 \cap X_2$. The empty set is denoted by $\emptyset$. The symbol $\forall$ means “for all.”

The set of real numbers (also referred to as scalars) is denoted by $\mathbb{R}$. The set of extended real numbers is denoted by $\mathbb{R}^*$:

$$\mathbb{R}^* = \mathbb{R} \cup \{ \infty, -\infty \}.$$  

We write $-\infty < x < \infty$ for all real numbers $x$, and $-\infty \leq x \leq \infty$ for all extended real numbers $x$. We denote by $[a, b]$ the set of (possibly extended) real numbers $x$ satisfying $a \leq x \leq b$. A rounded, instead of square, bracket denotes strict inequality in the definition. Thus $(a, b]$, $[a, b)$, and $(a, b)$ denote the set of all $x$ satisfying $a < x \leq b$, $a \leq x < b$, and $a < x < b$, respectively.

Generally, we adopt standard conventions regarding addition and multiplication in $\mathbb{R}^*$, except that we take

$$\infty - \infty = -\infty + \infty = \infty,$$
and we take the product of 0 and $\infty$ or $-\infty$ to be 0. In this way the sum and product of two extended real numbers is well-defined. Division by 0 or $\infty$ does not appear in our analysis. In particular, we adopt the following rules in calculations involving $\infty$ and $-\infty$:

$$
\alpha + \infty = \infty + \alpha = \infty, \quad \forall \alpha \in \mathbb{R}^*;
$$

$$
\alpha - \infty = -\infty + \alpha = -\infty, \quad \forall \alpha \in [-\infty, \infty);
$$

$$
\alpha \cdot \infty = \infty, \quad \alpha \cdot (-\infty) = -\infty, \quad \forall \alpha \in (0, \infty],
$$

$$
0 \cdot \infty = \infty \cdot 0 = 0 = 0 \cdot (-\infty) = (-\infty) \cdot 0 = (-\infty) = \infty.
$$

Under these rules, the following laws of arithmetic are still valid within $\mathbb{R}^*$:

$$
\alpha_1 + \alpha_2 = \alpha_2 + \alpha_1, \quad (\alpha_1 + \alpha_2) + \alpha_3 = \alpha_1 + (\alpha_2 + \alpha_3),
$$

$$
\alpha_1\alpha_2 = \alpha_2\alpha_1, \quad (\alpha_1\alpha_2)\alpha_3 = \alpha_1(\alpha_2\alpha_3).
$$

We also have

$$
\alpha(\alpha_1 + \alpha_2) = \alpha\alpha_1 + \alpha\alpha_2
$$

if either $\alpha \geq 0$ or else $(\alpha_1 + \alpha_2)$ is not of the form $\infty - \infty$.

**Inf and Sup Notation**

The *supremum* of a nonempty set $X \subset \mathbb{R}^*$, denoted by $\text{sup} \ X$, is defined as the smallest $y \in \mathbb{R}^*$ such that $y \geq x$ for all $x \in X$. Similarly, the *infimum* of $X$, denoted by $\text{inf} \ X$, is defined as the largest $y \in \mathbb{R}^*$ such that $y \leq x$ for all $x \in X$. For the empty set, we use the convention

$$
\text{sup} \ \emptyset = -\infty, \quad \text{inf} \ \emptyset = \infty.
$$

If $\text{sup} \ X$ is equal to an $\overline{x} \in \mathbb{R}^*$ that belongs to the set $X$, we say that $\overline{x}$ is the *maximum point* of $X$ and we write $\overline{x} = \text{max} \ X$. Similarly, if $\text{inf} \ X$ is equal to an $\underline{x} \in \mathbb{R}^*$ that belongs to the set $X$, we say that $\underline{x}$ is the *minimum point* of $X$ and we write $\underline{x} = \text{min} \ X$. Thus, when we write $\text{max} \ X$ (or $\text{min} \ X$, respectively), we do so just for emphasis: we indicate that it is either evident, or it is known through earlier analysis, or it is about to be shown that the maximum (or minimum, respectively) of the set $X$ is attained at one of its points.
A.2 FUNCTIONS

If \( f \) is a function, we use the notation \( f : X \mapsto Y \) to indicate the fact that \( f \) is defined on a nonempty set \( X \) (its domain) and takes values in a set \( Y \) (its range). Thus when using the notation \( f : X \mapsto Y \), we implicitly assume that \( X \) is nonempty. We will often use the unit function \( e : X \mapsto \mathbb{R} \), defined by

\[
e(x) = 1, \quad \forall x \in X.
\]

Given a set \( X \), we denote by \( \mathcal{R}(X) \) the set of real-valued functions \( J : X \mapsto \mathbb{R} \), and by \( \mathcal{E}(X) \) the set of all extended real-valued functions \( J : X \mapsto \mathbb{R}^+ \). For any collection \( \{J_\gamma \mid \gamma \in \Gamma\} \subset \mathcal{E}(X) \), parameterized by the elements of a set \( \Gamma \), we denote by \( \inf_{\gamma \in \Gamma} J_\gamma \) the function taking the value \( \inf_{\gamma \in \Gamma} J_\gamma(x) \) at each \( x \in X \).

For two functions \( J_1, J_2 \in \mathcal{E}(X) \), we use the shorthand notation \( J_1 \leq J_2 \) to indicate the pointwise inequality

\[
J_1(x) \leq J_2(x), \quad \forall x \in X.
\]

We use the shorthand notation \( \inf_{i \in I} J_i \) to denote the function obtained by pointwise infimum of a collection \( \{J_i \mid i \in I\} \subset \mathcal{E}(X) \), i.e.,

\[
\left( \inf_{i \in I} J_i \right)(x) = \inf_{i \in I} J_i(x), \quad \forall x \in X.
\]

We use similar notation for sup.

Given subsets \( S_1, S_2, S_3 \subset \mathcal{E}(X) \) and mappings \( T_1 : S_1 \mapsto S_3 \) and \( T_2 : S_2 \mapsto S_1 \), the composition of \( T_1 \) and \( T_2 \) is the mapping \( T_1 T_2 : S_2 \mapsto S_3 \) defined by

\[
(T_1 T_2)(x) = (T_1(T_2(x))), \quad \forall J \in S_2, \ x \in X.
\]

In particular, given a subset \( S \subset \mathcal{E}(X) \) and mappings \( T_1 : S \mapsto S \) and \( T_2 : S \mapsto S \), the composition of \( T_1 \) and \( T_2 \) is the mapping \( T_1 T_2 : S \mapsto S \) defined by

\[
(T_1 T_2)(x) = (T_1(T_2(x))), \quad \forall J \in S, \ x \in X.
\]

Similarly, given mappings \( T_k : S \mapsto S \), \( k = 1, \ldots, N \), their composition is the mapping \( (T_1 \cdots T_N) : S \mapsto S \) defined by

\[
(T_1 T_2 \cdots T_N)(x) = (T_1(T_2(\cdots(T_N(x)))))(x), \quad \forall J \in S, \ x \in X.
\]

In our notation involving compositions we minimize the use of parentheses, as long as clarity is not compromised. In particular, we write \( T_1 T_2 J \) instead of \( (T_1 T_2) J \) or \( T_1(T_2) J \) or \( T_1(T_2 J) \), but we write \( (T_1 T_2 J)(x) \) to indicate the value of \( T_1 T_2 J \) at \( x \in X \).

If \( X \) and \( Y \) are nonempty sets, a mapping \( T : S_1 \mapsto S_2 \), where \( S_1 \subset \mathcal{E}(X) \) and \( S_2 \subset E(Y) \), is said to be monotone if for all \( J, J' \in S_1 \),

\[
J \leq J' \implies T J \leq T J'.
\]
Sequences of Functions

For a sequence of functions \( \{J_k\} \subset \mathcal{E}(X) \) that converges pointwise, we denote by \( \lim_{k \to \infty} J_k \) the pointwise limit of \( \{J_k\} \). We denote by \( \limsup_{k \to \infty} J_k \) (or \( \liminf_{k \to \infty} J_k \)) the pointwise limit superior (or inferior, respectively) of \( \{J_k\} \). If \( \{J_k\} \subset \mathcal{E}(X) \) converges pointwise to \( J \), we write \( J_k \to J \). Note that we reserve this notation for pointwise convergence. To denote convergence with respect to a norm \( \| \cdot \| \), we write \( \|J_k - J\| \to 0 \).

A sequence of functions \( \{J_k\} \subset \mathcal{E}(X) \) is said to be **monotonically nonincreasing** (or **monotonically nondecreasing**) if \( J_{k+1} \leq J_k \) for all \( k \) (or \( J_{k+1} \geq J_k \) for all \( k \), respectively). Such a sequence always has a (pointwise) limit within \( \mathcal{E}(X) \). We write \( J_k \downarrow J \) (or \( J_k \uparrow J \)) to indicate that \( \{J_k\} \) is monotonically nonincreasing (or monotonically nonincreasing, respectively) and that its limit is \( J \).

Let \( \{J_{mn}\} \subset \mathcal{E}(X) \) be a double indexed sequence, which is monotonically nonincreasing separately for each index in the sense that

\[
J_{(m+1)n} \leq J_{mn}, \quad J_{m(n+1)} \leq J_{mn}, \quad \forall m, n = 0, 1, \ldots.
\]

For such sequences, a useful fact is that

\[
\lim_{m \to \infty} \left( \lim_{n \to \infty} J_{mn} \right) = \lim_{m \to \infty} J_{mm}.
\]

There is a similar fact for monotonically nondecreasing sequences.

Expected Values

Given a random variable \( w \) defined over a probability space \( \Omega \), the expected value of \( w \) is defined by

\[
E\{w\} = E\{w^+\} + E\{w^-\},
\]

where \( w^+ \) and \( w^- \) are the positive and negative parts of \( w \),

\[
w^+(\omega) = \max\{0, w(\omega)\}, \quad w^-(\omega) = \min\{0, w(\omega)\}.
\]

In this way, taking also into account the rule \( \infty - \infty = \infty \), the expected value \( E\{w\} \) is well-defined if \( \Omega \) is finite or countably infinite. In more general cases, \( E\{w\} \) is similarly defined by the appropriate form of integration, as will be discussed in more detail at specific points as needed.
APPENDIX B:

Contraction Mappings

B.1 CONTRACTION MAPPING FIXED POINT THEOREMS

The purpose of this appendix is to provide some background on contraction mappings and their properties. Let $Y$ be a real vector space with a norm $\| \cdot \|$, i.e., a real-valued function satisfying for all $y \in Y$, $\| y \| \geq 0$, $\| y \| = 0$ if and only if $y = 0$, and

$$\| ay \| = |a| \| y \|, \quad \forall a \in \mathbb{R}, \quad \| y + z \| \leq \| y \| + \| z \|, \quad \forall y, z \in Y.$$  

Let $\bar{Y}$ be a closed subset of $Y$. A function $F : \bar{Y} \to \bar{Y}$ is said to be a contraction mapping if for some $\rho \in (0, 1)$, we have

$$\| Fy - Fz \| \leq \rho \| y - z \|, \quad \forall y, z \in \bar{Y}.$$  

The scalar $\rho$ is called the modulus of contraction of $F$.

Example B.1 (Linear Contraction Mappings in $\mathbb{R}^n$)

Consider the case of a linear mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$Fy = b + Ay,$$

where $A$ is an $n \times n$ matrix and $b$ is a vector in $\mathbb{R}^n$. Let $\sigma(A)$ denote the spectral radius of $A$ (the largest modulus among the moduli of the eigenvalues of $A$). Then it can be shown that $A$ is a contraction mapping with respect to some norm if and only if $\sigma(A) < 1$.

Specifically, given $\epsilon > 0$, there exists a norm $\| \cdot \|_s$ such that

$$\| Ay \|_s \leq (\sigma(A) + \epsilon) \| y \|_s, \quad \forall y \in \mathbb{R}^n.$$  

(B.1)
Thus, if $\sigma(A) < 1$ we may select $\epsilon > 0$ such that $\rho = \sigma(A) + \epsilon < 1$, and obtain the contraction relation
\[
\|Fy - Fz\| = \|A(y - z)\| \leq \rho \|y - z\|, \quad \forall \ y, z \in \mathbb{R}^n. \tag{B.2}
\]
The norm $\| \cdot \|_s$ can be taken to be a weighted Euclidean norm, i.e., it may have the form $\|y\|_s = \|My\|$, where $M$ is a square invertible matrix, and $\| \cdot \|$ is the standard Euclidean norm, i.e., $\|x\| = \sqrt{x^T x}$. \footnote{We may show Eq. (B.1) by using the Jordan canonical form of $A$, which is denoted by $J$. In particular, if $P$ is a nonsingular matrix such that $P^{-1}AP = J$ and $D$ is the diagonal matrix with $1, \delta, \ldots, \delta^{n-1}$ along the diagonal, where $\delta > 0$, it is straightforward to verify that $D^{-1}P^{-1}APD = J$, where $J$ is the matrix that is identical to $J$ except that each nonzero off-diagonal term is replaced by $\delta$.}

Conversely, if Eq. (B.2) holds for some norm $\| \cdot \|_s$ and all real vectors $y, z$, it also holds for all complex vectors $y, z$ with the squared norm $\|e\|_s^2$ of a complex vector $e$ defined as the sum of the squares of the norms of the real and the imaginary components. Thus from Eq. (B.2), by taking $y - z = u$, where $u$ is an eigenvector corresponding to an eigenvalue $\lambda$ with $|\lambda| = \sigma(A)$, we have $\sigma(A)\|u\|_s = \|Au\|_s \leq \rho\|u\|_s$. Hence $\sigma(A) \leq \rho$, and it follows that if $F$ is a contraction with respect to a given norm, we must have $\sigma(A) < 1$.

A sequence $\{y_k\} \subset Y$ is said to be a Cauchy sequence if $\|y_m - y_n\| \to 0$ as $m, n \to \infty$, i.e., given any $\epsilon > 0$, there exists $N$ such that $\|y_m - y_n\| \leq \epsilon$ for all $m, n \geq N$. The space $Y$ is said to be complete under the norm $\| \cdot \|$ if every Cauchy sequence $\{y_k\} \subset Y$ is convergent, in the sense that for some $\hat{y} \in Y$, we have $\|y_k - \hat{y}\| \to 0$. Note that a Cauchy sequence is always bounded. Also, a Cauchy sequence of real numbers is convergent, implying that the real line is a complete space and so is every real finite-dimensional vector space. On the other hand, an infinite dimensional space may not be complete under some norms, while it may be complete under other norms.

When $Y$ is complete and $\bar{Y}$ is a closed subset of $Y$, an important property of a contraction mapping $F : \bar{Y} \to \bar{Y}$ is that it has a unique fixed point within $\bar{Y}$, i.e., the equation
\[
y = Fy
\]
has a unique solution $y^* \in \bar{Y}$, called the fixed point of $F$. Furthermore, the sequence $\{y_k\}$ generated by the iteration
\[
y_{k+1} = Fy_k
\]
Proposition B.1: (Contraction Mapping Fixed-Point Theorem) Let $\bar{Y}$ be a complete vector space and let $Y$ be a closed subset of $Y$. Then if $F : \bar{Y} \mapsto \bar{Y}$ is a contraction mapping with modulus $\rho \in (0, 1)$, there exists a unique $y^* \in \bar{Y}$ such that

$$y^* = F y^*.$$ 

Furthermore, the sequence $\{F^k y\}$ converges to $y^*$ for any $y \in \bar{Y}$, and we have

$$\|F^k y - y^*\| \leq \rho^k \|y - y^*\|, \quad k = 1, 2, \ldots.$$ 

Proof: Let $y \in \bar{Y}$ and consider the iteration $y_{k+1} = F y_k$ starting with $y_0 = y$. By the contraction property of $F$,

$$\|y_{k+1} - y_k\| \leq \rho \|y_k - y_{k-1}\|, \quad k = 1, 2, \ldots,$$

which implies that

$$\|y_{k+1} - y_k\| \leq \rho^k \|y_1 - y_0\|, \quad k = 1, 2, \ldots.$$ 

It follows that for every $k \geq 0$ and $m \geq 1$, we have

$$\|y_{k+m} - y_k\| \leq \sum_{i=1}^{m} \|y_{k+i} - y_{k+i-1}\|$$

$$\leq \rho^k (1 + \rho + \cdots + \rho^{m-1}) \|y_1 - y_0\|$$

$$\leq \frac{\rho^k}{1 - \rho} \|y_1 - y_0\|.$$ 

Therefore, $\{y_k\}$ is a Cauchy sequence in $\bar{Y}$ and must converge to a limit $y^* \in \bar{Y}$, since $\bar{Y}$ is complete and $\bar{Y}$ is closed. We have for all $k \geq 1$,

$$\|F y^* - y^*\| \leq \|F y^* - y_k\| + \|y_k - y^*\| \leq \rho \|y^* - y_{k-1}\| + \|y_k - y^*\|$$

and since $y_k$ converges to $y^*$, we obtain $F y^* = y^*$. Thus, the limit $y^*$ of $y_k$ is a fixed point of $F$. It is a unique fixed point because if $\tilde{y}$ were another fixed point, we would have

$$\|y^* - \tilde{y}\| = \|F y^* - F \tilde{y}\| \leq \rho \|y^* - \tilde{y}\|,$$

which implies that $y^* = \tilde{y}$. 

converges to $y^*$, starting from an arbitrary initial point $y_0$. 

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To show the convergence rate bound of the last part, note that
\[ \| F^k y - y^* \| = \| F^k y - F^k y^* \| \leq \rho \| F^{k-1} y - y^* \|. \]
Repeating this process for a total of \( k \) times, we obtain the desired result.
Q.E.D.

The convergence rate exhibited by \( F^k y \) in the preceding proposition is said to be \textit{geometric}, and \( F^k y \) is said to converge to its limit \( y^* \) \textit{geometrically}. This is in reference to the fact that the error \( \| F^k y - y^* \| \) converges to 0 faster than some geometric progression (\( \rho^k \| y - y^* \| \) in this case).

In some contexts of interest to us one may encounter mappings that are not contractions, but become contractions when iterated a finite number of times. In this case, one may use a slightly different version of the contraction mapping fixed point theorem, which we now present.

We say that a function \( F : \bar{Y} \mapsto \bar{Y} \) is an \( m \)-stage contraction mapping if there exists a positive integer \( m \) and some \( \rho < 1 \) such that
\[ \| F^m y - F^m y' \| \leq \rho \| y - y' \|, \quad \forall \ y, y' \in \bar{Y}, \]
where \( F^m \) denotes the composition of \( F \) with itself \( m \) times. Thus, \( F \) is an \( m \)-stage contraction if \( F^m \) is a contraction. Again, the scalar \( \rho \) is called the modulus of contraction. We have the following generalization of Prop. B.1.

**Proposition B.2: \( m \)-Stage Contraction Mapping Fixed-Point Theorem** Let \( Y \) be a complete vector space and let \( \bar{Y} \) be a closed subset of \( Y \). Then if \( F : \bar{Y} \mapsto \bar{Y} \) is an \( m \)-stage contraction mapping with modulus \( \rho \in (0,1) \), there exists a unique \( y^* \in \bar{Y} \) such that
\[ y^* = F y^*. \]
Furthermore, \( \{ F^k y \} \) converges to \( y^* \) for any \( y \in \bar{Y} \).

**Proof:** Since \( F^m \) maps \( \bar{Y} \) into \( \bar{Y} \) and is a contraction mapping, by Prop. B.1, it has a unique fixed point in \( \bar{Y} \), denoted \( y^* \). Applying \( F \) to both sides of the relation \( y^* = F^m y^* \), we see that \( F y^* \) is also a fixed point of \( F^m \), so by the uniqueness of the fixed point, we have \( y^* = F y^* \). Therefore \( y^* \) is a fixed point of \( F \). If \( F \) had another fixed point, say \( \hat{y} \), then we would have \( \hat{y} = F^m \hat{y} \), which by the uniqueness of the fixed point of \( F^m \) implies that \( \hat{y} = y^* \). Thus, \( y^* \) is the unique fixed point of \( F \).

To show the convergence of \( \{ F^k y \} \), note that by Prop. B.1, we have for all \( y \in \bar{Y} \),
\[ \lim_{k \to \infty} \| F^{mk} y - y^* \| = 0. \]
Using $F^\ell y$ in place of $y$, we obtain
\[
\lim_{k \to \infty} \|F^{mk+\ell}y - y^*\| = 0, \quad \ell = 0, 1, \ldots, m - 1,
\]
which proves the desired result. \text{Q.E.D.}

\section{B.2 Weighted Sup-Norm Contractions}

In this section, we will focus on contraction mappings within a specialized context that is particularly important in DP. Let $X$ be a set (typically the state space in DP), and let $v: X \to \mathbb{R}$ be a positive-valued function, $v(x) > 0$, $\forall x \in X$.

Let $B(X)$ denote the set of all functions $J: X \to \mathbb{R}$ such that $J(x)/v(x)$ is bounded as $x$ ranges over $X$. We define a norm on $B(X)$, called the \textit{weighted sup-norm}, by
\[
\|J\| = \sup_{x \in X} \frac{|J(x)|}{v(x)}.
\]  \hfill (B.3)

It is easily verified that $\|\cdot\|$ thus defined has the required properties for being a norm. Furthermore, $B(X)$ \textit{is complete under this norm}. To see this, consider a Cauchy sequence $\{J_k\} \subset B(X)$, and note that $\|J_m - J_n\| \to 0$ as $m, n \to \infty$ implies that for all $x \in X$, $\{J_k(x)\}$ is a Cauchy sequence of real numbers, so it converges to some $J^*(x)$. We will show that $J^* \in B(X)$ and that $\|J_k - J^*\| \to 0$. To this end, it will be sufficient to show that given any $\epsilon > 0$, there exists a $K$ such that
\[
\frac{|J_k(x) - J^*(x)|}{v(x)} \leq \epsilon, \quad \forall x \in X, \, k \geq K.
\]
This will imply that
\[
\sup_{x \in X} \frac{|J^*(x)|}{v(x)} \leq \epsilon + \|J_k\|, \quad \forall k \geq K,
\]
so that $J^* \in B(X)$, and will also imply that $\|J_k - J^*\| \leq \epsilon$, so that $\|J_k - J^*\| \to 0$. Assume the contrary, i.e., that there exists an $\epsilon > 0$ and a subsequence $\{x_{m_1}, x_{m_2}, \ldots\} \subset X$ such that $m_i < m_{i+1}$ and
\[
\epsilon < \frac{|J_{m_i}(x_{m_i}) - J^*(x_{m_i})|}{v(x_{m_i})}, \quad \forall i \geq 1.
\]
The right-hand side above is less or equal to
\[
\frac{|J_{m_i}(x_{m_i}) - J_n(x_{m_i})|}{v(x_{m_i})} + \frac{|J_n(x_{m_i}) - J^*(x_{m_i})|}{v(x_{m_i})}, \quad \forall n \geq 1, \, i \geq 1.
\]
The first term in the above sum is less than $\epsilon/2$ for $i$ and $n$ larger than some threshold; fixing $i$ and letting $n$ be sufficiently large, the second term can also be made less than $\epsilon/2$, so the sum is made less than $\epsilon$ - a contradiction. In conclusion, the space $\mathcal{B}(X)$ is complete, so the fixed point results of Props. B.1 and B.2 apply.

In our discussions, we will always assume that $\mathcal{B}(X)$ is equipped with the weighted sup-norm above, where the weight function $v$ will be clear from the context. There will be frequent occasions where the norm will be unweighted, i.e., $v(x) \equiv 1$ and $\|J\| = \sup_{x \in X} |J(x)|$, in which case we will explicitly state so.

Finite-Dimensional Cases

Let us now focus on the finite-dimensional case $X = \{1, \ldots, n\}$, in which case $\mathcal{R}(X)$ and $\mathcal{B}(X)$ can be identified with $\mathbb{R}^n$. Consider a linear mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ of the form

$$Fy = b + Ay,$$

where $A$ is an $n \times n$ matrix with components $a_{ij}$, and $b$ is a vector in $\mathbb{R}^n$ (cf. Example B.1). Then it can be shown (see the following proposition) that $F$ is a contraction with respect to the weighted sup-norm $\|y\| = \max_{i=1,\ldots,n} |y_i|/v(i)$ if and only if

$$\frac{\sum_{j=1}^n |a_{ij}| v(j)}{v(i)} < 1, \quad i = 1, \ldots, n.$$

Let us also denote by $|A|$ the matrix whose components are the absolute values of the components of $A$ and let $\sigma(|A|)$ denote the spectral radius of $|A|$. Then it can be shown that $F$ is a contraction with respect to some weighted sup-norm if and only if $\sigma(|A|) < 1$. A proof of this may be found in [BeT89], Ch. 2, Cor. 6.2. Thus any substochastic matrix $P$ ($p_{ij} \geq 0$ for all $i, j$, and $\sum_{j=1}^n p_{ij} \leq 1$, for all $i$) is a contraction with respect to some weighted sup-norm if and only if $\sigma(P) < 1$.

Finally, let us consider a nonlinear mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ that has the property

$$|Fy - Fz| \leq P |y - z|, \quad \forall \ y, z \in \mathbb{R}^n,$$

for some matrix $P$ with nonnegative components and $\sigma(P) < 1$. Here, we generically denote by $|w|$ the vector whose components are the absolute values of the components of $w$, and the inequality is componentwise. Then we claim that $F$ is a contraction with respect to some weighted sup-norm. To see this note that by the preceding discussion, $P$ is a contraction with respect to some weighted sup-norm $\|y\| = \max_{i=1,\ldots,n} |y_i|/v(i)$, and we have

$$\frac{(Fy - Fz)(i)}{v(i)} \leq \frac{(P |y - z|)(i)}{v(i)} \leq \alpha \|y - z\|, \quad \forall \ i = 1, \ldots, n,$$
for some \( \alpha \in (0, 1) \), where \( (|Fy - Fz|(i) \) and \( (P|y - z|(i) \) are the \( i \)th components of the vectors \(|Fy - Fz|\) and \( P|y - z|\), respectively. Thus, \( F\) is a contraction with respect to \( \| \cdot \| \). For additional discussion of linear and nonlinear contraction mapping properties and characterizations such as the one above, see the book [OrR70].

**Linear Mappings on Countable Spaces**

The case where \( X \) is countable (or, as a special case, finite) is frequently encountered in DP. The following proposition provides some useful criteria for verifying the contraction property of mappings that are either linear or are obtained via a parametric minimization of other contraction mappings.

**Proposition B.3:** Let \( X = \{1, 2, \ldots \} \).

(a) Let \( F : \mathcal{B}(X) \mapsto \mathcal{B}(X) \) be a linear mapping of the form

\[
(FJ)(i) = b_i + \sum_{j \in X} a_{ij} J(j), \quad i \in X,
\]

where \( b_i \) and \( a_{ij} \) are some scalars. Then \( F \) is a contraction with modulus \( \rho \) with respect to the weighted sup-norm (B.3) if and only if

\[
\sum_{j \in X} \frac{|a_{ij}| v(j)}{v(i)} \leq \rho, \quad i \in X. \tag{B.4}
\]

(b) Let \( F : \mathcal{B}(X) \mapsto \mathcal{B}(X) \) be a mapping of the form

\[
(FJ)(i) = \inf_{\mu \in M} (F_{\mu}J)(i), \quad i \in X,
\]

where \( M \) is parameter set, and for each \( \mu \in M \), \( F_{\mu} \) is a contraction mapping from \( \mathcal{B}(X) \) to \( \mathcal{B}(X) \) with modulus \( \rho \). Then \( F \) is a contraction mapping with modulus \( \rho \).

**Proof:** (a) Assume that Eq. (B.4) holds. For any \( J, J' \in \mathcal{B}(X) \), we have

\[
\|FJ - FJ'\| = \sup_{i \in X} \left| \sum_{j \in X} a_{ij} (J(j) - J'(j)) \right| v(i)
\]

\[
\leq \sup_{i \in X} \left( \sum_{j \in X} |a_{ij}| v(j) \left| (J(j) - J'(j))/v(j) \right| v(i) \right)
\]

\[
\leq \sup_{i \in X} \left( \sum_{j \in X} |a_{ij}| v(j) \right) \frac{\|J - J'\|}{v(i)}
\]
where the last inequality follows from the hypothesis.

Conversely, arguing by contradiction, let’s assume that Eq. (B.4) is violated for some \( i \in X \). Define \( J(j) = v(j) \sgn(a_{ij}) \) and \( J'(j) = 0 \) for all \( j \in X \). Then we have \( ||J - J'|| = ||J|| = 1 \), and

\[
\frac{|(FJ)(i) - (FJ')(i)|}{v(i)} = \sum_{j \in X} |a_{ij}| \frac{v(j)}{v(i)} > \rho = \rho ||J - J'||,
\]

showing that \( F \) is not a contraction of modulus \( \rho \).

(b) Since \( F_\mu \) is a contraction of modulus \( \rho \), we have for any \( J, J' \in \mathcal{B}(X) \),

\[
\frac{(F_\mu J)(i)}{v(i)} \leq \frac{(F_\mu J')(i)}{v(i)} + \rho ||J - J'||, \quad i \in X,
\]

so by taking the infimum over \( \mu \in M \),

\[
\frac{(FJ)(i)}{v(i)} \leq \frac{(FJ')(i)}{v(i)} + \rho ||J - J'||, \quad i \in X.
\]

Reversing the roles of \( J \) and \( J' \), we obtain

\[
\frac{|(FJ)(i) - (FJ')(i)|}{v(i)} \leq \rho ||J - J'||, \quad i \in X,
\]

and by taking the supremum over \( i \), the contraction property of \( F \) is proved.

Q.E.D.

The preceding proposition assumes that \( FJ \in \mathcal{B}(X) \) for all \( J \in \mathcal{B}(X) \). The following proposition provides conditions, particularly relevant to the DP context, which imply this assumption.

**Proposition B.4:** Let \( X = \{1, 2, \ldots \} \), let \( M \) be a parameter set, and for each \( \mu \in M \), let \( F_\mu \) be a linear mapping of the form

\[
(F_\mu J)(i) = b_i(\mu) + \sum_{j \in X} a_{ij}(\mu) J(j), \quad i \in X.
\]

(a) We have \( F_\mu J \in \mathcal{B}(X) \) for all \( J \in \mathcal{B}(X) \) provided \( b(\mu) \in \mathcal{B}(X) \) and \( V(\mu) \in \mathcal{B}(X) \), where

\[
b(\mu) = \{b_1(\mu), b_2(\mu), \ldots \}, \quad V(\mu) = \{V_1(\mu), V_2(\mu), \ldots \},
\]

with

\[
V_i(\mu) = \sum_{j \in X} |a_{ij}(\mu)| v(j), \quad i \in X.
\]
(b) Consider the mapping $F$

$$(FJ)(i) = \inf_{\mu \in M} (F\mu J)(i), \quad i \in X.$$  

We have $FJ \in B(X)$ for all $J \in B(X)$, provided $b \in B(X)$ and $V \in B(X)$, where

$$b = \{b_1, b_2, \ldots\}, \quad V = \{V_1, V_2, \ldots\},$$

with $b_i = \sup_{\mu \in M} b_i(\mu)$ and $V_i = \sup_{\mu \in M} V_i(\mu)$.

**Proof:** (a) For all $\mu \in M$, $J \in B(X)$ and $i \in X$, we have

$$(F\mu J)(i) \leq |b_i(\mu)| + \sum_{j \in X} |a_{ij}(\mu)| |J(j)/v(j)| v(j)$$

$$\leq |b_i(\mu)| + \|J\| \sum_{j \in X} |a_{ij}(\mu)| v(j)$$

$$= |b_i(\mu)| + \|J\| V_i(\mu),$$

and similarly $(F\mu J)(i) \geq -|b_i(\mu)| - \|J\| V_i(\mu)$. Thus

$$|(F\mu J)(i)| \leq |b_i(\mu)| + \|J\| V_i(\mu), \quad i \in X.$$  

By dividing this inequality with $v(i)$ and by taking the supremum over $i \in X$, we obtain

$$\|F\mu J\| \leq \|b\| + \|J\| \|V\| < \infty.$$  

(b) By doing the same as in (a), but after first taking the infimum of $(F\mu J)(i)$ over $\mu$, we obtain

$$\|FJ\| \leq \|b\| + \|J\| \|V\| < \infty.$$  

Q.E.D.
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