APPENDIX A:

Notation and Mathematical Conventions

In this appendix we collect our notation, and some related mathematical facts and conventions.

A.1 SET NOTATION AND CONVENTIONS

If $X$ is a set and $x$ is an element of $X$, we write $x \in X$. A set can be specified in the form $X = \{x \mid x \text{ satisfies } P\}$, as the set of all elements satisfying property $P$. The union of two sets $X_1$ and $X_2$ is denoted by $X_1 \cup X_2$, and their intersection by $X_1 \cap X_2$. The empty set is denoted by $\emptyset$. The symbol $\forall$ means “for all.”

The set of real numbers (also referred to as scalars) is denoted by $\mathbb{R}$. The set of extended real numbers is denoted by $\mathbb{R}^*$:

$$\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}.$$  

We write $-\infty < x < \infty$ for all real numbers $x$, and $-\infty \leq x \leq \infty$ for all extended real numbers $x$. We denote by $[a, b]$ the set of (possibly extended) real numbers $x$ satisfying $a \leq x \leq b$. A rounded, instead of square, bracket denotes strict inequality in the definition. Thus $(a, b]$, $[a, b)$, and $(a, b)$ denote the set of all $x$ satisfying $a < x \leq b$, $a \leq x < b$, and $a < x < b$, respectively.

Generally, we adopt standard conventions regarding addition and multiplication in $\mathbb{R}^*$, except that we take

$$\infty - \infty = -\infty + \infty = \infty,$$
and we take the product of 0 and \(\infty\) or \(-\infty\) to be 0. In this way the sum and product of two extended real numbers is well-defined. Division by 0 or \(\infty\) does not appear in our analysis. In particular, we adopt the following rules in calculations involving \(\infty\) and \(-\infty\):

\[
\begin{align*}
\alpha + \infty &= \infty + \alpha = \infty, & \forall \alpha \in \mathbb{R}^*, \\
\alpha - \infty &= -\infty + \alpha = -\infty, & \forall \alpha \in [-\infty, \infty), \\
\alpha \cdot \infty &= \infty, & \forall \alpha \in (0, \infty], \\
\alpha \cdot (\infty) &= \infty, & \forall \alpha \in (0, \infty], \\
\alpha \cdot -\infty &= -\infty, & \forall \alpha \in [-\infty, 0), \\
0 \cdot \infty &= \infty \cdot 0 = 0 = 0 \cdot (\infty) = (\infty) \cdot 0 = -\infty = \infty.
\end{align*}
\]

Under these rules, the following laws of arithmetic are still valid within \(\mathbb{R}^*\):

\[
\begin{align*}
\alpha_1 + \alpha_2 &= \alpha_2 + \alpha_1, & (\alpha_1 + \alpha_2) + \alpha_3 &= \alpha_1 + (\alpha_2 + \alpha_3), \\
\alpha_1\alpha_2 &= \alpha_2\alpha_1, & (\alpha_1\alpha_2)\alpha_3 &= \alpha_1(\alpha_2\alpha_3).
\end{align*}
\]

We also have

\[
\alpha(\alpha_1 + \alpha_2) = \alpha\alpha_1 + \alpha\alpha_2
\]

if either \(\alpha \geq 0\) or else \((\alpha_1 + \alpha_2)\) is not of the form \(\infty - \infty\).

### Inf and Sup Notation

The \textit{supremum} of a nonempty set \(X \subset \mathbb{R}^*\), denoted by \(\sup X\), is defined as the smallest \(y \in \mathbb{R}^*\) such that \(y \geq x\) for all \(x \in X\). Similarly, the \textit{infimum} of \(X\), denoted by \(\inf X\), is defined as the largest \(y \in \mathbb{R}^*\) such that \(y \leq x\) for all \(x \in X\). For the empty set, we use the convention

\[
\sup \emptyset = -\infty, \quad \inf \emptyset = \infty.
\]

If \(\sup X\) is equal to an \(\mathfrak{p} \in \mathbb{R}^*\) that belongs to the set \(X\), we say that \(\mathfrak{p}\) is the \textit{maximum point} of \(X\) and we write \(\mathfrak{p} = \max X\). Similarly, if \(\inf X\) is equal to an \(\mathfrak{p} \in \mathbb{R}^*\) that belongs to the set \(X\), we say that \(\mathfrak{p}\) is the \textit{minimum point} of \(X\) and we write \(\mathfrak{p} = \min X\). Thus, when we write \(\max X\) (or \(\min X\)) in place of \(\sup X\) (or \(\inf X\), respectively), we do so just for emphasis: we indicate that it is either evident, or it is known through earlier analysis, or it is about to be shown that the maximum (or minimum, respectively) of the set \(X\) is attained at one of its points.
A.2 FUNCTIONS

If $f$ is a function, we use the notation $f : X \mapsto Y$ to indicate the fact that $f$ is defined on a nonempty set $X$ (its domain) and takes values in a set $Y$ (its range). Thus when using the notation $f : X \mapsto Y$, we implicitly assume that $X$ is nonempty. We will often use the unit function $e : X \mapsto \mathbb{R}$, defined by

$$e(x) = 1, \quad \forall x \in X.$$  

Given a set $X$, we denote by $\mathcal{R}(X)$ the set of real-valued functions $J : X \mapsto \mathbb{R}$, and by $\mathcal{E}(X)$ the set of all extended real-valued functions $J : X \mapsto \mathbb{R}^\ast$. For any collection $\{J_\gamma \mid \gamma \in \Gamma\} \subset \mathcal{E}(X)$, parameterized by the elements of a set $\Gamma$, we denote by $\inf_{\gamma \in \Gamma} J_\gamma$ the function taking the value $\inf_{\gamma \in \Gamma} J_\gamma(x)$ at each $x \in X$.

For two functions $J_1, J_2 \in \mathcal{E}(X)$, we use the shorthand notation $J_1 \leq J_2$ to indicate the pointwise inequality

$$J_1(x) \leq J_2(x), \quad \forall x \in X.$$  

We use the shorthand notation $\inf_{i \in I} J_i$ to denote the function obtained by pointwise infimum of a collection $\{J_i \mid i \in I\} \subset \mathcal{E}(X)$, i.e.,

$$\left( \inf_{i \in I} J_i \right)(x) = \inf_{i \in I} J_i(x), \quad \forall x \in X.$$  

We use similar notation for sup.

Given subsets $S_1, S_2, S_3 \subset \mathcal{E}(X)$ and mappings $T_1 : S_1 \mapsto S_3$ and $T_2 : S_2 \mapsto S_1$, the composition of $T_1$ and $T_2$ is the mapping $T_1 T_2 : S_2 \mapsto S_3$ defined by

$$(T_1 T_2)(x) = (T_1(T_2)(x)), \quad \forall J \in S_2, x \in X.$$  

In particular, given a subset $S \subset \mathcal{E}(X)$ and mappings $T_1 : S \mapsto S$ and $T_2 : S \mapsto S$, the composition of $T_1$ and $T_2$ is the mapping $T_1 T_2 : S \mapsto S$ defined by

$$(T_1 T_2)(x) = (T_1(T_2)(x)), \quad \forall J \in S, x \in X.$$  

Similarly, given mappings $T_k : S \mapsto S$, $k = 1, \ldots, N$, their composition is the mapping $(T_1 \cdots T_N) : S \mapsto S$ defined by

$$(T_1 T_2 \cdots T_N)(x) = (T_1(T_2(\cdots (T_N)(x))))(x), \quad \forall J \in S, x \in X.$$  

In our notation involving compositions we minimize the use of parentheses, as long as clarity is not compromised. In particular, we write $T_1 T_2 J$ instead of $(T_1 T_2)(J)$ or $(T_1(T_2)(J)$ or $T_1(T_2)(J)$, but we write $(T_1 T_2 J)(x)$ to indicate the value of $T_1 T_2 J$ at $x \in X$.

If $X$ and $Y$ are nonempty sets, a mapping $T : S_1 \mapsto S_2$, where $S_1 \subset \mathcal{E}(X)$ and $S_2 \subset \mathcal{E}(Y)$, is said to be monotone if for all $J, J' \in S_1$,

$$J \leq J' \quad \Rightarrow \quad TJ \leq TJ'.$$
Sequences of Functions

For a sequence of functions \( \{J_k\} \subset \mathcal{E}(X) \) that converges pointwise, we denote by \( \lim_{k \to \infty} J_k \) the pointwise limit of \( \{J_k\} \). We denote by \( \limsup_{k \to \infty} J_k \) (or \( \liminf_{k \to \infty} J_k \)) the pointwise limit superior (or inferior, respectively) of \( \{J_k\} \). If \( \{J_k\} \subset \mathcal{E}(X) \) converges pointwise to \( J \), we write \( J_k \to J \). Note that we reserve this notation for pointwise convergence. To denote convergence with respect to a norm \( \| \cdot \| \), we write \( \| J_k - J \| \to 0 \).

A sequence of functions \( \{J_k\} \subset \mathcal{E}(X) \) is said to be monotonically nonincreasing (or monotonically nondecreasing) if \( J_{k+1} \leq J_k \) for all \( k \) (or \( J_{k+1} \geq J_k \) for all \( k \), respectively). Such a sequence always has a (pointwise) limit within \( \mathcal{E}(X) \). We write \( J_k \downarrow J \) (or \( J_k \uparrow J \)) to indicate that \( \{J_k\} \) is monotonically nonincreasing (or monotonically nonincreasing, respectively) and that its limit is \( J \).

Let \( \{J_{mn}\} \subset \mathcal{E}(X) \) be a double indexed sequence, which is monotonically nonincreasing separately for each index in the sense that

\[
J_{(m+1)n} \leq J_{mn}, \quad J_{m(n+1)} \leq J_{mn}, \quad \forall m, n = 0, 1, \ldots
\]

For such sequences, a useful fact is that

\[
\lim_{m \to \infty} \left( \lim_{n \to \infty} J_{mn} \right) = \lim_{n \to \infty} J_{nm}.
\]

There is a similar fact for monotonically nondecreasing sequences.

Expected Values

Given a random variable \( w \) defined over a probability space \( \Omega \), the expected value of \( w \) is defined by

\[
E\{w\} = E\{w^+\} + E\{w^-\},
\]

where \( w^+ \) and \( w^- \) are the positive and negative parts of \( w \),

\[
w^+(\omega) = \max \{0, w(\omega)\}, \quad w^-(\omega) = \min \{0, w(\omega)\}.
\]

In this way, taking also into account the rule \( -\infty - \infty = \infty \), the expected value \( E\{w\} \) is well-defined if \( \Omega \) is finite or countably infinite. In more general cases, \( E\{w\} \) is similarly defined by the appropriate form of integration, and more detail will be given at specific points as needed.
APPENDIX B:

Contraction Mappings

B.1 CONTRACTION MAPPING FIXED POINT THEOREMS

The purpose of this appendix is to provide some background on contraction mappings and their properties. Let $Y$ be a real vector space with a norm $\| \cdot \|$, i.e., a real-valued function satisfying for all $y \in Y$, $\|y\| \geq 0$, $\|y\| = 0$ if and only if $y = 0$, and

\[
\|ay\| = |a|\|y\|, \quad \forall \ a \in \mathbb{R}, \quad \|y + z\| \leq \|y\| + \|z\|, \quad \forall \ y, z \in Y.
\]

Let $\overline{Y}$ be a closed subset of $Y$. A function $F : \overline{Y} \mapsto \overline{Y}$ is said to be a contraction mapping if for some $\rho \in (0, 1)$, we have

\[
\|Fy - Fz\| \leq \rho\|y - z\|, \quad \forall \ y, z \in \overline{Y}.
\]

The scalar $\rho$ is called the modulus of contraction of $F$.

Example B.1 (Linear Contraction Mappings in $\mathbb{R}^n$)

Consider the case of a linear mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ of the form

\[
Fy = b + Ay,
\]

where $A$ is an $n \times n$ matrix and $b$ is a vector in $\mathbb{R}^n$. Let $\sigma(A)$ denote the spectral radius of $A$ (the largest modulus among the moduli of the eigenvalues of $A$). Then it can be shown that $A$ is a contraction mapping with respect to some norm if and only if $\sigma(A) < 1$.

Specifically, given $\epsilon > 0$, there exists a norm $\| \cdot \|_s$ such that

\[
\|Ay\|_s \leq (\sigma(A) + \epsilon)\|y\|_s, \quad \forall \ y \in \mathbb{R}^n.
\]  

(B.1)
Thus, if $\sigma(A) < 1$ we may select $\epsilon > 0$ such that $\rho = \sigma(A) + \epsilon < 1$, and obtain the contraction relation
\[
\|Fy - Fz\| = \|A(y - z)\|_s \leq \rho \|y - z\|_s, \quad \forall \ y, z \in \mathbb{R}^n. \tag{B.2}
\]
The norm $\| \cdot \|_s$ can be taken to be a weighted Euclidean norm, i.e., it may have the form $\|y\|_s = \|My\|$, where $M$ is a square invertible matrix, and $\| \cdot \|$ is the standard Euclidean norm, i.e., $\|x\| = \sqrt{x^T x}$.\(^\dagger\)

Conversely, if Eq. (B.2) holds for some norm $\| \cdot \|_s$ and all real vectors $y, z$, it also holds for all complex vectors $y, z$ with the squared norm $\|c\|_s^2$ of a complex vector $c$ defined as the sum of the squares of the norms of the real and the imaginary components. Thus from Eq. (B.2), by taking $y - z = u$, where $u$ is an eigenvector corresponding to an eigenvalue $\lambda$ with $|\lambda| = \sigma(A)$, we have $\sigma(A)\|u\|_s \leq \rho\|u\|_s$. Hence $\sigma(A) \leq \rho$, and it follows that if $F$ is a contraction with respect to a given norm, we must have $\sigma(A) < 1$.

Thus, if $\sigma(A) < 1$ we may select $\epsilon > 0$ such that $\rho = \sigma(A) + \epsilon < 1$, and obtain the contraction relation
\[
\|Fy - Fz\| = \|A(y - z)\|_s \leq \rho \|y - z\|_s, \quad \forall \ y, z \in \mathbb{R}^n. \tag{B.2}
\]
The norm $\| \cdot \|_s$ can be taken to be a weighted Euclidean norm, i.e., it may have the form $\|y\|_s = \|My\|$, where $M$ is a square invertible matrix, and $\| \cdot \|$ is the standard Euclidean norm, i.e., $\|x\| = \sqrt{x^T x}$.\(^\dagger\)

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A sequence $\{y_k\} \subset Y$ is said to be a Cauchy sequence if $\|y_m - y_n\| \to 0$ as $m, n \to \infty$, i.e., given any $\epsilon > 0$, there exists $N$ such that $\|y_m - y_n\| \leq \epsilon$ for all $m, n \geq N$. The space $Y$ is said to be complete under the norm $\| \cdot \|$ if every Cauchy sequence $\{y_k\} \subset Y$ is convergent, in the sense that for some $y \in Y$, we have $\|y_k - y\| \to 0$. Note that a Cauchy sequence is always bounded. Also, a Cauchy sequence of real numbers is convergent, implying that the real line is a complete space and so is every real finite-dimensional vector space. On the other hand, an infinite dimensional space may not be complete under some norms, while it may be complete under other norms.

When $Y$ is complete and $\overline{Y}$ is a closed subset of $Y$, an important property of a contraction mapping $F : \overline{Y} \to \overline{Y}$ is that it has a unique fixed point within $\overline{Y}$, i.e., the equation
\[
y = Fy
\]
has a unique solution $y^* \in \overline{Y}$, called the fixed point of $F$. Furthermore, the sequence $\{y_k\}$ generated by the iteration
\[
y_{k+1} = Fy_k
\]
\(^\dagger\) We may show Eq. (B.1) by using the Jordan canonical form of $A$, which is denoted by $J$. In particular, if $P$ is a nonsingular matrix such that $P^{-1}AP = J$ and $D$ is the diagonal matrix with $1, \delta, \ldots, \delta^{n-1}$ along the diagonal, where $\delta > 0$, it is straightforward to verify that $D^{-1}P^{-1}APD = J$, where $J$ is the matrix that is identical to $J$ except that each nonzero off-diagonal term is replaced by $\delta$.

Defining $\hat{P} = PD$, we have $A = \hat{P}J\hat{P}^{-1}$. Now if $\| \cdot \|$ is the standard Euclidean norm, we note that for some $\beta > 0$, we have $\|Jz\| \leq (\sigma(A) + \beta\delta)\|z\|$ for all $z \in \mathbb{R}^n$ and $\delta \in (0, 1]$. For a given $\delta \in (0, 1]$, consider the weighted Euclidean norm $\| \cdot \|_s$ defined by $\|y\|_s = \|\hat{P}^{-1}y\|$. Then we have for all $y \in \mathbb{R}^n$,
\[
\|Ay\|_s = \|\hat{P}^{-1}Ay\| = \|\hat{P}^{-1}\hat{P}J\hat{P}^{-1}y\| = \|\hat{J}\hat{P}^{-1}y\| \leq (\sigma(A) + \beta\delta)\|\hat{P}^{-1}y\|,
\]
so that $\|Ay\|_s \leq (\sigma(A) + \beta\delta)\|y\|_s$, for all $y \in \mathbb{R}^n$. For a given $\epsilon > 0$, we choose $\delta = \epsilon/\beta$, so the preceding relation yields Eq. (B.1).
Proposition B.1: (Contraction Mapping Fixed-Point Theorem) Let $Y$ be a complete vector space and let $\overline{Y}$ be a closed subset of $Y$. Then if $F : \overline{Y} \rightarrow \overline{Y}$ is a contraction mapping with modulus $\rho \in (0, 1)$, there exists a unique $y^* \in \overline{Y}$ such that

$$y^* = F y^*.$$ 

Furthermore, the sequence $\{F^k y\}$ converges to $y^*$ for any $y \in \overline{Y}$, and we have

$$\|F^k y - y^*\| \leq \rho^k \|y - y^*\|, \quad k = 1, 2, \ldots.$$ 

Proof: Let $y \in \overline{Y}$ and consider the iteration $y_{k+1} = F y_k$ starting with $y_0 = y$. By the contraction property of $F$,

$$\|y_{k+1} - y_k\| \leq \rho \|y_k - y_{k-1}\|, \quad k = 1, 2, \ldots,$$

which implies that

$$\|y_{k+1} - y_k\| \leq \rho^k \|y_1 - y_0\|, \quad k = 1, 2, \ldots.$$ 

It follows that for every $k \geq 0$ and $m \geq 1$, we have

$$\|y_{k+m} - y_k\| \leq \sum_{i=1}^{m} \|y_{k+i} - y_{k+i-1}\| \leq \rho^k (1 + \rho + \cdots + \rho^{m-1}) \|y_1 - y_0\| \leq \frac{\rho^k}{1 - \rho} \|y_1 - y_0\|.$$ 

Therefore, $\{y_k\}$ is a Cauchy sequence in $\overline{Y}$ and must converge to a limit $y^* \in \overline{Y}$, since $Y$ is complete and $\overline{Y}$ is closed. We have for all $k \geq 1$,

$$\|F y^* - y^*\| = \|F y^* - F y_k\| + \|y_k - y^*\| \leq \rho \|y^* - y_{k-1}\| + \|y_k - y^*\|$$

and since $y_k$ converges to $y^*$, we obtain $F y^* = y^*$. Thus, the limit $y^*$ of $y_k$ is a fixed point of $F$. It is a unique fixed point because if $\tilde{y}$ were another fixed point, we would have

$$\|y^* - \tilde{y}\| = \|F y^* - F \tilde{y}\| \leq \rho \|y^* - \tilde{y}\|,$$

which implies that $y^* = \tilde{y}$. 

converges to $y^*$, starting from an arbitrary initial point $y_0$. 

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To show the convergence rate bound of the last part, note that
\[ \| F^k y - y^* \| = \| F^k y - Fy^* \| \leq \rho \| F^{k-1} y - y^* \|. \]

Repeating this process for a total of \( k \) times, we obtain the desired result.

Q.E.D.

The convergence rate exhibited by \( F^k y \) in the preceding proposition is said to be geometric, and \( F^k y \) is said to converge to its limit \( y^* \) geometrically. This is in reference to the fact that the error \( \| F^k y - y^* \| \) converges to 0 faster than some geometric progression (\( \rho^k \| y - y^* \| \) in this case).

In some contexts of interest to us one may encounter mappings that are not contractions, but become contractions when iterated a finite number of times. In this case, one may use a slightly different version of the contraction mapping fixed point theorem, which we now present.

We say that a function \( F : Y \rightarrow Y \) is an \( m \)-stage contraction mapping if there exists a positive integer \( m \) and some \( \rho < 1 \) such that
\[ \| F^m y - F^m y' \| \leq \rho \| y - y' \|, \quad \forall y, y' \in Y, \]
where \( F^m \) denotes the composition of \( F \) with itself \( m \) times. Thus, \( F \) is an \( m \)-stage contraction if \( F^m \) is a contraction. Again, the scalar \( \rho \) is called the modulus of contraction. We have the following generalization of Prop. B.1.

**Proposition B.2: (m-Stage Contraction Mapping Fixed-Point Theorem)** Let \( Y \) be a complete vector space and let \( \overline{Y} \) be a closed subset of \( Y \). Then if \( F : \overline{Y} \rightarrow \overline{Y} \) is an \( m \)-stage contraction mapping with modulus \( \rho \in (0, 1) \), there exists a unique \( y^* \in \overline{Y} \) such that
\[ y^* = Fy^*. \]
Furthermore, \( \{ F^k y \} \) converges to \( y^* \) for any \( y \in \overline{Y} \).

**Proof:** Since \( F^m \) maps \( \overline{Y} \) into \( \overline{Y} \) and is a contraction mapping, by Prop. B.1, it has a unique fixed point in \( \overline{Y} \), denoted \( y^* \). Applying \( F \) to both sides of the relation \( y^* = F^my^* \), we see that \( Fy^* \) is also a fixed point of \( F^m \), so by the uniqueness of the fixed point, we have \( y^* = Fy^* \). Therefore \( y^* \) is a fixed point of \( F \). If \( F \) had another fixed point, say \( \hat{y} \), then we would have \( \hat{y} = F^m \hat{y} \), which by the uniqueness of the fixed point of \( F^m \) implies that \( \hat{y} = y^* \). Thus, \( y^* \) is the unique fixed point of \( F \).

To show the convergence of \( \{ F^k y \} \), note that by Prop. B.1, we have for all \( y \in \overline{Y} \),
\[ \lim_{k \to \infty} \| F^{mk} y - y^* \| = 0. \]
Using $F^\ell y$ in place of $y$, we obtain
\[
\lim_{k \to \infty} \|F^{mk+\ell} y - y^*\| = 0, \quad \ell = 0, 1, \ldots, m - 1,
\]
which proves the desired result. \textit{Q.E.D.}

\section{Weighted Sup-Norm Contractions}

In this section, we will focus on contraction mappings within a specialized context that is particularly important in DP. Let $X$ be a set (typically the state space in DP), and let $v : X \mapsto \mathbb{R}$ be a positive-valued function,
\[ v(x) > 0, \quad \forall \ x \in X. \]
Let $B(X)$ denote the set of all functions $J : X \mapsto \mathbb{R}$ such that $J(x)/v(x)$ is bounded as $x$ ranges over $X$. We define a norm on $B(X)$, called the \textit{weighted sup-norm}, by
\[ \|J\| = \sup_{x \in X} \left| \frac{J(x)}{v(x)} \right|. \quad (B.3) \]
It is easily verified that $\|\cdot\|$ thus defined has the required properties for being a norm. Furthermore, $B(X)$ \textit{is complete under this norm}. To see this, consider a Cauchy sequence $\{J_k\} \subset B(X)$, and note that $\|J_m - J_n\| \to 0$ as $m, n \to \infty$ implies that for all $x \in X$, $\{J_k(x)\}$ is a Cauchy sequence of real numbers, so it converges to some $J^*(x)$. We will show that $J^* \in B(X)$ and that $\|J_k - J^*\| \to 0$. To this end, it will be sufficient to show that given any $\epsilon > 0$, there exists an integer $K$ such that
\[ \left| \frac{J_k(x) - J^*(x)}{v(x)} \right| \leq \epsilon, \quad \forall \ x \in X, \ k \geq K. \]
This will imply that
\[ \sup_{x \in X} \left| \frac{J^*(x)}{v(x)} \right| \leq \epsilon + \|J_k\|, \quad \forall \ k \geq K, \]
so that $J^* \in B(X)$, and will also imply that $\|J_k - J^*\| \leq \epsilon$, so that $\|J_k - J^*\| \to 0$. Assume the contrary, i.e., that there exists an $\epsilon > 0$ and a subsequence $\{x_{m_1}, x_{m_2}, \ldots\} \subset X$ such that $m_i < m_{i+1}$ and
\[ \epsilon < \left| \frac{J_{m_i}(x_{m_i}) - J^*(x_{m_i})}{v(x_{m_i})} \right|, \quad \forall \ i \geq 1. \]
The right-hand side above is less or equal to
\[ \left| \frac{J_{m_i}(x_{m_i}) - J_n(x_{m_i})}{v(x_{m_i})} \right| + \left| \frac{J_n(x_{m_i}) - J^*(x_{m_i})}{v(x_{m_i})} \right|, \quad \forall \ n \geq 1, \ i \geq 1. \]
The first term in the above sum is less than $\epsilon/2$ for $i$ and $n$ larger than some threshold; fixing $i$ and letting $n$ be sufficiently large, the second term can also be made less than $\epsilon/2$, so the sum is made less than $\epsilon$ - a contradiction. In conclusion, the space $\mathcal{B}(X)$ is complete, so the fixed point results of Props. B.1 and B.2 apply.

In our discussions, unless we specify otherwise, we will assume that $\mathcal{B}(X)$ is equipped with the weighted sup-norm above, where the weight function $v$ will be clear from the context. There will be frequent occasions where the norm will be unweighted, i.e., $v(x) \equiv 1$ and $\|J\| = \sup_{x \in X} |J(x)|$, in which case we will explicitly state so.

**Finite-Dimensional Cases**

Let us now focus on the finite-dimensional case $X = \{1, \ldots, n\}$, in which case $\mathcal{R}(X)$ and $\mathcal{B}(X)$ can be identified with $\mathbb{R}^n$. We first consider a linear mapping (cf. Example B.1). We have the following proposition.

**Proposition B.3:** Consider a linear mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ of the form

$$Fy = b + Ay,$$

where $A$ is an $n \times n$ matrix with components $a_{ij}$, and $b$ is a vector in $\mathbb{R}^n$. Denote by $|A|$ the matrix whose components are the absolute values of the components of $A$ and let $\sigma(A)$ and $\sigma(|A|)$ denote the spectral radii of $A$ and $|A|$, respectively. Then:

(a) $|A|$ has a real eigenvalue $\lambda$, which is equal to its spectral radius, and an associated nonnegative eigenvector.

(b) $F$ is a contraction with respect to some weighted sup-norm if and only if $\sigma(|A|) < 1$. In particular, any substochastic matrix $P$ ($p_{ij} \geq 0$ for all $i, j$, and $\sum_{j=1}^n p_{ij} \leq 1$, for all $i$) is a contraction with respect to some weighted sup-norm if and only if $\sigma(P) < 1$.

(c) $F$ is a contraction with respect to the weighted sup-norm

$$\|y\| = \max_{i=1, \ldots, n} \frac{|y_i|}{v(i)}$$

if and only if

$$\frac{\sum_{j=1}^n |a_{ij}| v(j)}{v(i)} < 1, \quad i = 1, \ldots, n.$$
Proof: (a) This is the Perron-Frobenius Theorem; see e.g., [BeT89], Chapter 2, Prop. 6.6.
(b) This follows from the Perron-Frobenius Theorem; see [BeT89], Ch. 2, Cor. 6.2.
(c) This is proved in more general form in the following Prop. B.4. Q.E.D.

Consider next a nonlinear mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ that has the property

$$|Fy - Fz| \leq P |y - z|, \quad \forall y, z \in \mathbb{R}^n,$$

for some matrix $P$ with nonnegative components and $\sigma(P) < 1$. Here, we generically denote by $|w|$ the vector whose components are the absolute values of the components of $w$, and the inequality is componentwise. Then we claim that $F$ is a contraction with respect to some weighted sup-norm. To see this note that by the preceding discussion, $P$ is a contraction with respect to some weighted sup-norm $\|y\| = \max_{i=1, \ldots, n} |y_i|/v(i)$, and we have

$$\frac{(|Fy - Fz|)(i)}{v(i)} \leq \frac{(P |y - z|)(i)}{v(i)} \leq \alpha \|y - z\|, \quad \forall i = 1, \ldots, n,$$

for some $\alpha \in (0, 1)$, where $(|Fy - Fz|)(i)$ and $(P |y - z|)(i)$ are the $i$th components of the vectors $|Fy - Fz|$ and $P |y - z|$, respectively. Thus, $F$ is a contraction with respect to $\| \cdot \|$. For additional discussion of linear and nonlinear contraction mapping properties and characterizations such as the one above, see the book [OrR70].

Linear Mappings on Countable Spaces

The case where $X$ is countable (or, as a special case, finite) is frequently encountered in DP. The following proposition provides some useful criteria for verifying the contraction property of mappings that are either linear or are obtained via a parametric minimization of other contraction mappings.

**Proposition B.4:** Let $X = \{1, 2, \ldots\}$.

(a) Let $F : \mathcal{B}(X) \mapsto \mathcal{B}(X)$ be a linear mapping of the form

$$(FJ)(i) = b_i + \sum_{j \in X} a_{ij} J(j), \quad i \in X,$$
where \( b_i \) and \( a_{ij} \) are some scalars. Then \( F \) is a contraction with modulus \( \rho \) with respect to the weighted sup-norm (B.3) if and only if

\[
\sum_{j \in X} |a_{ij}| \frac{v(j)}{v(i)} \leq \rho, \quad i \in X. \tag{B.4}
\]

(b) Let \( F : \mathcal{B}(X) \mapsto \mathcal{B}(X) \) be a mapping of the form

\[
(FJ)(i) = \inf_{\mu \in M} (F_{\mu}J)(i), \quad i \in X,
\]

where \( M \) is parameter set, and for each \( \mu \in M \), \( F_{\mu} \) is a contraction mapping from \( \mathcal{B}(X) \) to \( \mathcal{B}(X) \) with modulus \( \rho \). Then \( F \) is a contraction mapping with modulus \( \rho \).

**Proof:** (a) Assume that Eq. (B.4) holds. For any \( J, J' \in \mathcal{B}(X) \), we have

\[
\|FJ - FJ'\| = \sup_{i \in X} \left| \sum_{j \in X} a_{ij} (J(j) - J'(j)) \right| \frac{v(j)}{v(i)} \leq \sup_{i \in X} \sum_{j \in X} a_{ij} v(j) \frac{|J(j) - J'(j)|}{v(i)} \leq \rho \|J - J'\|
\]

where the last inequality follows from the hypothesis.

Conversely, arguing by contradiction, let’s assume that Eq. (B.4) is violated for some \( i \in X \). Define \( J(j) = v(j) \text{sgn}(a_{ij}) \) and \( J'(j) = 0 \) for all \( j \in X \). Then we have \( \|J - J'\| = \|J\| = 1 \), and

\[
\left(\frac{(FJ)(i) - (FJ')(i)}{v(i)} \right) = \sum_{j \in X} a_{ij} \frac{v(j)}{v(i)} > \rho = \rho \|J - J'\|
\]

showing that \( F \) is not a contraction of modulus \( \rho \).

(b) Since \( F_{\mu} \) is a contraction of modulus \( \rho \), we have for any \( J, J' \in \mathcal{B}(X) \),

\[
\frac{(F_{\mu}J)(i)}{v(i)} \leq \frac{(F_{\mu}J')(i)}{v(i)} + \rho \|J - J'\|, \quad i \in X,
\]

so by taking the infimum over \( \mu \in M \),

\[
\frac{(FJ)(i)}{v(i)} \leq \frac{(FJ')(i)}{v(i)} + \rho \|J - J'\|, \quad i \in X.
\]
Sec. B.2  Weighted Sup-Norm Contractions

Reversing the roles of $J$ and $J'$, we obtain

$$\frac{|(FJ)(i) - (FJ')(i)|}{v(i)} \leq \rho \|J - J'\|, \quad i \in X,$$

and by taking the supremum over $i$, the contraction property of $F$ is proved.

Q.E.D.

The preceding proposition assumes that $FJ \in \mathcal{B}(X)$ for all $J \in \mathcal{B}(X)$. The following proposition provides conditions, particularly relevant to the DP context, which imply this assumption.

**Proposition B.5:** Let $X = \{1, 2, \ldots\}$, let $M$ be a parameter set, and for each $\mu \in M$, let $F_\mu$ be a linear mapping of the form

$$(F_\mu J)(i) = b_i(\mu) + \sum_{j \in X} a_{ij}(\mu) J(j), \quad i \in X.$$

(a) We have $F_\mu J \in \mathcal{B}(X)$ for all $J \in \mathcal{B}(X)$ provided $b(\mu) \in \mathcal{B}(X)$ and $V(\mu) \in \mathcal{B}(X)$, where

$$b(\mu) = \{b_1(\mu), b_2(\mu), \ldots\}, \quad V(\mu) = \{V_1(\mu), V_2(\mu), \ldots\},$$

with

$$V_i(\mu) = \sum_{j \in X} |a_{ij}(\mu)| v(j), \quad i \in X.$$

(b) Consider the mapping $F$

$$(FJ)(i) = \inf_{\mu \in M} (F_\mu J)(i), \quad i \in X.$$

We have $FJ \in \mathcal{B}(X)$ for all $J \in \mathcal{B}(X)$, provided $b \in \mathcal{B}(X)$ and $V \in \mathcal{B}(X)$, where

$$b = \{b_1, b_2, \ldots\}, \quad V = \{V_1, V_2, \ldots\},$$

with $b_i = \sup_{\mu \in M} b_i(\mu)$ and $V_i = \sup_{\mu \in M} V_i(\mu)$.

**Proof:** (a) For all $\mu \in M$, $J \in \mathcal{B}(X)$ and $i \in X$, we have

$$(F_\mu J)(i) \leq |b_i(\mu)| + \sum_{j \in X} |a_{ij}(\mu)| |J(j)/v(j)| v(j)$$
\[ \leq |b_i(\mu)| + \|J\| \sum_{j \in X} |a_{ij}(\mu)| v(j) \]

\[ = |b_i(\mu)| + \|J\| V_i(\mu), \]

and similarly \((F_\mu J)(i) \geq -|b_i(\mu)| - \|J\| V_i(\mu)\). Thus

\[ |(F_\mu J)(i)| \leq |b_i(\mu)| + \|J\| V_i(\mu), \quad i \in X. \]

By dividing this inequality with \(v(i)\) and by taking the supremum over \(i \in X\), we obtain

\[ \|F_\mu J\| \leq \|b\| + \|J\| \|V\| < \infty. \]

(b) By doing the same as in (a), but after first taking the infimum of \((F_\mu J)(i)\) over \(\mu\), we obtain

\[ \|F J\| \leq \|b\| + \|J\| \|V\| < \infty. \]

Q.E.D.
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