APPENDIX A:

Notation and Mathematical Conventions

In this appendix we collect our notation, and some related mathematical facts and conventions.

A.1 SET NOTATION AND CONVENTIONS

If $X$ is a set and $x$ is an element of $X$, we write $x \in X$. A set can be specified in the form $X = \{ x \mid x \text{ satisfies } P \}$, as the set of all elements satisfying property $P$. The union of two sets $X_1$ and $X_2$ is denoted by $X_1 \cup X_2$, and their intersection by $X_1 \cap X_2$. The empty set is denoted by $\emptyset$. The symbol $\forall$ means “for all.”

The set of real numbers (also referred to as scalars) is denoted by $\mathbb{R}$. The set of extended real numbers is denoted by $\mathbb{R}^*$:

$$\mathbb{R}^* = \mathbb{R} \cup \{ \infty, -\infty \}.$$ 

We write $-\infty < x < \infty$ for all real numbers $x$, and $-\infty \leq x \leq \infty$ for all extended real numbers $x$. We denote by $[a, b]$ the set of (possibly extended) real numbers $x$ satisfying $a \leq x \leq b$. A rounded, instead of square, bracket denotes strict inequality in the definition. Thus $[a, b]$, $(a, b)$, and $(a, b)$ denote the set of all $x$ satisfying $a < x \leq b$, $a \leq x < b$, and $a < x < b$, respectively.

Generally, we adopt standard conventions regarding addition and multiplication in $\mathbb{R}^*$, except that we take

$$\infty - \infty = -\infty \cdot \infty = \infty,$$
and we take the product of 0 and $\infty$ or $-\infty$ to be 0. In this way the sum and product of two extended real numbers is well-defined. Division by 0 or $\infty$ does not appear in our analysis. In particular, we adopt the following rules in calculations involving $\infty$ and $-\infty$

\[
\alpha + \infty = \infty + \alpha = \infty, \quad \forall \alpha \in \mathbb{R}^*
\]

\[
\alpha - \infty = -\infty + \alpha = -\infty, \quad \forall \alpha \in [-\infty, \infty),
\]

\[
\alpha \cdot \infty = \infty, \quad \alpha \cdot (-\infty) = -\infty, \quad \forall \alpha \in (0, \infty],
\]

\[
\alpha \cdot -\infty = -\infty, \quad \alpha \cdot (-\infty) = \infty, \quad \forall \alpha \in [-\infty, 0),
\]

\[
0 \cdot \infty = \infty \cdot 0 = 0 = 0 \cdot (-\infty) = (-\infty) \cdot 0 = -\infty = \infty.
\]

Under these rules, the following laws of arithmetic are still valid within $\mathbb{R}^*$:

\[
\alpha_1 + \alpha_2 = \alpha_2 + \alpha_1, \quad (\alpha_1 + \alpha_2) + \alpha_3 = \alpha_1 + (\alpha_2 + \alpha_3),
\]

\[
\alpha_1\alpha_2 = \alpha_2\alpha_1, \quad (\alpha_1\alpha_2)\alpha_3 = \alpha_1(\alpha_2\alpha_3).
\]

We also have

\[
\alpha(\alpha_1 + \alpha_2) = \alpha\alpha_1 + \alpha\alpha_2
\]

if either $\alpha \geq 0$ or else $(\alpha_1 + \alpha_2)$ is not of the form $\infty - \infty$.

**Inf and Sup Notation**

The *supremum* of a nonempty set $X \subset \mathbb{R}^*$, denoted by $\sup X$, is defined as the smallest $y \in \mathbb{R}^*$ such that $y \geq x$ for all $x \in X$. Similarly, the *infimum* of $X$, denoted by $\inf X$, is defined as the largest $y \in \mathbb{R}^*$ such that $y \leq x$ for all $x \in X$. For the empty set, we use the convention

\[
\sup \emptyset = -\infty, \quad \inf \emptyset = \infty.
\]

If $\sup X$ is equal to an $\overline{x} \in \mathbb{R}^*$ that belongs to the set $X$, we say that $\overline{x}$ is the *maximum point* of $X$ and we write $\overline{x} = \max X$. Similarly, if $\inf X$ is equal to an $\underline{x} \in \mathbb{R}^*$ that belongs to the set $X$, we say that $\underline{x}$ is the *minimum point* of $X$ and we write $\underline{x} = \min X$. Thus, when we write $\max X$ (or $\min X$) in place of $\sup X$ (or $\inf X$, respectively), we do so just for emphasis: we indicate that it is either evident, or it is known through earlier analysis, or it is about to be shown that the maximum (or minimum, respectively) of the set $X$ is attained at one of its points.
A.2 FUNCTIONS

If $f$ is a function, we use the notation $f : X \mapsto Y$ to indicate the fact that $f$ is defined on a nonempty set $X$ (its domain) and takes values in a set $Y$ (its range). Thus when using the notation $f : X \mapsto Y$, we implicitly assume that $X$ is nonempty. We will often use the unit function $e : X \mapsto \mathbb{R}$, defined by

$$e(x) = 1, \quad \forall x \in X.$$ 

Given a set $X$, we denote by $\mathcal{R}(X)$ the set of real-valued functions $J : X \mapsto \mathbb{R}$, and by $\mathcal{E}(X)$ the set of all extended real-valued functions $J : X \mapsto \mathbb{R}^\ast$. For any collection $\{J_\gamma \mid \gamma \in \Gamma\} \subset \mathcal{E}(X)$, parameterized by the elements of a set $\Gamma$, we denote by $\inf_{\gamma \in \Gamma} J_\gamma$ the function taking the value $\inf_{\gamma \in \Gamma} J_\gamma(x)$ at each $x \in X$.

For two functions $J_1, J_2 \in \mathcal{E}(X)$, we use the shorthand notation $J_1 \leq J_2$ to indicate the pointwise inequality

$$J_1(x) \leq J_2(x), \quad \forall x \in X.$$ 

We use the shorthand notation $\inf_{i \in I} J_i$ to denote the function obtained by pointwise infimum of a collection $\{J_i \mid i \in I\} \subset \mathcal{E}(X)$, i.e.,

$$\left(\inf_{i \in I} J_i\right)(x) = \inf_{i \in I} J_i(x), \quad \forall x \in X.$$ 

We use similar notation for sup.

Given subsets $S_1, S_2, S_3 \subset \mathcal{E}(X)$ and mappings $T_1 : S_1 \mapsto S_3$ and $T_2 : S_2 \mapsto S_1$, the composition of $T_1$ and $T_2$ is the mapping $T_1T_2 : S_2 \mapsto S_3$ defined by

$$(T_1T_2)(J)(x) = (T_1(T_2J))(x), \quad \forall J \in S_2, \ x \in X.$$ 

In particular, given a subset $S \subset \mathcal{E}(X)$ and mappings $T_1 : S \mapsto S$ and $T_2 : S \mapsto S$, the composition of $T_1$ and $T_2$ is the mapping $T_1T_2 : S \mapsto S$ defined by

$$(T_1T_2)(J)(x) = (T_1(T_2J))(x), \quad \forall J \in S, \ x \in X.$$ 

Similarly, given mappings $T_k : S \mapsto S$, $k = 1, \ldots, N$, their composition is the mapping $(T_1 \cdots T_N) : S \mapsto S$ defined by

$$(T_1T_2 \cdots T_N)(J)(x) = (T_1(T_2(\cdots (T_NJ))))(x), \quad \forall J \in S, \ x \in X.$$ 

In our notation involving compositions we minimize the use of parentheses, as long as clarity is not compromised. In particular, we write $T_1T_2J$ instead of $(T_1T_2J)$ or $(T_1T_2)J$ or $T_1(T_2J)$, but we write $(T_1T_2J)(x)$ to indicate the value of $T_1T_2J$ at $x \in X$.

If $X$ and $Y$ are nonempty sets, a mapping $T : S_1 \mapsto S_2$, where $S_1 \subset \mathcal{E}(X)$ and $S_2 \subset E(Y)$, is said to be monotone if for all $J, J' \in S_1$,

$$J \leq J' \quad \Rightarrow \quad TJ \leq TJ'.$$
Sequences of Functions

For a sequence of functions \( \{ J_k \} \subset \mathcal{E}(X) \) that converges pointwise, we denote by \( \lim_{k \to \infty} J_k \) the pointwise limit of \( \{ J_k \} \). We denote by \( \limsup_{k \to \infty} J_k \) (or \( \liminf_{k \to \infty} J_k \)) the pointwise limit superior (or inferior, respectively) of \( \{ J_k \} \). If \( \{ J_k \} \subset \mathcal{E}(X) \) converges pointwise to \( J \), we write \( J_k \to J \). Note that we reserve this notation for pointwise convergence. To denote convergence with respect to a norm \( \| \cdot \| \), we write \( \| J_k - J \| \to 0 \).

A sequence of functions \( \{ J_k \} \subset \mathcal{E}(X) \) is said to be monotonically nonincreasing (or monotonically nondecreasing) if \( J_{k+1} \leq J_k \) for all \( k \) (or \( J_{k+1} \geq J_k \) for all \( k \), respectively). Such a sequence always has a (pointwise) limit within \( \mathcal{E}(X) \). We write \( J_k \downarrow J \) (or \( J_k \uparrow J \)) to indicate that \( \{ J_k \} \) is monotonically nonincreasing (or monotonically nondecreasing, respectively) and that its limit is \( J \).

Let \( \{ J_{mn} \} \subset \mathcal{E}(X) \) be a double indexed sequence, which is monotonically nonincreasing separately for each index in the sense that

\[
J_{(m+1)n} \leq J_{mn}, \quad J_{m(n+1)} \leq J_{mn}, \quad \forall m, n = 0, 1, \ldots
\]

For such sequences, a useful fact is that

\[
\lim_{m \to \infty} \left( \lim_{n \to \infty} J_{mn} \right) = \lim_{m \to \infty} J_{mm}.
\]

There is a similar fact for monotonically nondecreasing sequences.

Expected Values

Given a random variable \( w \) defined over a probability space \( \Omega \), the expected value of \( w \) is defined by

\[
E\{w\} = E\{w^+\} + E\{w^-\},
\]

where \( w^+ \) and \( w^- \) are the positive and negative parts of \( w \),

\[
w^+(\omega) = \max \{ 0, w(\omega) \}, \quad w^-(\omega) = \min \{ 0, w(\omega) \}.
\]

In this way, taking also into account the rule \( -\infty - \infty = \infty \), the expected value \( E\{w\} \) is well-defined if \( \Omega \) is finite or countably infinite. In more general cases, \( E\{w\} \) is similarly defined by the appropriate form of integration, and more detail will be given at specific points as needed.
APPENDIX B:

Contraction Mappings

B.1 CONTRACTION MAPPING FIXED POINT THEOREMS

The purpose of this appendix is to provide some background on contraction mappings and their properties. Let $Y$ be a real vector space with a norm $\| \cdot \|$, i.e., a real-valued function satisfying for all $y \in Y$, $\|y\| \geq 0$, $\|y\| = 0$ if and only if $y = 0$, and

$$
\|ay\| = |a|\|y\|, \quad \forall a \in \mathbb{R}, \quad \|y + z\| \leq \|y\| + \|z\|, \quad \forall y, z \in Y.
$$

Let $\overline{Y}$ be a closed subset of $Y$. A function $F : \overline{Y} \rightarrow \overline{Y}$ is said to be a contraction mapping if for some $\rho \in (0, 1)$, we have

$$
\|Fy - Fz\| \leq \rho\|y - z\|, \quad \forall y, z \in \overline{Y}.
$$

The scalar $\rho$ is called the modulus of contraction of $F$.

Example B.1 (Linear Contraction Mappings in $\mathbb{R}^n$)

Consider the case of a linear mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ of the form

$$
Fy = b + Ay,
$$

where $A$ is an $n \times n$ matrix and $b$ is a vector in $\mathbb{R}^n$. Let $\sigma(A)$ denote the spectral radius of $A$ (the largest modulus among the moduli of the eigenvalues of $A$). Then it can be shown that $A$ is a contraction mapping with respect to some norm if and only if $\sigma(A) < 1$.

Specifically, given $\epsilon > 0$, there exists a norm $\| \cdot \|_s$ such that

$$
\|Ay\|_s \leq (\sigma(A) + \epsilon)\|y\|_s, \quad \forall y \in \mathbb{R}^n. \quad \text{(B.1)}
$$

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Thus, if \( \sigma(A) < 1 \) we may select \( \epsilon > 0 \) such that \( \rho = \sigma(A) + \epsilon < 1 \), and obtain the contraction relation

\[
\| Fy - Fz \| = \| A(y - z) \| = \rho \| y - z \|, \quad \forall \ y, z \in \mathbb{R}^n.
\] (B.2)

The norm \( \| \cdot \| \) can be taken to be a weighted Euclidean norm, i.e., it may have the form \( \| y \|_s = \| My \| \), where \( M \) is a square invertible matrix, and \( \| \cdot \| \) is the standard Euclidean norm, i.e., \( \| x \| = \sqrt{x^T x} \).

Conversely, if Eq. (B.2) holds for some norm \( \| \cdot \| \) and all real vectors \( y, z \), it also holds for all complex vectors \( y, z \) with the squared norm \( \| e \|_s^2 \) of a complex vector \( e \) defined as the sum of the squares of the norms of the real and the imaginary components. Thus from Eq. (B.2), by taking \( y - z = u \), where \( u \) is an eigenvector corresponding to an eigenvalue \( |\lambda| = \sigma(A) \), we have \( \sigma(A) \| u \|_s = \| Au \|_s \leq \rho \| u \|_s \). Hence \( \sigma(A) \leq \rho \), and it follows that if \( F \) is a contraction with respect to a given norm, we must have \( \sigma(A) < 1 \).

A sequence \( \{ y_k \} \subset Y \) is said to be a \textit{Cauchy sequence} if \( \| y_m - y_n \| \to 0 \) as \( m, n \to \infty \), i.e., given any \( \epsilon > 0 \), there exists \( N \) such that \( \| y_m - y_n \| \leq \epsilon \) for all \( m, n \geq N \). The space \( Y \) is said to be \textit{complete} under the norm \( \| \cdot \| \) if every Cauchy sequence \( \{ y_k \} \subset Y \) is convergent, in the sense that for some \( y \in Y \), we have \( \| y_k - y \| \to 0 \). Note that a Cauchy sequence is always bounded. Also, a Cauchy sequence of real numbers is convergent, implying that the real line is a complete space and so is every real finite-dimensional vector space. On the other hand, an infinite dimensional space may not be complete under some norms, while it may be complete under other norms.

When \( Y \) is complete and \( \overline{Y} \) is a closed subset of \( Y \), an important property of a contraction mapping \( F : \overline{Y} \to \overline{Y} \) is that it has a unique fixed point within \( \overline{Y} \), i.e., the equation

\[
y = Fy
\]

has a unique solution \( y^* \in \overline{Y} \), called the \textit{fixed point} of \( F \). Furthermore, the sequence \( \{ y_k \} \) generated by the iteration

\[
y_{k+1} = Fy_k
\]

has a unique solution \( y^* \in \overline{Y} \), called the \textit{fixed point} of \( F \). Furthermore, the sequence \( \{ y_k \} \) generated by the iteration

\[
y_{k+1} = Fy_k
\]
Converges to $y^*$, starting from an arbitrary initial point $y_0$.

**Proposition B.1: (Contraction Mapping Fixed-Point Theorem)** Let $Y$ be a complete vector space and let $\overline{Y}$ be a closed subset of $Y$. Then if $F : \overline{Y} \to \overline{Y}$ is a contraction mapping with modulus $\rho \in (0, 1)$, there exists a unique $y^* \in \overline{Y}$ such that

$$y^* = Fy^*.$$  

Furthermore, the sequence $\{F^ky\}$ converges to $y^*$ for any $y \in \overline{Y}$, and we have

$$\|F^ky - y^*\| \leq \rho^k\|y - y^*\|, \quad k = 1, 2, \ldots.$$  

**Proof:** Let $y \in \overline{Y}$ and consider the iteration $y_{k+1} = Fy_k$ starting with $y_0 = y$. By the contraction property of $F$,

$$\|y_{k+1} - y_k\| \leq \rho\|y_k - y_{k-1}\|, \quad k = 1, 2, \ldots,$$

which implies that

$$\|y_{k+1} - y_k\| \leq \rho^k\|y_1 - y_0\|, \quad k = 1, 2, \ldots.$$  

It follows that for every $k \geq 0$ and $m \geq 1$, we have

$$\|y_{k+m} - y_k\| \leq \sum_{i=1}^{m} \|y_{k+i} - y_{k+i-1}\| \leq \rho^k(1 + \rho + \cdots + \rho^{m-1})\|y_1 - y_0\| \leq \frac{\rho^k}{1 - \rho}\|y_1 - y_0\|.$$  

Therefore, $\{y_k\}$ is a Cauchy sequence in $\overline{Y}$ and must converge to a limit $y^* \in \overline{Y}$, since $Y$ is complete and $\overline{Y}$ is closed. We have for all $k \geq 1$,

$$\|Fy^* - y^*\| \leq \|Fy^* - y_k\| + \|y_k - y^*\| \leq \rho\|y^* - y_{k-1}\| + \|y_k - y^*\|$$

and since $y_k$ converges to $y^*$, we obtain $Fy^* = y^*$. Thus, the limit $y^*$ of $y_k$ is a fixed point of $F$. It is a unique fixed point because if $\tilde{y}$ were another fixed point, we would have

$$\|y^* - \tilde{y}\| = \|Fy^* - F\tilde{y}\| \leq \rho\|y^* - \tilde{y}\|,$$

which implies that $y^* = \tilde{y}$. 

To show the convergence rate bound of the last part, note that
\[ \|F^k y - y^*\| = \|F^k y - F y^*\| \leq \rho \|F^{k-1} y - y^*\|. \]

Repeating this process for a total of \( k \) times, we obtain the desired result.
Q.E.D.

The convergence rate exhibited by \( F^k y \) in the preceding proposition is said to be geometric, and \( F^k y \) is said to converge to its limit \( y^* \) geometrically. This is in reference to the fact that the error \( \|F^k y - y^*\| \) converges to 0 faster than some geometric progression (\( \rho^k \|y - y^*\| \) in this case).

In some contexts of interest to us one may encounter mappings that are not contractions, but become contractions when iterated a finite number of times. In this case, one may use a slightly different version of the contraction mapping fixed point theorem, which we now present.

We say that a function \( F : Y \mapsto Y \) is an \( m \)-stage contraction mapping if there exists a positive integer \( m \) and some \( \rho < 1 \) such that
\[ \|F^m y - F^m y'\| \leq \rho \|y - y'\|, \quad \forall \ y, y' \in Y, \]
where \( F^m \) denotes the composition of \( F \) with itself \( m \) times. Thus, \( F \) is an \( m \)-stage contraction if \( F^m \) is a contraction. Again, the scalar \( \rho \) is called the modulus of contraction. We have the following generalization of Prop. B.1.

**Proposition B.2: (m-Stage Contraction Mapping Fixed-Point Theorem)** Let \( Y \) be a complete vector space and let \( \overline{Y} \) be a closed subset of \( Y \). Then if \( F : \overline{Y} \mapsto \overline{Y} \) is an \( m \)-stage contraction mapping with modulus \( \rho \in (0, 1) \), there exists a unique \( y^* \in \overline{Y} \) such that
\[ y^* = F y^*. \]
Furthermore, \( \{F^k y\} \) converges to \( y^* \) for any \( y \in \overline{Y} \).

**Proof:** Since \( F^m \) maps \( \overline{Y} \) into \( \overline{Y} \) and is a contraction mapping, by Prop. B.1, it has a unique fixed point in \( \overline{Y} \), denoted \( y^* \). Applying \( F \) to both sides of the relation \( y^* = F^m y^* \), we see that \( F y^* \) is also a fixed point of \( F^m \), so by the uniqueness of the fixed point, we have \( y^* = F y^* \). Therefore \( y^* \) is a fixed point of \( F \). If \( F \) had another fixed point, say \( \hat{y} \), then we would have \( \hat{y} = F^m \hat{y} \), which by the uniqueness of the fixed point of \( F^m \) implies that \( \hat{y} = y^* \). Thus, \( y^* \) is the unique fixed point of \( F \).

To show the convergence of \( \{F^k y\} \), note that by Prop. B.1, we have for all \( y \in \overline{Y} \),
\[ \lim_{k \to \infty} \|F^{mk} y - y^*\| = 0. \]
Sec. B.2  Weighted Sup-Norm Contractions

Using \( F^\ell y \) in place of \( y \), we obtain

\[
\lim_{k \to \infty} \| F^{mk+\ell} y - y^* \| = 0, \quad \ell = 0, 1, \ldots, m - 1,
\]

which proves the desired result.  Q.E.D.

B.2 WEIGHTED SUP-NORM CONTRACTIONS

In this section, we will focus on contraction mappings within a specialized context that is particularly important in DP. Let \( X \) be a set (typically the state space in DP), and let \( v : X \to \mathbb{R} \) be a positive-valued function, \( v(x) > 0, \quad \forall x \in X \).

Let \( B(X) \) denote the set of all functions \( J : X \to \mathbb{R} \) such that \( J(x)/v(x) \) is bounded as \( x \) ranges over \( X \). We define a norm on \( B(X) \), called the weighted sup-norm, by

\[
\| J \| = \sup_{x \in X} \frac{|J(x)|}{v(x)}.
\]

It is easily verified that \( \| \cdot \| \) thus defined has the required properties for being a norm. Furthermore, \( B(X) \) is complete under this norm. To see this, consider a Cauchy sequence \( \{ J_k \} \subset B(X) \), and note that \( \| J_m - J_n \| \to 0 \) as \( m, n \to \infty \) implies that for all \( x \in X \), \( \{ J_k(x) \} \) is a Cauchy sequence of real numbers, so it converges to some \( J^*(x) \). We will show that \( J^* \in B(X) \) and that \( \| J_k - J^* \| \to 0 \). To this end, it will be sufficient to show that given any \( \epsilon > 0 \), there exists an integer \( K \) such that

\[
\frac{|J_k(x) - J^*(x)|}{v(x)} \leq \epsilon, \quad \forall x \in X, \; k \geq K.
\]

This will imply that

\[
\sup_{x \in X} \frac{|J^*(x)|}{v(x)} \leq \epsilon + \| J_k \|, \quad \forall k \geq K,
\]

so that \( J^* \in B(X) \), and will also imply that \( \| J_k - J^* \| \leq \epsilon \), so that \( \| J_k - J^* \| \to 0 \). Assume the contrary, i.e., that there exists an \( \epsilon > 0 \) and a subsequence \( \{ x_{m_1}, x_{m_2}, \ldots \} \subset X \) such that \( m_i < m_{i+1} \) and

\[
\epsilon < \frac{|J_{m_i}(x_{m_i}) - J^*(x_{m_i})|}{v(x_{m_i})}, \quad \forall i \geq 1.
\]

The right-hand side above is less or equal to

\[
\frac{|J_{m_i}(x_{m_i}) - J_n(x_{m_i})|}{v(x_{m_i})} + \frac{|J_n(x_{m_i}) - J^*(x_{m_i})|}{v(x_{m_i})}, \quad \forall n \geq 1, \; i \geq 1.
\]
The first term in the above sum is less than $\epsilon/2$ for $i$ and $n$ larger than some threshold; fixing $i$ and letting $n$ be sufficiently large, the second term can also be made less than $\epsilon/2$, so the sum is made less than $\epsilon$ - a contradiction.

In conclusion, the space $\mathcal{B}(X)$ is complete, so the fixed point results of Props. B.1 and B.2 apply.

In our discussions, unless we specify otherwise, we will assume that $\mathcal{B}(X)$ is equipped with the weighted sup-norm above, where the weight function $v$ will be clear from the context. There will be frequent occasions where the norm will be unweighted, i.e., $v(x) \equiv 1$ and $\|J\| = \sup_{x \in X} |J(x)|$, in which case we will explicitly state so.

**Finite-Dimensional Cases**

Let us now focus on the finite-dimensional case $X = \{1, \ldots, n\}$, in which case $\mathcal{R}(X)$ and $\mathcal{B}(X)$ can be identified with $\mathbb{R}^n$. We first consider a linear mapping (cf. Example B.1). We have the following proposition.

**Proposition B.3:** Consider a linear mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ of the form

$$Fy = b + Ay,$$

where $A$ is an $n \times n$ matrix with components $a_{ij}$, and $b$ is a vector in $\mathbb{R}^n$. Denote by $|A|$ the matrix whose components are the absolute values of the components of $A$ and let $\sigma(A)$ and $\sigma(|A|)$ denote the spectral radii of $A$ and $|A|$, respectively. Then:

(a) $|A|$ has a real eigenvalue $\lambda$, which is equal to its spectral radius, and an associated nonnegative eigenvector.

(b) $F$ is a contraction with respect to some weighted sup-norm if and only if $\sigma(|A|) < 1$. In particular, any substochastic matrix $P$ ($p_{ij} \geq 0$ for all $i, j$, and $\sum_{j=1}^n p_{ij} \leq 1$, for all $i$) is a contraction with respect to some weighted sup-norm if and only if $\sigma(P) < 1$.

(c) $F$ is a contraction with respect to the weighted sup-norm

$$\|y\| = \max_{i=1, \ldots, n} \frac{|y_i|}{v(i)}$$

if and only if

$$\sum_{j=1}^n \frac{|a_{ij}| v(j)}{v(i)} < 1, \quad i = 1, \ldots, n.$$
Proof: (a) This is the Perron-Frobenius Theorem; see e.g., [BeT89], Chapter 2, Prop. 6.6.

(b) This follows from the Perron-Frobenius Theorem; see [BeT89], Ch. 2, Cor. 6.2.

(c) This is proved in more general form in the following Prop. B.4. Q.E.D.

Consider next a nonlinear mapping \( F : \mathbb{R}^n \mapsto \mathbb{R}^n \) that has the property
\[
|Fy - Fz| \leq P |y - z|, \quad \forall y, z \in \mathbb{R}^n,
\]
for some matrix \( P \) with nonnegative components and \( \sigma(P) < 1 \). Here, we generically denote by \(|w|\) the vector whose components are the absolute values of the components of \( w \), and the inequality is componentwise. Then we claim that \( F \) is a contraction with respect to some weighted sup-norm. To see this note that by the preceding discussion, \( P \) is a contraction with respect to some weighted sup-norm \( \|y\| = \max_{i=1,...,n} |y_i|/v(i) \), and we have
\[
\frac{(|Fy - Fz|)(i)}{v(i)} \leq \frac{(P |y - z|)(i)}{v(i)} \leq \alpha \|y - z\|, \quad \forall i = 1,\ldots,n,
\]
for some \( \alpha \in (0, 1) \), where \((|Fy - Fz|)(i)\) and \((P |y - z|)(i)\) are the \( i \)th components of the vectors \(|Fy - Fz|\) and \(P |y - z|\), respectively. Thus, \( F \) is a contraction with respect to \( \| \cdot \| \). For additional discussion of linear and nonlinear contraction mapping properties and characterizations such as the one above, see the book [OrR70].

Linear Mappings on Countable Spaces

The case where \( X \) is countable (or, as a special case, finite) is frequently encountered in DP. The following proposition provides some useful criteria for verifying the contraction property of mappings that are either linear or are obtained via a parametric minimization of other contraction mappings.

**Proposition B.4:** Let \( X = \{1, 2, \ldots\} \).

(a) Let \( F : \mathcal{B}(X) \mapsto \mathcal{B}(X) \) be a linear mapping of the form
\[
(FJ)(i) = b_i + \sum_{j \in X} a_{ij}J(j), \quad i \in X,
\]
where $b_i$ and $a_{ij}$ are some scalars. Then $F$ is a contraction with modulus $\rho$ with respect to the weighted sup-norm (B.3) if and only if
\[
\sum_{j \in X} |a_{ij}| \frac{v(j)}{v(i)} \leq \rho, \quad i \in X.
\] (B.4)

(b) Let $F : \mathcal{B}(X) \mapsto \mathcal{B}(X)$ be a mapping of the form
\[
(FJ)(i) = \inf_{\mu \in M} (F \mu J)(i), \quad i \in X,
\]
where $M$ is parameter set, and for each $\mu \in M$, $F \mu$ is a contraction mapping from $\mathcal{B}(X)$ to $\mathcal{B}(X)$ with modulus $\rho$. Then $F$ is a contraction mapping with modulus $\rho$.

**Proof:** (a) Assume that Eq. (B.4) holds. For any $J, J' \in \mathcal{B}(X)$, we have
\[
\|FJ - FJ'\| = \sup_{i \in X} \left| \sum_{j \in X} a_{ij} (J(j) - J'(j)) \right| \frac{v(j)}{v(i)} \\
\leq \sup_{i \in X} \left| \sum_{j \in X} a_{ij} v(j) \left| (J(j) - J'(j))/v(j) \right| \right| \frac{v(j)}{v(i)} \\
\leq \sup_{i \in X} \left| \sum_{j \in X} a_{ij} v(j) \right| \left| J - J' \right| / v(i) \\
\leq \rho \|J - J'\|,
\]
where the last inequality follows from the hypothesis.

Conversely, arguing by contradiction, let’s assume that Eq. (B.4) is violated for some $i \in X$. Define $J(j) = v(j) \text{sgn}(a_{ij})$ and $J'(j) = 0$ for all $j \in X$. Then we have $\|J - J'\| = \|J\| = 1$, and
\[
\left| (FJ)(i) - (FJ')(i) \right| / v(i) = \sum_{j \in X} a_{ij} |v(j)| / v(i) > \rho = \rho \|J - J'\|,
\]
showing that $F$ is not a contraction of modulus $\rho$.

(b) Since $F \mu$ is a contraction of modulus $\rho$, we have for any $J, J' \in \mathcal{B}(X)$,
\[
\frac{(F \mu J)(i)}{v(i)} \leq \frac{(F \mu J')(i)}{v(i)} + \rho \|J - J'\|, \quad i \in X,
\]
so by taking the infimum over $\mu \in M$,
\[
\left( FJ \right)(i) / v(i) \leq \left( FJ' \right)(i) / v(i) + \rho \|J - J'\|, \quad i \in X.
\]
Reversing the roles of \( J \) and \( J' \), we obtain

\[
\frac{|(FJ)(i) - (FJ')(i)|}{v(i)} \leq \rho \|J - J'\|, \quad i \in X,
\]

and by taking the supremum over \( i \), the contraction property of \( F \) is proved. Q.E.D.

The preceding proposition assumes that \( FJ \in \mathcal{B}(X) \) for all \( J \in \mathcal{B}(X) \). The following proposition provides conditions, particularly relevant to the DP context, which imply this assumption.

**Proposition B.5:** Let \( X = \{1, 2, \ldots\} \), let \( M \) be a parameter set, and for each \( \mu \in M \), let \( F_\mu \) be a linear mapping of the form

\[
(F_\mu J)(i) = b_i(\mu) + \sum_{j \in X} a_{ij}(\mu) J(j), \quad i \in X.
\]

(a) We have \( F_\mu J \in \mathcal{B}(X) \) for all \( J \in \mathcal{B}(X) \) provided \( b(\mu) \in \mathcal{B}(X) \) and \( V(\mu) \in \mathcal{B}(X) \), where

\[
b(\mu) = \{b_1(\mu), b_2(\mu), \ldots\}, \quad V(\mu) = \{V_1(\mu), V_2(\mu), \ldots\},
\]

with

\[
V_i(\mu) = \sum_{j \in X} |a_{ij}(\mu)| v(j), \quad i \in X.
\]

(b) Consider the mapping \( F \)

\[
(FJ)(i) = \inf_{\mu \in M} (F_\mu J)(i), \quad i \in X.
\]

We have \( FJ \in \mathcal{B}(X) \) for all \( J \in \mathcal{B}(X) \), provided \( b \in \mathcal{B}(X) \) and \( V \in \mathcal{B}(X) \), where

\[
b = \{b_1, b_2, \ldots\}, \quad V = \{V_1, V_2, \ldots\},
\]

with \( b_i = \sup_{\mu \in M} b_i(\mu) \) and \( V_i = \sup_{\mu \in M} V_i(\mu) \).

**Proof:** (a) For all \( \mu \in M \), \( J \in \mathcal{B}(X) \) and \( i \in X \), we have

\[
(F_\mu J)(i) \leq |b_i(\mu)| + \sum_{j \in X} |a_{ij}(\mu)| |J(j)/v(j)| v(j)
\]
\[ \leq |b_i(\mu)| + \|J\| \sum_{j \in X} |a_{ij}(\mu)| v(j) \]

\[ = |b_i(\mu)| + \|J\| V_i(\mu), \]

and similarly \((F_\mu J)(i) \geq -|b_i(\mu)| - \|J\| V_i(\mu)\). Thus

\[ |(F_\mu J)(i)| \leq |b_i(\mu)| + \|J\| V_i(\mu), \quad i \in X. \]

By dividing this inequality with \(v(i)\) and by taking the supremum over \(i \in X\), we obtain

\[ \|F_\mu J\| \leq \|b_\mu\| + \|J\| \|V_\mu\| < \infty. \]

(b) By doing the same as in (a), but after first taking the infimum of \((F_\mu J)(i)\) over \(\mu\), we obtain

\[ \|FJ\| \leq \|b\| + \|J\| \|V\| < \infty. \]

Q.E.D.
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