Abstract Dynamic Programming

by

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Chapter 4
Noncontractive Models

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This is an updated version of Chapter 4 of the author’s research monograph “Abstract Dynamic Programming,” Athena Scientific, 2013. It replaces Sections 4.4-4.7 of the original with new material. The chapter will be periodically updated, and represents “work in progress.” It may contain errors (hopefully not serious ones). Your comments and suggestions to the author at dimitrib@mit.edu are welcome.
Noncontractive Models

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In this chapter, we consider abstract DP models similar to the ones of the earlier chapters, but without assuming any contraction-like property. We will discuss both finite and infinite horizon models, and introduce just enough assumptions (including monotonicity) to obtain some minimal results, which we will strengthen as we go along.

In Section 4.2, we discuss a general type of finite horizon problem. Under some reasonable assumptions, we show the standard results that one may expect in an abstract setting.

In Section 4.3, we consider an infinite horizon problem that is motivated by the well-known positive and negative DP models (see [Ber12a], Chapter 4). These are the special cases of the infinite horizon stochastic optimal control problem of Example 1.2.1, where the cost per stage $g$ is uniformly nonpositive or uniformly nonnegative. For these models there is interesting theory (the validity of Bellman’s equation and the availability of optimality conditions in a DP context), which we discuss in Section 4.3.1. There are also interesting computational methods, patterned after the value and policy iteration methods, which we discuss in Sections 4.3.2 and 4.3.3. However, the performance guarantees for these methods are not as powerful as in the contractive case, and their validity hinges upon certain additional assumptions.

In Section 4.4, we will extend the notion of regularity of Section 3.2 so that it applies more broadly, including situations where nonstationary policies need to be considered. The mathematical reason for considering nonstationary policies is that for some of the noncontractive models of Section 4.3, stationary policies are insufficient in the sense that there may not exist $\epsilon$-optimal policies that are stationary. In this section, we also discuss some applications, including some general types of optimal control problems with nonnegative cost per stage. Principal results here are that $J^*$ is the unique solution of Bellman’s equation within a large class of functions, and related results regarding the convergence of the VI algorithm.

In Section 4.5, we discuss a nonnegative cost deterministic optimal control problem, which combines elements of the noncontractive models of Section 4.3 and the semicontractive models of Chapter 3. Within this setting we explore the structure and the multiplicity of solutions of Bellman’s equation. The main difference from Section 4.4 is that we use a perturbation-based line of analysis, similar to the one of Section 3.4. In particular, the starting point is a perturbed version of the mapping $T_\mu$ that defines the “stable” policies, in place of a subset $S$ that defines the $S$-regular policies. Here, the “well-behaved” policies are the ones that are stable rather than the ones that are $S$-regular.

Finally, in Section 4.6, we extend the ideas of Section 4.5 to stochastic optimal control problems, by generalizing the notion of a proper policy to the case of infinite state and control spaces.
4.1 NONCONTRACTIVE MODELS

Throughout this chapter we will continue to use the model of Chapter 3, which involves the set of extended real numbers
\[ \mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\} \]
and the set \( \mathcal{E}(X) \) the set of all extended real-valued functions \( J : X \to \mathbb{R}^* \). We denote by \( \mathcal{R}(X) \) the set of real-valued functions \( J : X \to \mathbb{R} \), and by \( \mathcal{B}(X) \) the set of real-valued functions \( J : X \to \mathbb{R} \) that are bounded with respect to a given weighted sup-norm. The operations with \( \infty \) and \( -\infty \) are standard and are summarized in Appendix A.

To repeat some of the basic definitions, we have a set \( X \) of states and a set \( U \) of controls, and for each \( x \in X \), the nonempty control constraint set \( U(x) \subset U \). We denote by \( \mathcal{M} \) the set of all functions \( \mu : X \to U \) with \( \mu(x) \in U(x) \), for all \( x \in X \), and by \( \Pi \) the set of “nonstationary policies” \( \pi = \{\mu_0, \mu_1, \ldots\} \), with \( \mu_k \in \mathcal{M} \) for all \( k \). We refer to a stationary policy \( \{\mu, \mu, \ldots\} \) simply as \( \mu \).

We introduce a mapping \( H : X \times U \times \mathcal{E}(X) \to \mathbb{R}^* \), and we define the mapping \( T : \mathcal{E}(X) \to \mathcal{E}(X) \) by
\[
(T J)(x) = \inf_{u \in U(x)} H(x, u, J), \quad \forall x \in X, J \in \mathcal{E}(X),
\]
and for each \( \mu \in \mathcal{M} \) the mapping \( T_\mu : \mathcal{E}(X) \to \mathcal{E}(X) \) by
\[
(T_\mu J)(x) = H(x, \mu(x), J), \quad \forall x \in X, J \in \mathcal{E}(X).
\]
We continue to use the following assumption throughout this chapter, without mentioning it explicitly in various propositions.

**Assumption 4.1.1: (Monotonicity)** If \( J, J' \in \mathcal{E}(X) \) and \( J \leq J' \), then
\[
H(x, u, J) \leq H(x, u, J'), \quad \forall x \in X, u \in U(x).
\]

A fact that we will be using frequently is that for each \( J \in \mathcal{E}(X) \) and scalar \( \epsilon > 0 \), there exists a \( \mu_\epsilon \in \mathcal{M} \) such that for all \( x \in X \),
\[
(T_{\mu_\epsilon} J)(x) \leq \begin{cases} (T J)(x) + \epsilon & \text{if } (T J)(x) > -\infty, \\ -(1/\epsilon) & \text{if } (T J)(x) = -\infty. \end{cases}
\]
In particular, if \( J \) is such that \( (T J)(x) > -\infty \) for all \( x \in X \), then for each \( \epsilon > 0 \), there exists a \( \mu_\epsilon \in \mathcal{M} \) such that
\[
(T_{\mu_\epsilon} J)(x) \leq (T J)(x) + \epsilon, \quad \forall x \in X.
\]
We will often use in our analysis the unit function $e$, defined by $e(x) \equiv 1$, so for example, we will write the above relation in shorthand as

$$T_{\mu_e} J \leq T J + \epsilon e.$$ 

We define cost functions for policies consistently with Chapters 2 and 3. In particular, we are given a function $J \in \mathcal{E}(X)$, and we consider for every policy $\pi = \{\mu_0, \mu_1, \ldots\} \in \Pi$ and positive integer $N$ the function $J_{N,\pi} \in \mathcal{E}(X)$ defined by

$$J_{N,\pi}(x) = (T_{\mu_0} \cdots T_{\mu_{N-1}} J)(x), \quad \forall \ x \in X,$$

and the function $J_\pi \in \mathcal{E}(X)$ defined by

$$J_\pi(x) = \limsup_{k \to \infty} (T_{\mu_0} \cdots T_{\mu_k} J)(x), \quad \forall \ x \in X.$$

We refer to $J_{N,\pi}$ as the $N$-stage cost function of $\pi$ and to $J_\pi$ as the infinite horizon cost function of $\pi$ (or just “cost function” if the length of the horizon is clearly implied by the context). For a stationary policy $\pi = \{\mu, \mu, \ldots\}$ we also write $J_\pi$ as $J_{\mu}$.

In Section 4.2, we consider the $N$-stage optimization problem

$$\text{minimize} \quad J_{N,\pi}(x)$$

subject to $\pi \in \Pi$, (4.1)

while in Section 4.3 we discuss its infinite horizon version

$$\text{minimize} \quad J_\pi(x)$$

subject to $\pi \in \Pi$. (4.2)

For a fixed $x \in X$, we denote by $J_N^*(x)$ and $J_\pi^*$ the optimal costs for these problems, i.e.,

$$J_N^*(x) = \inf_{\pi \in \Pi} J_{N,\pi}(x), \quad J_\pi^*(x) = \inf_{\pi \in \Pi} J_\pi(x), \quad \forall \ x \in X.$$

We say that a policy $\pi^* \in \Pi$ is $N$-stage optimal if

$$J_{N,\pi^*}(x) = J_N^*(x), \quad \forall \ x \in X,$$

and (infinite horizon) optimal if

$$J_{\pi^*}(x) = J_\pi^*(x), \quad \forall \ x \in X.$$

For a given $\epsilon > 0$, we say that $\pi_\epsilon$ is $N$-stage $\epsilon$-optimal if

$$J_{\pi_\epsilon}(x) \leq \begin{cases} J_N^*(x) + \epsilon & \text{if } J_N^*(x) > -\infty, \\ -(1/\epsilon) & \text{if } J_N^*(x) = -\infty, \end{cases}$$

and we say that $\pi_\epsilon$ is $\epsilon$-optimal if

$$J_{\pi_\epsilon}(x) \leq \begin{cases} J^*(x) + \epsilon & \text{if } J^*(x) > -\infty, \\ -(1/\epsilon) & \text{if } J^*(x) = -\infty. \end{cases}$$
4.2 FINITE HORIZON PROBLEMS

Consider the $N$-stage problem (4.1), where the cost function $J_{N,\pi}$ is defined by

$$J_{N,\pi}(x) = (T \mu_0 \cdots T \mu_{N-1} \bar{J})(x), \quad \forall \ x \in X.$$ 

Based on the theory of finite horizon DP, we expect that (at least under some conditions) the optimal cost function $J_N^*$ is obtained by $N$ successive applications of the DP mapping $T$ on the initial function $\bar{J}$, i.e.,

$$J_N^* = \inf_{\pi \in \Pi} J_{N,\pi} = T^N \bar{J}.$$ 

This is the analog of Bellman’s equation for the finite horizon problem in a DP context.

The Case Where Uniformly $N$-Stage Optimal Policies Exist

A favorable case where the analysis is simplified and we can easily show that $J_N^* = T^N \bar{J}$ is when the finite horizon DP algorithm yields an optimal policy during its execution. By this we mean that the algorithm that starts with $\bar{J}$, and sequentially computes $T \bar{J}, T^2 \bar{J}, \ldots, T^N \bar{J}$, also yields corresponding $\mu^*_1, \mu^*_2, \ldots, \mu^*_N \in M$ such that

$$T \mu^*_k T^{N-k-1} \bar{J} = T^{N-k} \bar{J}, \quad k = 0, \ldots, N - 1. \quad (4.3)$$

While $\mu^*_1, \ldots, \mu^*_N \in M$ satisfying this relation need not exist (because the corresponding infimum in the definition of $T$ is not attained), if they do exist, they both form an optimal policy and also guarantee that

$$J_N^* = T^N \bar{J}.$$ 

The proof is simple: we have for every $\pi = \{\mu_0, \mu_1, \ldots\} \in \Pi$

$$J_{N,\pi} = T \mu_0 \cdots T \mu_{N-1} \bar{J} \geq T^N \bar{J} = T \mu^*_0 \cdots T \mu^*_{N-1} \bar{J},$$

where the inequality follows from the monotonicity assumption and the definition of $T$, and the last equality follows from Eq. (4.3). Thus $\{\mu^*_0, \mu^*_1, \ldots\}$ has no worse $N$-stage cost function than every other policy, so it is $N$-stage optimal and $J_N^* = T \mu^*_0 \cdots T \mu^*_{N-1} \bar{J}$. By taking the infimum of the left-hand side over $\pi \in \Pi$, we obtain $J_N^* = T^N \bar{J}$.

The preceding argument can also be used to show that $\{\mu^*_k, \mu^*_{k+1}, \ldots\}$ is $(N-k)$-stage optimal for all $k = 0, \ldots, N - 1$. Such a policy is called uniformly $N$-stage optimal. The fact that the finite horizon DP algorithm provides an optimal solution of all the $k$-stage problems for $k = 1, \ldots, N$, rather than just the last one, is a manifestation of the classical principle
of optimality, expounded by Bellman in the early days of DP (the tail portion of an optimal policy obtained by DP minimizes the corresponding tail portion of the finite horizon cost). Note, however, that there may exist an \( N \)-stage optimal policy that is not \( k \)-stage optimal for some \( k < N \).

We state the result just derived as a proposition.

**Proposition 4.2.1:** Suppose that a policy \( \{\mu^*_0, \mu^*_1, \ldots\} \) satisfies the condition (4.3). Then this policy is uniformly \( N \)-stage optimal, and we have \( J^*_N = T^N \bar{J} \).

While the preceding result is theoretically limited, it is very useful in practice, because the existence of a policy satisfying the condition (4.3) can often be established with a simple analysis. For example, this condition is trivially satisfied if the control space is finite. The following proposition provides a generalization.

**Proposition 4.2.2:** Let the control space \( U \) be a metric space, and assume that for each \( x \in X \), \( \lambda \in \mathbb{R} \), and \( k = 0, 1, \ldots, N - 1 \), the set

\[
U_k(x, \lambda) = \{u \in U(x) \mid H(x, u, T^k \bar{J}) \leq \lambda\}
\]

is compact. Then there exists a uniformly \( N \)-stage optimal policy.

**Proof:** We will show that the infimum in the relation

\[
(T^{k+1} \bar{J})(x) = \inf_{u \in U(x)} H(x, u, T^k \bar{J})
\]

is attained for all \( x \in X \) and \( k \). Indeed if \( H(x, u, T^k \bar{J}) = \infty \) for all \( u \in U(x) \), then every \( u \in U(x) \) attains the infimum. If for a given \( x \in X \),

\[
\inf_{u \in U(x)} H(x, u, T^k \bar{J}) < \infty,
\]

the corresponding part of the proof of Lemma 3.3.1 applies and shows that the above infimum is attained. The result now follows from Prop. 4.2.1.

Q.E.D.

**The General Case**

We now consider the case where there is no uniformly \( N \)-stage optimal policy. By using the definitions of \( J^*_N \) and \( T^N \bar{J} \), the equation \( J^*_N = T^N \bar{J} \)
Sec. 4.2 Finite Horizon Problems

can be equivalently written as

\[ \inf_{\mu_0, \ldots, \mu_{N-1} \in \mathcal{M}} T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J} = \inf_{\mu_0 \in \mathcal{M}} T_{\mu_0} \left( \inf_{\mu_1 \in \mathcal{M}} T_{\mu_1} \left( \cdots \inf_{\mu_{N-1} \in \mathcal{M}} T_{\mu_{N-1}} \bar{J} \right) \right). \]

Thus we have \( J_N^* = T^N \bar{J} \) if the operations \( \inf \) and \( T_{\mu} \) can be interchanged in the preceding equation. We will introduce two alternative assumptions, which guarantee that this interchange is valid. Our first assumption is a form of continuity from above of \( H \) with respect to \( J \).

**Assumption 4.2.1:** For each sequence \( \{J_m\} \subset \mathcal{E}(X) \) with \( J_m \downarrow J \) and \( H(x, u, J_0) < \infty \) for all \( x \in X \) and \( u \in U(x) \), we have

\[ \lim_{m \to \infty} H(x, u, J_m) = H(x, u, J), \quad \forall x \in X, \ u \in U(x). \quad (4.4) \]

Note that if \( \{J_m\} \) is monotonically nonincreasing, the same is true for \( \{T_{\mu} J_m\} \). It follows that

\[ \inf_m J_m = \lim_{m \to \infty} J_m, \quad \inf_m (T_{\mu} J_m) = \lim_{m \to \infty} (T_{\mu} J_m), \]

so for all \( \mu \in \mathcal{M} \), Eq. (4.4) implies that

\[ \inf_m (T_{\mu} J_m) = \lim_{m \to \infty} (T_{\mu} J_m) = T_{\mu} \left( \lim_{m \to \infty} J_m \right) = T_{\mu} \left( \inf_m J_m \right). \]

This equality can be extended for any \( \mu_1, \ldots, \mu_k \in \mathcal{M} \) as follows:

\[ \inf_m (T_{\mu_1} \cdots T_{\mu_k} J_m) = T_{\mu_1} \left( \inf_m (T_{\mu_1} \cdots T_{\mu_k} J_m) \right) \]

\[ = \cdots \]

\[ = T_{\mu_1} T_{\mu_1} \cdots T_{\mu_{k-1}} \left( \inf_m (T_{\mu_k} J_m) \right) \]

\[ = T_{\mu_1} \cdots T_{\mu_k} \left( \inf_m J_m \right). \quad (4.5) \]

We use this relation to prove the following proposition.

**Proposition 4.2.3:** Let Assumption 4.2.1 hold, and assume further that \( J_{k,\pi}(x) < \infty \), for all \( x \in X, \ \pi \in \Pi, \) and \( k \geq 1 \). Then \( J_N^* = T^N \bar{J} \).

**Proof:** We select for each \( k = 0, \ldots, N-1 \), a sequence \( \{\mu_k^m\} \subset \mathcal{M} \) such that

\[ \lim_{m \to \infty} T_{\mu_k^m} (T^{N-k-1} \bar{J}) \downarrow T^{N-k} \bar{J}. \]
Since $J_N^* \leq T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J}$ for all $\mu_0, \ldots, \mu_{N-1} \in M$, we have using also Eq. (4.5) and the assumption $J_{k,\pi}(x) < \infty$, for all $k$, $\pi$, and $x$,

$$J_N^* \leq \inf_{m_0} \cdots \inf_{m_{N-1}} T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J}$$

$$= \inf_{m_0} \cdots \inf_{m_{N-2}} T_{\mu_0} \cdots T_{\mu_{N-2}} \left( \inf_{m_{N-1}} T_{\mu_{N-1}} \bar{J} \right)$$

$$= \inf_{m_0} \cdots \inf_{m_{N-2}} T_{\mu_0} \cdots T_{\mu_{N-2}} \bar{J}$$

$$= \cdots$$

$$= \inf_{m_0} T_{\mu_0} (T^{N-1} \bar{J})$$

$$= T^N \bar{J}.$$ 

On the other hand, it is clear from the definitions that $T^N \bar{J} \leq J_{N,\pi}$ for all $N$ and $\pi \in \Pi$, so that $T^N \bar{J} \leq J_N^*$. Thus, $J_N^* = T^N \bar{J}$. Q.E.D.

We now introduce an alternative assumption, which in addition to $J_N^* = T^N \bar{J}$, guarantees the existence of an $\epsilon$-optimal policy.

**Assumption 4.2.2:** We have

$$J_N^* (x) > -\infty, \quad \forall \ x \in X, \ k = 1, \ldots, N.$$

Moreover, there exists a scalar $\alpha \in (0, \infty)$ such that for all scalars $r \in (0, \infty)$ and functions $J \in \mathcal{E}(X)$, we have

$$H(x, u, J + r e) \leq H(x, u, J) + \alpha r, \quad \forall \ x \in X, \ u \in U(x). \quad (4.6)$$

**Proposition 4.2.4:** Let Assumption 4.2.2 hold. Then $J_N^* = T^N \bar{J}$, and for every $\epsilon > 0$, there exists an $\epsilon$-optimal policy.

**Proof:** Note that since by assumption, $J_N^* (x) > -\infty$ for all $x \in X$, an $N$-stage $\epsilon$-optimal policy $\pi_\epsilon \in \Pi$ is one for which

$$J_N^* \leq J_{N,\pi_\epsilon} \leq J_N^* + \epsilon e.$$

We use induction. The result clearly holds for $N = 1$. Assume that it holds for $N = k$, i.e., $J_k^* = T^k \bar{J}$ and for a given $\epsilon > 0$, there is a $\pi_\epsilon \in \Pi$
with $J_{k,\pi} \leq J^*_k + \epsilon e$. Using Eq. (4.6), we have for all $\mu \in \mathcal{M},$

$$J^*_{k+1} \leq T_{\mu}J_{k,\pi} \leq T_{\mu}J^*_k + \alpha \epsilon e.$$ 

Taking the infimum over $\mu$ and then the limit as $\epsilon \to 0$, we obtain

$$J^*_{k+1} \leq \inf_{\mu \in \mathcal{M}} T_{\mu}J^*_k + \alpha \epsilon e.$$ 

We now turn to the existence of an $\epsilon$-optimal policy part of the induction argument. Using the assumption $J^*_k(x) > -\infty$ for all $x \in X$, for any $\epsilon > 0$, we can choose $\pi = \{\pi_0, \pi_1, \ldots\}$ such that

$$J^*_{k+1,\pi} \leq T_{\mu}J^*_k + \frac{\epsilon}{2} e.$$ 

Let $\pi _\tau = \{\pi, \pi_0, \pi_1, \ldots\}$. Then

$$J^*_{k+1,\pi _\tau} = T_{\pi}J_{k,\pi} \leq T_{\pi}J^*_k + \frac{\epsilon}{2} e \leq T_{\pi}J^*_k + \tau e = J^*_{k+1} + \tau e,$$

where the first inequality is obtained by applying $T_{\pi}$ to Eq. (4.7) and using Eq. (4.6). The induction is complete. Q.E.D.

We now provide some counterexamples showing that the conditions of the preceding propositions are necessary, and that for exceptional (but otherwise very simple) problems, the Bellman equation $J^*_N = T^N \tilde{J}$ may not hold and/or there may not exist an $\epsilon$-optimal policy.

**Example 4.2.1 (Counterexample to Bellman’s Equation I)**

Let

$$X = \{0\}, \quad U(0) = (-1, 0], \quad \tilde{J}(0) = 0,$$

$$H(0, u, J) = \begin{cases} u & \text{if } -1 < J(0), \\ J(0) + u & \text{if } J(0) \leq -1. \end{cases}$$

Then

$$(T_{\mu_0} \cdots T_{\mu_{N-1}} J)(0) = \mu_0(0),$$

and $J^*_N(0) = -1$, while $(T^N \tilde{J})(0) = -N$ for every $N$. Here Assumption 4.2.1, and the condition (4.6) (cf. Assumption 4.2.2) are violated, even though the condition $J^*_k(x) > -\infty$ for all $x \in X$ (cf. Assumption 4.2.2) is satisfied.
Example 4.2.2 (Counterexample to Bellman’s Equation II)

Let

\[ X = \{0, 1\}, \quad U(0) = U(1) = (-\infty, 0], \quad \bar{J}(0) = \bar{J}(1) = 0, \]

\[ H(0, u, J) = \begin{cases} u & \text{if } J(1) = -\infty, \\ 0 & \text{if } J(1) > -\infty, \end{cases} \quad H(1, u, J) = u. \]

Then

\[ (T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J})(0) = 0, \quad (T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J})(1) = \mu(1), \quad \forall \ N \geq 1. \]

It can be seen that for \( N \geq 2 \), we have \( J^*_N(0) = 0 \) and \( J^*_N(1) = -\infty \), but \( (T^N \bar{J})(0) = (T^N \bar{J})(1) = -\infty \). Here Assumption 4.2.1, and the condition \( J^*_k(x) > -\infty \) for all \( x \in X \) (cf. Assumption 4.2.2) are violated, even though the condition (4.6) of Assumption 4.2.2 is satisfied.

In the preceding two examples, the anomalies are due to discontinuity of the mapping \( H \) with respect to \( J \). In classical finite horizon DP, the mapping \( H \) is generally continuous when it takes finite values, but counterexamples arise in unusual problems where infinite values occur. The next example is a simple stochastic optimal control problem, which involves some infinite expected values of random variables and we have \( J^*_2 \neq T^2 \bar{J} \).

Example 4.2.3 (Counterexample to Bellman’s Equation III)

Let

\[ X = \{0, 1\}, \quad U(0) = U(1) = \mathbb{R}, \quad \bar{J}(0) = \bar{J}(1) = 0, \]

let \( w \) be a real-valued random variable with \( E\{w\} = \infty \), and let

\[ H(x, u, J) = \begin{cases} E\{w + J(1)\} & \text{if } x = 0, \\ u + J(1) & \text{if } x = 1, \end{cases} \quad \forall \ x \in X, \ u \in U(x). \]

Then if \( J_m \) is real-valued for all \( m \), and \( J_m(1) \downarrow J(1) = -\infty \), we have

\[ \lim_{m \to \infty} H(0, u, J_m) = \lim_{m \to \infty} E\{w + J_m(1)\} = \infty, \]

while

\[ H\left(0, u, \lim_{m \to \infty} J_m\right) = E\{w + J(1)\} = -\infty, \]

so Assumption 4.2.1 is violated. Indeed, the reader may verify with a straightforward calculation that \( J^*_2(0) = \infty \), \( J^*_2(1) = -\infty \), while \( (T^2 \bar{J})(0) = -\infty \), \( (T^2 \bar{J})(1) = -\infty \), so \( J^*_2 \neq T^2 \bar{J} \). Note that Assumption 4.2.2 is also violated because \( J^*_2(1) = -\infty \).

In the next counterexample, Bellman’s equation holds, but there is no \( \epsilon \)-optimal policy. This is an undiscounted deterministic optimal control problem of the type discussed in Section 1.1, where \( J^*_k(x) = -\infty \) for some \( x \) and \( k \), so Assumption 4.2.2 is violated. We use the notation introduced there.
Example 4.2.4 (Counterexample to Existence of an \( \epsilon \)-Optimal Policy)

Let \( \alpha = 1 \) and \( N = 2, \) \( X = \{0, 1, \ldots\}, \) \( U(x) = (0, \infty), \) \( J(x) = 0, \) \( \forall x \in X, \)

\[
f(x, u) = 0, \quad \forall x \in X, \ u \in U(x),
\]

\[
g(x, u) = \begin{cases} -u & \text{if } x = 0, \\ x & \text{if } x \neq 0, \end{cases} \quad \forall u \in U(x),
\]

so that

\[
H(x, u, J) = g(x, u) + J(0).
\]

Then for \( \pi \in \Pi \) and \( x \neq 0, \) we have \( J_{2, \pi}(x) = x - \mu_1(0), \) so that \( J_{2}^*(x) = -\infty \)
for all \( x \in X. \) Here Assumption 4.2.1, as well as Eq. (4.6) (cf. Assumption 4.2.2) are satisfied, and indeed we have \( J_{2}^*(x) = (T^2J)(x) = -\infty \)
for all \( x \in X. \) However, the condition \( J_{k}^*(x) \geq -\infty \) for all \( x \) and \( k \) (cf. Assumption 4.2.2) is violated, and it is seen that there does not exist a two-stage \( \epsilon \)-optimal policy for any \( \epsilon > 0, \) since an \( \epsilon \)-optimal policy \( \pi = \{\mu_0, \mu_1\} \) must satisfy

\[
J_{2, \pi}(x) = x - \mu_1(0) \leq -\frac{1}{\epsilon}, \quad \forall x \in X,
\]

[in view of \( J_{2}^*(x) = -\infty \) for all \( x \in X \)], which is impossible.

4.3 INFINITE HORIZON PROBLEMS

Consider the infinite horizon problem (4.2), where the cost function of a policy \( \pi = \{\mu_0, \mu_1, \ldots\} \) is

\[
J_{\pi}(x) = \limsup_{k \to \infty} (T_{\mu_0} \cdots T_{\mu_k} J)(x), \quad \forall x \in X.
\]

In this section one of the following two assumptions will be in effect.

**Assumption I: (Monotone Increase)**

(a) We have

\[
-\infty < \bar{J}(x) \leq H(x, u, J), \quad \forall x \in X, \ u \in U(x).
\]

(b) For each sequence \( \{J_m\} \subset \mathcal{E}(X) \) with \( J_m \uparrow \bar{J} \) and \( \bar{J} \leq J_m \) for all \( m \geq 0, \) we have

\[
\lim_{m \to \infty} H(x, u, J_m) = H(x, u, J), \quad \forall x \in X, \ u \in U(x).
\]
There exists a scalar $\alpha \in (0, \infty)$ such that for all scalars $r \in (0, \infty)$ and functions $J \in \mathcal{E}(X)$ with $\bar{J} \leq J$, we have

$$H(x, u, J + re) \leq H(x, u, J) + \alpha r, \quad \forall x \in X, u \in U(x).$$

### Assumption D: (Monotone Decrease)

(a) We have

$$\bar{J}(x) \geq H(x, u, J), \quad \forall x \in X, u \in U(x).$$

(b) For each sequence $\{J_m\} \subset \mathcal{E}(X)$ with $J_m \downarrow \bar{J}$ and $J_m \leq \bar{J}$ for all $m \geq 0$, we have

$$\lim_{m \to \infty} H(x, u, J_m) = H(x, u, J), \quad \forall x \in X, u \in U(x).$$

Assumptions I and D apply to the classes of the negative and positive DP models, respectively (see [Ber12a], Chapter 4). These are the special cases of the infinite horizon stochastic optimal control problem of Example 1.2.1, where $\bar{J}(x) \equiv 0$ and the cost per stage $g$ is uniformly nonnegative or uniformly nonpositive, respectively. The latter occurs often when we want to maximize positive rewards.

It is important to note that Assumptions I and D allow $J_\pi$ to be defined as a limit rather than as a lim sup. In particular, part (a) of the assumptions and the monotonicity of $H$ imply that

$$\bar{J} \leq T_{\mu_0} \bar{J} \leq T_{\mu_0} T_{\mu_1} \bar{J} \leq \cdots \leq T_{\mu_0} \cdots T_{\mu_k} \bar{J} \leq \cdots$$

under Assumption I, and

$$\bar{J} \geq T_{\mu_0} \bar{J} \geq T_{\mu_0} T_{\mu_1} \bar{J} \geq \cdots \geq T_{\mu_0} \cdots T_{\mu_k} \bar{J} \geq \cdots$$

under Assumption D. Thus we have

$$J_\pi(x) = \lim_{k \to \infty} (T_{\mu_0} \cdots T_{\mu_k} \bar{J})(x), \quad \forall x \in X,$$

with the limit being a real number or $\infty$ or $-\infty$. 
Sec. 4.3 Infinite Horizon Problems

Figure 4.3.1. Illustration of the consequences of lack of continuity of $T_\mu$ from below or from above [cf. part (b) of Assumption I or D, respectively]. In the figure on the left, we have $J \leq T_\mu J$ but $T_\mu$ is discontinuous from below at $J_\mu$, so Assumption I does not hold, and $J_\mu$ is not a fixed point of $T_\mu$. In the figure on the right, we have $J \geq T_\mu J$ but $T_\mu$ is discontinuous from above at $J_\mu$, so Assumption D does not hold, and $J_\mu$ is not a fixed point of $T_\mu$.

The conditions of part (b) of Assumptions I and D are continuity assumptions designed to preclude some of the anomalies of the type encountered also in Chapter 3, and addressed with the use of $S$-regular policies. In particular, these conditions are essential for making a connection with fixed point theory: they ensure that $J_\mu$ is a fixed point of $T_\mu$, as shown in the following proposition.

Proposition 4.3.1: Let Assumption I or Assumption D hold. Then for every policy $\mu \in \mathcal{M}$, we have

$$J_\mu = T_\mu J_\mu.$$ 

Proof: Let Assumption I hold. Then for all $k \geq 0$,

$$(T_\mu^{k+1} \tilde{J})(x) = H(x, \mu(x), T_\mu^k \tilde{J}), \quad x \in X,$$

and by taking the limit as $k \to \infty$, and using part (b) of Assumption I, and the fact $T_\mu^k \tilde{J} \uparrow J_\mu$, we have for all $x \in X$,

$$J_\mu(x) = \lim_{k \to \infty} H(x, \mu(x), T_\mu^k \tilde{J}) = H(x, \mu(x), \lim_{k \to \infty} T_\mu^k \tilde{J}) = H(x, \mu(x), J_\mu),$$

or equivalently $J_\mu = T_\mu J_\mu$. The proof for the case of Assumption D is similar. Q.E.D.
Figure 4.3.1 illustrates how $J_\mu$ may fail to be a fixed point of $T_\mu$ if part (b) of Assumption I or D is violated. Note also that continuity of $T_\mu$ does not imply continuity of $T$, and for example, under Assumption I, $T$ may be discontinuous from below. We will see later that as a result, the value iteration sequence $\{T^k J\}$ may fail to converge to $J^*$ in the absence of additional conditions (see Section 4.3.2). Part (c) of Assumption I is a technical condition that facilitates the analysis, and assures the existence of $\epsilon$-optimal policies.

Despite the similarities between Assumptions I and D, the corresponding results that one may obtain involve some substantial differences. An important fact, which breaks the symmetry between the two cases, is that $J^*$ is approached by $T^k J$ from below in the case of Assumption I and from above in the case of Assumption D. Another important fact is that since the condition $\hat{J}(x) > -\infty$ for all $x \in X$ is part of Assumption I, all the functions $J$ encountered in the analysis under this assumption (such as $T^k J$, $J_\pi$, and $J^*$) also satisfy $J(x) > -\infty$, for all $x \in X$. In particular, if $J \geq \hat{J}$, we have

$$(TJ)(x) \geq (T\hat{J})(x) > -\infty, \quad \forall \ x \in X,$$

and for every $\epsilon > 0$ there exists $\mu_\epsilon \in \mathcal{M}$ such that

$$T_{\mu_\epsilon} J \leq TJ + \epsilon.$$

This property is not available under Assumption D and accounts in part for the different character of the results that can be obtained under the two assumptions.

### 4.3.1 Fixed Point Properties and Optimality Conditions

We first consider the question whether the optimal cost function $J^*$ is a fixed point of $T$. This is indeed true, but the lines of proof are different under the Assumptions I and D. We begin with the proof under Assumption I, and as a preliminary step we show the following result, which is of independent interest.

**Proposition 4.3.2:** Let Assumption I hold. Then given any $\epsilon > 0$, there exists a policy $\pi_\epsilon \in \Pi$ such that

$$J^* \leq J_{\pi_\epsilon} \leq J^* + \epsilon.$$  

Furthermore, if the scalar $\alpha$ in part (c) of Assumption I satisfies $\alpha < 1$, the policy $\pi_\epsilon$ can be taken to be stationary.
Proof: Let \( \{ \epsilon_k \} \) be a sequence such that \( \epsilon_k > 0 \) for all \( k \) and

\[
\sum_{k=0}^{\infty} \alpha^k \epsilon_k = \epsilon. \tag{4.8}
\]

For each \( x \in X \), consider a sequence of policies \( \{ \pi_k[x] \} \subset \Pi \) of the form

\[
\pi_k[x] = \{ \mu^0_k[x], \mu^1_k[x], \ldots \}, \tag{4.9}
\]

such that for \( k = 0, 1, \ldots \),

\[
J_{\pi_k[x]}(x) \leq J^*(x) + \epsilon_k. \tag{4.10}
\]

Such a sequence exists, since we have assumed that \( \bar{J}(x) > -\infty \), and therefore \( J^*(x) > -\infty \), for all \( x \in X \).

The preceding notation should be interpreted as follows. The policy \( \pi_k[x] \) of Eq. (4.9) is associated with \( x \). Thus \( \mu^i_k[x] \) denotes for each \( x \) and \( k \), a function in \( \mathcal{M} \), while \( \mu^i_k[x](z) \) denotes the value of \( \mu^i_k[x] \) at an element \( z \in X \). In particular, \( \mu^i_k[x](x) \) denotes the value of \( \mu^i_k[x] \) at \( x \in X \).

Consider the functions \( \overline{\mu}_k \) defined by

\[
\overline{\mu}_k(x) = \mu^0_k[x](x), \quad \forall x \in X, \tag{4.11}
\]

and the functions \( \overline{J}_k \) defined by

\[
\overline{J}_k(x) = H \left( x, \overline{\mu}_k(x), \lim_{m \to \infty} T_{\mu^1_k[x]} \cdots T_{\mu^m_k[x]} \overline{J} \right), \quad \forall x \in X, k = 0, 1, \ldots. \tag{4.12}
\]

By using Eqs. (4.10), (4.11), and part (b) of Assumption I, we obtain for all \( x \in X \) and \( k = 0, 1, \ldots \)

\[
\overline{J}_k(x) = \lim_{m \to \infty} (T_{\mu^1_k[x]} \cdots T_{\mu^m_k[x]} \overline{J})(x)
\]

\[
= J_{\pi_k[x]}(x)
\]

\[
\leq J^*(x) + \epsilon_k. \tag{4.13}
\]

From Eqs. (4.12), (4.13), and part (c) of Assumption I, we have for all \( x \in X \) and \( k = 1, 2, \ldots \),

\[
(T_{\overline{\mu}_{k-1}} \overline{J}_k)(x) = H \left( x, \overline{\mu}_{k-1}(x), \overline{J}_k \right)
\]

\[
\leq H \left( x, \overline{\mu}_{k-1}(x), J^* + \epsilon_k \right)
\]

\[
\leq H \left( x, \overline{\mu}_{k-1}(x), J^* \right) + \alpha \epsilon_k
\]

\[
\leq H \left( x, \overline{\mu}_{k-1}(x), \lim_{m \to \infty} T_{\mu^1_{k-1}[x]} \cdots T_{\mu^m_{k-1}[x]} \overline{J} \right) + \alpha \epsilon_k
\]

\[
= \overline{J}_{k-1}(x) + \alpha \epsilon_k,
\]

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and finally
\[ T_{\pi_{k-1}} \tilde{J}_k \leq \tilde{J}_{k-1} + \alpha \epsilon_k e, \quad k = 1, 2, \ldots. \]

Using this inequality and part (c) of Assumption I, we obtain
\[
T_{\pi_{k-2}} T_{\pi_{k-1}} \tilde{J}_k \leq T_{\pi_{k-2}} (\tilde{J}_{k-1} + \alpha \epsilon_k e)
\leq T_{\pi_{k-2}} \tilde{J}_{k-1} + \alpha^2 \epsilon_k e
\leq \tilde{J}_{k-2} + (\alpha \epsilon_{k-1} + \alpha^2 \epsilon_k) e.
\]

Continuing in the same manner, we have for \( k = 1, 2, \ldots, \)
\[
T_{\pi_0} \cdots T_{\pi_{k-1}} \tilde{J}_k \leq \tilde{J}_0 + (\alpha \epsilon_1 + \cdots + \alpha^k \epsilon_k) e \leq J^* + \left( \sum_{i=0}^{k} \alpha^i \epsilon_i \right) e.
\]

Since \( \tilde{J} \leq \tilde{J}_k, \) it follows that
\[
T_{\pi_0} \cdots T_{\pi_{k-1}} \tilde{J} \leq \tilde{J}_0 + (\alpha \epsilon_1 + \cdots + \alpha^k \epsilon_k) e \leq J^* + \left( \sum_{i=0}^{k} \alpha^i \epsilon_i \right) e.
\]

Denote \( \pi_\epsilon = \{\overline{\pi}_0, \overline{\pi}_1, \ldots\}. \) Then by taking the limit in the preceding inequality and using Eq. (4.8), we obtain
\[
J_{\pi_\epsilon} \leq J^* + \epsilon e.
\]

If \( \alpha < 1, \) we take \( \epsilon_k = \epsilon(1-\alpha) \) for all \( k, \) and \( \pi_k[x] = \{\mu_0[x], \mu_1[x], \ldots\} \) in Eq. (4.10). The stationary policy \( \pi_\epsilon = \{\overline{\pi}, \overline{\pi}, \ldots\}, \) where \( \overline{\pi}(x) = \mu_0[x](x) \) for all \( x \in X, \) satisfies \( J_{\pi_\epsilon} \leq J^* + \epsilon e. \) Q.E.D.

Note that the assumption \( \alpha < 1 \) is essential in order to be able to take \( \pi_\epsilon \) stationary in the preceding proposition. As an example, let \( X = \{0\}, \)
\[ U(0) = (0, \infty), \quad J(0) = 0, \quad H(0, u, J) = u + J(0). \]
Then \( J'(0) = 0, \) but for any \( \mu \in M, \) we have \( J_\mu(0) = \infty. \)

By using Prop. 4.3.2 we can prove the following.

**Proposition 4.3.3:** Let Assumption I hold. Then
\[
J^* = TJ^*.
\]

Furthermore, if \( J' \in \mathcal{E}(X) \) is such that \( J' \geq \tilde{J} \) and \( J' \geq TJ', \) then \( J' \geq J^*. \)
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Proof: For every \( \pi = \{ \mu_0, \mu_1, \ldots \} \in \Pi \) and \( x \in X \), we have using part (b) of Assumption I,

\[
J_\pi(x) = \lim_{k \to \infty} (T_{\mu_0} T_{\mu_1} \cdots T_{\mu_k} \bar{J})(x) \\
= T_{\mu_0} \left( \lim_{k \to \infty} T_{\mu_1} \cdots T_{\mu_k} \bar{J} \right)(x) \\
\geq (T_{\mu_0} J^*)(x) \\
\geq (T^* J^*)(x).
\]

By taking the infimum of the left-hand side over \( \pi \in \Pi \), we obtain

\[
J^* \geq T^* J^*.
\]

To prove the reverse inequality, let \( \epsilon_1 \) and \( \epsilon_2 \) be any positive scalars, and let \( \overline{\pi} = \{ \overline{\mu}_0, \overline{\mu}_1, \ldots \} \) be such that

\[
T_{\overline{\mu}_0} J^* \leq T^* J^* + \epsilon_1 e, \quad J_{\overline{\pi}_1} \leq J^* + \epsilon_2 e,
\]

where \( \overline{\pi}_1 = \{ \overline{\mu}_1, \overline{\mu}_2, \ldots \} \) (such a policy exists by Prop. 4.3.2). The sequence \( \{ T_{\overline{\mu}_0} T_{\overline{\mu}_1} \cdots T_{\overline{\mu}_k} \bar{J} \} \) is monotonically nondecreasing, so by using the preceding relations and part (c) of Assumption I, we have

\[
T_{\overline{\mu}_0} T_{\overline{\mu}_1} \cdots T_{\overline{\mu}_k} \bar{J} \leq T_{\overline{\mu}_0} \left( \lim_{k \to \infty} T_{\overline{\mu}_1} \cdots T_{\overline{\mu}_k} \bar{J} \right) \\
= T_{\overline{\mu}_0} J_{\overline{\pi}_1} \\
\leq T_{\overline{\mu}_0} J^* + \alpha \epsilon_2 e \\
\leq T^* J^* + (\epsilon_1 + \alpha \epsilon_2) e.
\]

Taking the limit as \( k \to \infty \), we obtain

\[
J^* \leq J_{\overline{\pi}} = \lim_{k \to \infty} T_{\overline{\mu}_0} T_{\overline{\mu}_1} \cdots T_{\overline{\mu}_k} \bar{J} \leq T^* J^* + (\epsilon_1 + \alpha \epsilon_2) e.
\]

Since \( \epsilon_1 \) and \( \epsilon_2 \) can be taken arbitrarily small, it follows that

\[
J^* \leq T^* J^*.
\]

Hence \( J^* = T^* J^* \).

Assume that \( J' \in \mathcal{E}(X) \) satisfies \( J' \geq \bar{J} \) and \( J' \geq T^* J' \). Let \( \{ \epsilon_k \} \) be any sequence with \( \epsilon_k > 0 \) for all \( k \), and consider a policy \( \overline{\pi} = \{ \overline{\mu}_0, \overline{\mu}_1, \ldots \} \in \Pi \) such that

\[
T_{\overline{\mu}_k} J' \leq T^* J' + \epsilon_k e, \quad k = 0, 1, \ldots.
\]
We have from part (c) of Assumption I

\[ J^* = \inf_{\pi \in \Pi} \lim_{k \to \infty} T_{\mu_0} \cdots T_{\mu_k} J \]

\leq \inf_{\pi \in \Pi} \liminf_{k \to \infty} T_{\mu_0} \cdots T_{\mu_k} J'

\leq \liminf_{k \to \infty} T_{\mu_0} \cdots T_{\mu_{k-1}} (T J' + \epsilon_k e)

\leq \liminf_{k \to \infty} T_{\mu_0} \cdots T_{\mu_{k-1}} (J' + \epsilon_k e)

\leq \liminf_{k \to \infty} (T_{\mu_0} \cdots T_{\mu_{k-1}} J' + \alpha \epsilon_k e)

\vdots

\leq \lim_{k \to \infty} \left( T J' + \left( \sum_{i=0}^{k-1} \alpha^i \epsilon_i \right) e \right)

\leq J' + \left( \sum_{i=0}^{k} \alpha^i \epsilon_i \right) e.

Since we may choose \( \sum_{i=0}^{k} \alpha^i \epsilon_i \) as small as desired, it follows that \( J^* \leq J' \).

Q.E.D.

The following counterexamples show that parts (b) and (c) of Assumption I are essential for the preceding proposition to hold.

**Example 4.3.1 (Counterexample to Bellman’s Equation I)**

Let

\( X = \{0, 1\}, \quad U(0) = U(1) = (-1, 0), \quad \bar{J}(0) = \bar{J}(1) = -1, \)

\( H(0, u, \cdot) = \begin{cases} u & \text{if } J(1) \leq -1, \\ 0 & \text{if } J(1) > -1, \end{cases} \quad H(1, u, J) = u. \)

Then for \( N \geq 1, \)

\( (T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J})(0) = 0, \quad (T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J})(1) = \mu_0(1). \)

Thus

\( J^*(0) = 0, \quad J^*(1) = -1, \quad (T J^*)(0) = -1, \quad (T J^*)(1) = -1, \)

and hence \( J^* \neq T J^* \). Notice also that \( \bar{J} \) is a fixed point of \( T \), while \( \bar{J} \leq J^* \) and \( \bar{J} \neq J^* \), so the second part of Prop. 4.3.3 fails when \( \bar{J} = J^* \). Here parts (a) and (b) of Assumption I are satisfied, but part (c) is violated, since \( H(0, u, \cdot) \) is discontinuous at \( J = -1 \) when \( u < 0 \).
Example 4.3.2 (Counterexample to Bellman’s Equation II)

Let

\[ X = \{0, 1\}, \quad U(0) = U(1) = \{0\}, \quad J(0) = J(1) = 0, \]

\[ H(0, 0, J) = \begin{cases} 0 & \text{if } J(1) < \infty, \\ \infty & \text{if } J(1) = \infty, \end{cases} \quad H(1, 0, J) = J(1) + 1. \]

Here there is only one policy, which we denote by \( \mu \). For all \( N \geq 1 \), we have

\[ (T^N_\mu J)(0) = 0, \quad (T^N_\mu J)(1) = N, \]

so \( J^*(0) = 0, J^*(1) = \infty \). On the other hand, we have \((TJ^*)(0) = (TJ^*)(1) = \infty \) and \( J^* \neq TJ^* \). Here parts (a) and (c) of Assumption I are satisfied, but part (b) is violated.

As a corollary to Prop. 4.3.3 we obtain the following.

**Proposition 4.3.4:** Let Assumption I hold. Then for every \( \mu \in M \), we have

\[ J_\mu = T_\mu J_\mu. \]

Furthermore, if \( J' \in \mathcal{E}(X) \) is such that \( J' \geq J \) and \( J' \geq T_\mu J' \), then \( J' \geq J_\mu \).

**Proof:** Consider the variant of the infinite horizon problem where the control constraint set is \( U_\mu(x) = \{\mu(x)\} \) rather than \( U(x) \) for all \( x \in X \). Application of Prop. 4.3.3 yields the result. \( \text{Q.E.D.} \)

We now provide the counterpart of Prop. 4.3.3 under Assumption D. We first prove a preliminary result regarding the convergence of the value iteration method, which is of independent interest (we will see later that this result need not hold under Assumption I).

**Proposition 4.3.5:** Let Assumption D hold. Then \( T^N J = J^*_N \), where \( J^*_N \) is the optimal cost function for the \( N \)-stage problem. Moreover

\[ J^* = \lim_{N \to \infty} J^*_N. \]

**Proof:** By repeating the proof of Prop. 4.2.3, we have \( T^N J = J^*_N \) [part (b) of Assumption D is essentially identical to the assumption of that proposition]. Clearly we have \( J^* \leq J^*_N \) for all \( N \), and hence \( J^* \leq \lim_{N \to \infty} J^*_N \).
Also for all \( \pi = \{ \mu_0, \mu_1, \ldots \} \in \Pi \), we have

\[
T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J} \geq J_N^*,
\]

so by taking the limit of both sides as \( N \to \infty \), we obtain \( J_\pi \geq \lim_{N \to \infty} J_N^* \), and by taking infimum over \( \pi \), \( J^* \geq \lim_{N \to \infty} J_N^* \). Thus \( J^* = \lim_{N \to \infty} J_N^* \).

Q.E.D.

**Proposition 4.3.6:** Let Assumption D hold. Then

\[
J^* = TJ^*.
\]

Furthermore, if \( J' \in \mathcal{E}(X) \) is such that \( J' \leq \bar{J} \) and \( J' \leq TJ' \), then \( J' \leq J^* \).

**Proof:** For any \( \pi = \{ \mu_0, \mu_1, \ldots \} \in \Pi \), we have

\[
J_\pi = \lim_{k \to \infty} T_{\mu_0} T_{\mu_1} \cdots T_{\mu_k} \bar{J} \geq \lim_{k \to \infty} T_{\mu_0} T^k \bar{J} \geq T_{\mu_0} J^*,
\]

where the last inequality follows from the fact \( T^k \bar{J} \downarrow J^* \) (cf. Prop. 4.3.5). Taking the infimum of both sides over \( \pi \in \Pi \), we obtain \( J^* \geq TJ^* \).

To prove the reverse inequality, we select any \( \mu \in M \), and we apply \( T_\mu \) to both sides of the equation \( J^* = \lim_{N \to \infty} T^N \bar{J} \) (cf. Prop. 4.3.5). By using part (b) of assumption D, we obtain

\[
T_\mu J^* = T_\mu \left( \lim_{N \to \infty} T^N \bar{J} \right) = \lim_{N \to \infty} T_\mu T^N \bar{J} \geq \lim_{N \to \infty} T^{N+1} \bar{J} = J^*.
\]

Taking the infimum of the left-hand side over \( \mu \in M \), we obtain \( TJ^* \geq J^* \), showing that \( TJ^* = J^* \).

To complete the proof, let \( J' \in \mathcal{E}(X) \) be such that \( J' \leq \bar{J} \) and \( J' \leq TJ' \). Then we have

\[
J^* = \inf_{\pi \in \Pi} \lim_{N \to \infty} T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J}
\]
\[
\geq \lim_{N \to \infty} \inf_{\pi \in \Pi} T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J}
\]
\[
\geq \lim_{N \to \infty} \inf_{\pi \in \Pi} T_{\mu_0} \cdots T_{\mu_{N-1}} J'
\]
\[
\geq \lim_{N \to \infty} T^N J'
\]
\[
\geq J',
\]

where the last inequality follows from the hypothesis \( J' \leq TJ' \). Thus \( J^* \geq J' \). Q.E.D.
Counterexamples to Bellman’s equation can be readily constructed if part (b) of Assumption D (continuity from above) is violated. In particular, in Examples 4.2.1 and 4.2.2, part (a) of Assumption D is satisfied but part (b) is not. In both cases we have \( J^* \neq T J^* \), as the reader can verify with a straightforward calculation.

Similar to Prop. 4.3.4, we obtain the following.

**Proposition 4.3.7:** Let Assumption D hold. Then for every \( \mu \in \mathcal{M} \), we have
\[
J_{\mu} = T_{\mu} J_{\mu}.
\]
Furthermore, if \( J' \in \mathcal{E}(X) \) is such that \( J' \leq \bar{J} \) and \( J' \leq T_{\mu} J' \), then \( J' \leq J_{\mu} \).

**Proof:** Consider the variation of our problem where the control constraint set is \( U_{\mu}(x) = \{ \mu(x) \} \) rather than \( U(x) \) for all \( x \in X \). Application of Prop. 4.3.6 yields the result. \( \mathrm{Q.E.D.} \)

An examination of the proof of Prop. 4.3.6 shows that the only point where we need part (b) of Assumption D was in establishing the relations
\[
\lim_{N \to \infty} T J^*_N = T \left( \lim_{N \to \infty} J^*_N \right)
\]
and \( J^*_N = T^N \bar{J} \). If these relations can be established independently, then the result of Prop. 4.3.6 follows. In this manner we obtain the following proposition.

**Proposition 4.3.8:** Let part (a) of Assumption D hold, assume that \( X \) is a finite set, and that \( J^*(x) > -\infty \) for all \( x \in X \). Assume further that there exists a scalar \( \alpha \in (0, \infty) \) such that for all scalars \( r \in (0, \infty) \) and functions \( J \in \mathcal{E}(X) \) with \( J \leq \bar{J} \), we have
\[
H(x, u, J) - \alpha r \leq H(x, u, J - r e), \quad \forall \ x \in X, \ u \in U(x). \quad (4.14)
\]
Then
\[
J^* = T J^*.
\]
Furthermore, if \( J' \in \mathcal{E}(X) \) is such that \( J' \leq \bar{J} \) and \( J' \leq T J' \), then \( J' \leq J^* \).

**Proof:** A nearly verbatim repetition of Prop. 4.2.4 shows that under our
assumptions we have $J^*_N = T^N J$ for all $N$. We will show that

$$
\lim_{N \to \infty} H(x, u, J^*_N) \leq H \left( x, u, \lim_{N \to \infty} J^*_N \right), \quad \forall \ x \in X, \ u \in U(x).
$$

Then the result follows as in the proof of Prop. 4.3.6.

Assume the contrary, i.e., that for some $\bar{x} \in X$, $\bar{u} \in U(\bar{x})$, and $\epsilon > 0$, there holds

$$
H(\bar{x}, \bar{u}, J^*_k) - \epsilon > H \left( \bar{x}, \bar{u}, \lim_{N \to \infty} J^*_N \right), \quad k = 1, 2, \ldots.
$$

From the finiteness of $X$ and the fact $J^*(x) = \lim_{N \to \infty} J^*_N(x) > -\infty$ for all $x$, we know that for some integer $\bar{k} > 0$

$$
J^*_k - (\epsilon/\alpha) e \leq \lim_{N \to \infty} J^*_N, \quad \forall \ k \geq \bar{k}.
$$

By using the condition (4.14), we obtain for all $k \geq \bar{k}$

$$
H(\bar{x}, \bar{u}, J^*_k) - \epsilon \leq H(\bar{x}, \bar{u}, J^*_k - (\epsilon/\alpha) e) \leq H \left( \bar{x}, \bar{u}, \lim_{N \to \infty} J^*_N \right),
$$

which contradicts the earlier inequality. \textbf{Q.E.D.}

**Characterization of Optimal Policies**

We now provide necessary and sufficient conditions for optimality of a stationary policy. These conditions are markedly different under Assumptions I and D.

**Proposition 4.3.9:** Let Assumption I hold. Then a stationary policy $\mu$ is optimal if and only if

$$
T_\mu J^* = TJ^*.
$$

**Proof:** If $\mu$ is optimal, then $J_\mu = J^*$ so that the equation $J^* = TJ^*$ (cf. Prop. 4.3.3) implies that $J_\mu = T J_\mu$. Since $J_\mu = T_\mu J_\mu$ (cf. Prop. 4.3.4), it follows that $T_\mu J^* = TJ^*$.

Conversely, if $T_\mu J^* = TJ^*$, then since $J^* = TJ^*$, it follows that $T_\mu J^* = J^*$. By Prop. 4.3.4, it follows that $J_\mu \leq J^*$, so $\mu$ is optimal. \textbf{Q.E.D.}
Proposition 4.3.10: Let Assumption D hold. Then a stationary  
 policy \( \mu \) is optimal if and only if
\[
T_{\mu}J_{\mu} = T J_{\mu}.
\]

**Proof:** If \( \mu \) is optimal, then \( J_{\mu} = J^* \), so that the equation \( J^* = T J^* \)  
(cf. Prop. 4.3.6) can be written as \( J_{\mu} = T J_{\mu} \). Since \( J_{\mu} = T J_{\mu} \)  
(cf. Prop. 4.3.4), it follows that \( T_{\mu}J_{\mu} = T J_{\mu} \).

Conversely, if \( T_{\mu}J_{\mu} = T J_{\mu} \), then since \( J_{\mu} = T J_{\mu} \), it follows that \( J_{\mu} = T J_{\mu} \). By Prop. 4.3.7, it follows that \( J_{\mu} \leq J^* \), so \( \mu \) is optimal. Q.E.D.

An example showing that under Assumption I, the condition \( T_{\mu}J_{\mu} = T J_{\mu} \) does not guarantee optimality of \( \mu \) is given in Exercise 4.3. Under Assumption D, we note that by Prop. 4.3.1 or 4.3.7, we have \( J_{\mu} = T_{\mu}J_{\mu} \) for all \( \mu \), so if \( \mu \) is a stationary optimal policy, the fixed point equation
\[
J^*(x) = \inf_{u \in U(x)} H(x, u, J^*), \quad \forall x \in X,
\]
and the optimality condition of Prop. 4.3.10, yield
\[
T J^* = J^* = T_{\mu}J_{\mu} = T_{\mu}J^*.
\]

Thus under D, a stationary optimal policy attains the infimum in the fixed point Eq. (4.15) for all \( x \). However, there may exist nonoptimal stationary policies also attaining the infimum for all \( x \); an example is the shortest path problem of Section 3.1.1 for the case where \( a = 0 \) and \( b = 1 \). Moreover, it is possible that this infimum is attained but no optimal policy exists, as shown by Fig. 4.3.2.

Proposition 4.3.9 shows that under Assumption I, there exists a stationary optimal policy if and only if the infimum in the optimality equation
\[
J^*(x) = \inf_{u \in U(x)} H(x, u, J^*)
\]
is attained for every \( x \in X \). When the infimum is not attained for some \( x \in X \), this optimality equation can still be used to yield an \( \epsilon \)-optimal policy, which can be taken to be stationary whenever the scalar \( \alpha \) in Assumption I(c) is strictly less than 1. This is shown in the following proposition.
Figure 4.3.2. An example where nonstationary policies are dominant under Assumption D. Here there is only one state and \( S = \mathbb{R} \). There are two stationary policies \( \mu \) and \( \bar{\pi} \) with cost functions \( J_\mu \) and \( J_{\bar{\pi}} \) as shown. However, by considering a nonstationary policy of the form \( \pi_k = \{\bar{\pi}, \ldots, \bar{\pi}, \mu, \mu, \ldots\} \), with a number \( k \) of policies \( \bar{\pi} \), we can obtain a sequence \( \{J_{\pi_k}\} \) that converges to the value \( J^* \) shown. Note that here there is no optimal policy, stationary or not.

Proposition 4.3.11: Let Assumption I hold. Then:

(a) If \( \epsilon > 0 \), the sequence \( \{\epsilon_k\} \) satisfies \( \sum_{k=0}^{\infty} \alpha^k \epsilon_k = \epsilon \), and \( \epsilon_k > 0 \) for all \( k \), and the policy \( \pi^* = \{\mu^*_0, \mu^*_1, \ldots\} \in \Pi \) is such that

\[
T_{\mu^*_k} J^* \leq T J^* + \epsilon_k \epsilon, \quad \forall \ k = 0, 1, \ldots,
\]

then

\[
J^* \leq J_{\pi^*} \leq J^* + \epsilon \epsilon.
\]

(b) If \( \epsilon > 0 \), the scalar \( \alpha \) in part (c) of Assumption I is strictly less than 1, and \( \mu^* \in \mathcal{M} \) is such that

\[
T_{\mu^*} J^* \leq T J^* + \epsilon (1 - \alpha) \epsilon,
\]

then

\[
J^* \leq J_{\mu^*} \leq J^* + \epsilon \epsilon.
\]
Sec. 4.3 Infinite Horizon Problems

Proof: (a) Since \( T J^* = J^* \), we have

\[ T_{\mu_k^*}^* J^* \leq J^* + \epsilon_k e, \]

and applying \( T_{\mu_k^*}^* \) to both sides, we obtain

\[ T_{\mu_k^*}^* T_{\mu_{k-1}^*}^* J^* \leq T_{\mu_{k-1}^*}^* J^* + \alpha \epsilon_k e \leq J^* + (\epsilon_{k-1} + \alpha \epsilon_k) e. \]

Applying \( T_{\mu_{k-2}^*}^* \) throughout and repeating the process, we obtain for every \( k = 1, 2, \ldots \),

\[ T_{\mu_0^*}^* \cdots T_{\mu_k^*}^* J^* \leq J^* + \left( \sum_{i=0}^{k} \alpha^i \epsilon_i \right) e, \quad k = 1, 2, \ldots. \]

Since \( \bar{J} \leq J^* \), it follows that

\[ T_{\mu_0^*}^* \cdots T_{\mu_k^*}^* \bar{J} \leq J^* + \left( \sum_{i=0}^{k} \alpha^i \epsilon_i \right) e, \quad k = 1, 2, \ldots. \]

By taking the limit as \( k \to \infty \), we obtain \( J_{\pi^*} = J^* + \epsilon e \).

(b) This part is proved by taking \( \epsilon_k = \epsilon (1 - \alpha) \) and \( \mu_k^* = \mu^* \) for all \( k \) in the preceding argument. Q.E.D.

Under Assumption D, the existence of an \( \epsilon \)-optimal policy is harder to establish, and requires some restrictive conditions.

Proposition 4.3.12: Let Assumption D hold, and let the additional assumptions of Prop. 4.3.8 hold. Then for any \( \epsilon > 0 \), there exists an \( \epsilon \)-optimal policy.

Proof: For each \( N \), denote

\[ \epsilon_N = \frac{\epsilon}{2(1 + \alpha + \cdots + \alpha^{N-1})}. \]

and let

\[ \pi_N = \{ \mu_0^N, \mu_1^N, \ldots, \mu_{N-1}^N, \mu, \mu, \ldots \} \]

be such that \( \mu \in \mathcal{M} \), and for \( k = 0, \ldots, N - 1 \), \( \mu_k^N \in \mathcal{M} \) and

\[ T_{\mu_k^N}^* T^{N-k-1} \bar{J} = T^{N-k} \bar{J} + \epsilon_N e. \]
We have $T_{N_{N-1}}^N \bar{J} \leq T \bar{J} + \epsilon_N e$, and applying $T_{N_{N-2}}^N$ to both sides, we obtain

$$T_{N_{N-2}}^N T_{N_{N-1}}^N \bar{J} \leq T_{N_{N-2}}^N T \bar{J} + \alpha \epsilon_N e \leq T^2 \bar{J} + (1 + \alpha) \epsilon_N e.$$ 

Continuing in the same manner, we have

$$T_{N_{N-1}}^N \cdots T_{N_{N-1}}^N \bar{J} \leq T^N \bar{J} + (1 + \alpha + \cdots + \alpha^{N-1}) \epsilon_N e,$$

from which we obtain for $N = 0, 1, \ldots$,

$$J_{\pi_N} \leq T^N \bar{J} + (\epsilon/2) e.$$

By Prop. 4.3.5, we have $J^* = \lim_{N \to \infty} T^N \bar{J}$, so let $\bar{N}$ be such that

$$T^{\bar{N}} \bar{J} \leq J^* + (\epsilon/2) e$$

[such a $\bar{N}$ exists using the assumptions of finiteness of $X$ and $J^*(x) > -\infty$ for all $x \in X$]. Then we obtain $J_{\pi_{\bar{N}}} \leq J^* + \epsilon e$, and $\pi_{\bar{N}}$ is the desired policy. \textbf{Q.E.D.}

4.3.2 Value Iteration

We will now discuss algorithms for abstract DP under Assumptions I and D. We first consider the VI algorithm, which consists of successively generating $T \bar{J}, T^2 \bar{J}, \ldots$. Note that because $T$ need not be a contraction, it may have multiple fixed points $J$ all of which satisfy $J \geq J^*$ under Assumption I (cf. Prop. 4.3.3) or $J \leq J^*$ under Assumption D (cf. Prop. 4.3.6). Thus, in the absence of additional conditions (to be discussed in Sections 4.4 and 4.5), it is essential to start VI with $\bar{J}$ or an initial $J_0$ such that $\bar{J} \leq J_0 \leq J^*$ under Assumption I or $\bar{J} \geq J_0 \geq J^*$ under Assumption D. In the next two propositions, we show that for such initial conditions, we have convergence of VI to $J^*$ under Assumption D, and with an additional compactness condition, under Assumption I.

**Proposition 4.3.13:** Let Assumption D hold, and assume that $J_0 \in \mathcal{E}(X)$ is such that $\bar{J} \geq J_0 \geq J^*$. Then

$$\lim_{k \to \infty} T^k J_0 = J^*.$$

**Proof:** The condition $\bar{J} \geq J_0 \geq J^*$ implies that $T^k \bar{J} \geq T^k J_0 \geq J^*$ for all $k$. By Prop. 4.3.5, $T^k \bar{J} \to J^*$, and the result follows. \textbf{Q.E.D.}
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The convergence of VI under I requires an additional compactness condition, which is satisfied in particular if $U(x)$ is a finite set for all $x \in X$.

**Proposition 4.3.14:** Let Assumption I hold, let $U$ be a metric space, and assume that the sets

\[ U_k(x, \lambda) = \{ u \in U(x) \mid H(x, u, T^k \bar{J}) \leq \lambda \} \quad (4.16) \]

are compact for every $x \in X$, $\lambda \in \mathbb{R}$, and for all $k$ greater than some integer $\bar{k}$. Assume that $J_0 \in \mathcal{E}(X)$ is such that $\bar{J} \leq J_0 \leq J^*$. Then

\[ \lim_{k \to \infty} T^k J_0 = J^*. \]

Furthermore, there exists a stationary optimal policy.

**Proof:** Similar to the proof of Prop. 4.3.13, it will suffice to show that $T^k \bar{J} \to J^*$. Since $\bar{J} \leq J^*$, we have $T^k \bar{J} \leq T^k J^* = J^*$, so that

\[ \bar{J} \leq T \bar{J} \leq \cdots \leq T^k \bar{J} \leq \cdots \leq J^*. \]

Thus we have $T^k \bar{J} \uparrow J_{\infty}$ for some $J_{\infty} \in \mathcal{E}(X)$ satisfying $T^k \bar{J} \leq J_{\infty} \leq J^*$ for all $k$. Applying $T$ to this relation, we obtain

\[ (T^{k+1} \bar{J})(x) = \min_{u \in U(x)} H(x, u, T^k \bar{J}) \leq (T J_{\infty})(x), \]

and by taking the limit as $k \to \infty$, it follows that

\[ J_{\infty} \leq T J_{\infty}. \]

Assume to arrive at a contradiction that there exists a state $\tilde{x} \in X$ such that

\[ J_{\infty}(\tilde{x}) < (T J_{\infty})(\tilde{x}). \quad (4.17) \]

Similar to Lemma 3.3.1, there exists a point $u_k$ attaining the minimum in

\[ (T^{k+1} \bar{J})(\tilde{x}) = \inf_{u \in U(\tilde{x})} H(\tilde{x}, u, T^k \bar{J}); \]

i.e., $u_k$ is such that

\[ (T^{k+1} \bar{J})(\tilde{x}) = H(\tilde{x}, u_k, T^k \bar{J}). \]

Clearly, by Eq. (4.17), we must have $J_{\infty}(\tilde{x}) < \infty$. For every $k$, consider the set

\[ U_k(\tilde{x}, J_{\infty}(\tilde{x})) = \{ u \in U(\tilde{x}) \mid H(\tilde{x}, u_k, T^k \bar{J}) \leq J_{\infty}(\tilde{x}) \}, \]
and the sequence \( \{ u_i \}_{i=1}^{\infty} \). Since \( T^k \tilde{J} \uparrow J_\infty \), it follows that for all \( i \geq k \),
\[
H(\tilde{x}, u_i, T^k \tilde{J}) \leq H(\tilde{x}, u_i, T^i \tilde{J}) \leq J_\infty(\tilde{x}).
\]
Therefore \( \{ u_i \}_{i=1}^{\infty} \subset U_k(\tilde{x}, J_\infty(\tilde{x})) \), and since \( U_k(\tilde{x}, J_\infty(\tilde{x})) \) is compact, all the limit points of \( \{ u_i \}_{i=1}^{\infty} \) belong to \( U_k(\tilde{x}, J_\infty(\tilde{x})) \) and at least one such limit point exists. Hence the same is true of the limit points of the whole sequence \( \{ u_i \} \). It follows that if \( \tilde{u} \) is a limit point of \( \{ u_i \} \) then
\[
\tilde{u} \in \cap_{k=0}^{\infty} U_k(\tilde{x}, J_\infty(\tilde{x})).
\]
By Eq. (4.16), this implies that for all \( k \geq \overline{k} \)
\[
J_\infty(\tilde{x}) \geq H(\tilde{x}, \tilde{u}, T^k \tilde{J}) \geq (T^{k+1} \tilde{J})(\tilde{x}).
\]
Taking the limit as \( k \to \infty \), and using part (b) of Assumption I, we obtain
\[
J_\infty(\tilde{x}) \geq H(\tilde{x}, \tilde{u}, J_\infty(\tilde{x})) \geq (T J_\infty)(\tilde{x}).
\] (4.18)
which contradicts Eq. (4.17). Hence \( J_\infty = T J_\infty \), which implies that \( J_\infty \geq J^* \) in view of Prop. 4.3.3. Combined with the inequality \( J_\infty \leq J^* \), which was shown earlier, we have \( J_\infty = J^* \).

To show that there exists an optimal stationary policy, observe that the relation \( J^* = J_\infty = T J_\infty \) and Eq. (4.18) [whose proof is valid for all \( \tilde{x} \in X \) such that \( J^*(\tilde{x}) < \infty \)] imply that \( \tilde{u} \) attains the infimum in
\[
J^*(\tilde{x}) = \inf_{u \in U(\tilde{x})} H(\tilde{x}, u, J^*)
\]
for all \( \tilde{x} \in X \) with \( J^*(\tilde{x}) < \infty \). For \( \tilde{x} \in X \) such that \( J^*(\tilde{x}) = \infty \), every \( u \in U(\tilde{x}) \) attains the preceding minimum. Hence by Prop. 4.3.9 an optimal stationary policy exists. Q.E.D.

The reader may verify by inspection of the preceding proof that if \( \mu_k(\tilde{x}) \), \( k = 0, 1, \ldots \), attains the infimum in the relation
\[
(T^{k+1} \tilde{J})(\tilde{x}) = \inf_{u \in U(\tilde{x})} H(\tilde{x}, u, T^k \tilde{J}),
\]
and \( \mu^*(\tilde{x}) \) is a limit point of \( \{ \mu_k(\tilde{x}) \} \), for every \( \tilde{x} \in X \), then the stationary policy \( \mu^* \) is optimal. Furthermore, \( \{ \mu_k(\tilde{x}) \} \) has at least one limit point for every \( \tilde{x} \in X \) for which \( J^*(\tilde{x}) < \infty \). Thus the VI algorithm under the assumption of Prop. 4.3.14 yields in the limit not only the optimal cost function \( J^* \) but also an optimal stationary policy.

On the other hand, under Assumption I but in the absence of the compactness condition (4.16), \( T^k \tilde{J} \) need not converge to \( J^* \). What is happening here is that while the mappings \( T_\mu \) are continuous from below as required by Assumption I(b), \( T \) may not be, and a phenomenon like the one illustrated in the left-hand side of Fig. 4.3.1 may occur, whereby
\[
\lim_{k \to \infty} T^k \tilde{J} \leq T \left( \lim_{k \to \infty} T^k \tilde{J} \right),
\]
with strict inequality for some \( x \in X \). This can happen even in simple deterministic optimal control problems, as shown by the following example.
Example 4.3.3 (Counterexample to Convergence of VI)

Let
\[ X = [0, \infty), \quad U(x) = (0, \infty), \quad \bar{J}(x) = 0, \quad \forall x \in X, \]

and
\[ H(x, u, J) = \min \left\{ 1, x + J(2x + u) \right\}, \quad \forall x \in X, u \in U(x). \]

Then it can be verified that for all \( x \in X \) and policies \( \mu \), we have \( J_{\mu}(x) = 1 \), as well as \( J^*(x) = 1 \), while it can be seen by induction that starting with \( \bar{J} \), the VI algorithm yields
\[ (T_k \bar{J})(x) = \min \{ 1, (1 + 2^{k-1})x \}, \quad \forall x \in X, k = 1, 2, \ldots. \]

Thus we have \( 0 = \lim_{k \to \infty} (T_k \bar{J})(0) \neq J^*(0) = 1. \)

The range of convergence of VI may be expanded under additional assumptions. In particular, in Chapter 3, under various conditions involving the existence of optimal \( S \)-regular policies, we showed that VI converges to \( J^* \) assuming that the initial condition \( J_0 \) satisfies \( J_0 \geq J^* \). Thus if the assumptions of Prop. 4.3.14 hold in addition, we are guaranteed convergence of VI starting from any \( J \) satisfying \( J \geq \bar{J} \). Results of this type will be obtained in Sections 4.4 and 4.5, where semicontractive models satisfying Assumption I will be discussed.

Asynchronous Value Iteration

The concepts of asynchronous VI that we developed in Section 2.6.1 apply also under the Assumptions I and D of this section. Under Assumption I, if \( J^* \) is real-valued, we may apply Prop. 2.6.1 with the sets \( S(k) \) defined by
\[ S(k) = \{ J \mid T_k \bar{J} \leq J \leq J^* \}, \quad k = 0, 1, \ldots. \]

Assuming that \( T_k \bar{J} \to J^* \) (cf. Prop. 4.3.14), it follows that the asynchronous form of VI converges pointwise to \( J^* \) starting from any function in \( S(0) \). This result can also be shown for the case where \( J^* \) is not real-valued, by using a simple extension of Prop. 2.6.1, where the set of real-valued functions \( R(X) \) is replaced by the set of all \( J \in \mathcal{E}(X) \) with \( \bar{J} \leq J \leq J^* \).

Under Assumption D similar conclusions hold for the asynchronous version of VI that starts with a function \( J \) with \( J^* \leq J \leq \bar{J} \). Asynchronous pointwise convergence to \( J^* \) can be shown, based on an extension of the asynchronous convergence theorem (Prop. 2.6.1), where \( R(X) \) is replaced by the set of all \( J \in \mathcal{E}(X) \) with \( J^* \leq J \leq \bar{J} \).
4.3.3 Policy Iteration

Unfortunately, in the absence of further conditions, the PI algorithm is not guaranteed to yield the optimal cost function and/or an optimal policy under either Assumption I or D. However, there are convergence results for nonoptimistic and optimistic variants of PI under some conditions. In what follows in this section we will provide an analysis of various types of PI, mainly under Assumption D. The analysis of PI under Assumption I will be given primarily in the next two sections, as it requires different assumptions and methods of proof, and will be coupled with regularity ideas relating to the semicontractive models of Chapter 3.

Optimistic Policy Iteration Under D

A surprising fact under Assumption D is that nonoptimistic/exact PI may generate a policy that is strictly inferior over the preceding one. Moreover there may be an oscillation between nonoptimal policies even when the state and control spaces are finite. For a mathematical explanation, note that under Assumption D, we may have \( T^* \mu J^* = T J^* \) without \( \mu \) being optimal, so starting from an optimal policy, we may obtain a nonoptimal policy by PI. An illustrative example is the shortest path example of Section 3.1.1, where it can be verified that exact PI may oscillate between the policy that moves to the destination from node 1 and the policy that does not.

On the other hand optimistic PI under Assumption D has much better convergence properties, because it embodies the mechanism of VI, which is convergent to \( J^* \) as we saw in the preceding subsection. Indeed, let us consider an optimistic PI algorithm that generates a sequence \( \{J_k, \mu_k\} \) according to

\[
T_{\mu_k} J_k = T J_k, \quad J_{k+1} = T_{\mu_k}^{m_k} J_k, \tag{4.19}
\]

where \( m_k \) is a positive integer. We assume that the algorithm starts with a function \( J_0 \in \mathcal{E}(X) \) that satisfies \( \bar{J} \geq J_0 \geq T J_0 \) and \( J_0 \geq J^* \). For example, we may choose \( J_0 = \bar{J} \). We have the following proposition.

**Proposition 4.3.15:** Let Assumption D hold and let \( \{J_k, \mu_k\} \) be a sequence generated by the optimistic PI algorithm (4.19), assuming that \( \bar{J} \geq J_0 \geq J^* \) and \( J_0 \geq T J_0 \). Then \( J_k \downarrow J^* \).

**Proof:** We have

\[
J_0 \geq T_{\mu_0} J_0 \geq T_{\mu_0}^{m_0} J_0 = J_1 \geq T_{\mu_0}^{m_0+1} J_0 = T_{\mu_0} J_1 \geq T J_1 = T_{\mu_1} J_1 \geq \cdots \geq J_2;
\]

† As with all PI algorithms in this book, we assume that the policy improvement operation is well-defined, in the sense that there exists \( \mu_k \) such that \( T_{\mu_k} J_k = T J_k \) for all \( k \).
where the first, second, and third inequalities hold because the assumption $J_0 \geq TJ_0 = T_{\mu^0}J_0$ implies that $T^m_{\mu^0}J_0 \geq T^{m+1}_{\mu^0}J_0$ for all $m \geq 0$. Continuing similarly we obtain

$$J_k \geq TJ_k \geq J_{k+1}, \quad \forall k \geq 0.$$  \hfill (4.20)

Moreover, we can show by induction that $J_k \geq J^*$. Indeed this is true for $k = 0$ by assumption. If $J_k \geq J^*$, we have

$$J_{k+1} = T^{m_k}_{\mu^k}J_k \geq T^{m_k}J_k \geq T^{m_k}J^* = J^*,$$  \hfill (4.21)

where the last equality follows from the fact $TJ^* = J^*$ (cf. Prop. 4.3.6), thus completing the induction. Thus, by combining the preceding two relations, we have

$$J_k \geq TJ_k \geq J_{k+1} \geq J^*, \quad \forall k \geq 0.$$  \hfill (4.22)

We will now show by induction that

$$T^k J_0 \geq J_k \geq J^*, \quad \forall k \geq 0.$$  \hfill (4.23)

Indeed this relation holds by assumption for $k = 0$, and assuming that it holds for some $k \geq 0$, we have by applying $T$ to it and by using Eq. (4.22),

$$T^{k+1} J_0 \geq TJ_k \geq J_{k+1} \geq J^*,$$

thus completing the induction. By applying Prop. 4.3.13 to Eq. (4.23), we obtain $J_k \downarrow J^*$. Q.E.D.

**$\lambda$-Policy Iteration Under D**

We now consider an alternative optimistic PI algorithm, called $\lambda$-PI. It involves a scalar $\lambda \in (0, 1)$ and a corresponding multistep mapping, which bears a relation to temporal differences and the proximal algorithm (cf. Section 1.2.5). For undiscounted finite-state MDP, the algorithm can be implemented by using matrix inversion, just like nonoptimistic PI for discounted finite-state MDP. This can be an advantage in some settings, including approximate simulation-based implementations. The $\lambda$-PI algorithm is defined by

$$T^k J_k = TJ_k, \quad J_{k+1} = T^{(\lambda)} J_k,$$  \hfill (4.24)

where for any policy $\mu$ and scalar $\lambda \in (0, 1)$, $T^{(\lambda)}$ is the mapping defined by

$$T^{(\lambda)} J = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t T^{t+1}_\mu J.$$  \hfill (4.25)
Here we assume that \( T_\mu \) maps \( \mathcal{R}(X) \) to \( \mathcal{R}(X) \), and that for all \( \mu \in \mathcal{M} \) and \( J \in \mathcal{R}(X) \), the limit of the series above is well-defined as a function in \( \mathcal{R}(X) \).

We also assume a linearity property for \( T_\mu \), whereby we have

\[
T_\mu(T_\mu^{(\lambda)} J) = T_\mu^{(\lambda)} (T_\mu J), \quad \forall \mu \in \mathcal{M}, J \in \mathcal{R}(X).
\]

This assumption is commonly satisfied in DP problems where \( T_\mu \) is linear, such as the stochastic optimal control problem of Example 1.2.1.

To compare the \( \lambda \)-PI method with the exact PI algorithm, note that by the analysis of Section 1.2.5 (see also Exercise 1.2), the mapping \( T_\mu^{(\lambda)} \) is an extrapolated version of the proximal mapping for solving the fixed point equation \( J = T_\mu J \). Thus in \( \lambda \)-PI, the policy evaluation phase is done approximately with a single iteration of the (extrapolated) proximal algorithm.

To compare the \( \lambda \)-PI method with the optimistic PI method (4.19), we note that both mappings \( T_\mu^{(\mu_k)} \) and \( T_\mu^{(\lambda)} \) appearing in Eqs. (4.19) and (4.24), respectively, involve multiple applications of the VI mapping \( T_\mu \): a fixed number \( m_k \) in the former case, and a geometrically weighted infinite number in the latter case [cf. Eq. (4.25)]. Thus \( \lambda \)-PI and optimistic PI are similar: they just use the mapping \( T_\mu \) to apply VI in different ways.

Since \( \lambda \)-PI is a form of optimistic PI, it is not surprising that it has the same type of convergence properties as the earlier optimistic PI method (4.19). Similar to Prop. 4.3.15, we have the following.

**Proposition 4.3.16:** Let Assumption D hold and let \( \{J_k, \mu_k\} \) be a sequence generated by the \( \lambda \)-PI algorithm (4.24), assuming Eq. (4.26), and that \( \bar{J} \geq J_0 \geq J^* \) and \( J_0 \geq T J_0 \). Then \( J_k \downarrow J^* \).

**Proof:** As in the proof of Prop. 4.3.15, by using Assumption D, the monotonicity of \( T_\mu \), and the hypothesis \( J_0 \geq T J_0 \), we have

\[
J_0 \geq T J_0 = T_{\mu^0} J_0 \geq T^{(\lambda)}_{\mu^0} J_0 = J_1 \geq T_{\mu^0} J_1 \geq T_{\mu^1} J_1 \geq T^{(\lambda)}_{\mu^1} J_0 = J_2,
\]

where for the third inequality, we use the relation \( J_0 \geq T_{\mu^0} J_0 \), the definition of \( J_1 \), and the assumption (4.26). Continuing in the same manner,

\[
J_k \geq T J_k \geq J_{k+1}, \quad \forall k \geq 0.
\]

Similar to the proof of Prop. 4.3.15, we show by induction that \( J_k \geq J^* \), using the fact that if \( J_k \geq J^* \), then

\[
J_{k+1} = T^{(\lambda)}_{\mu_k} J_k \geq T^{(\lambda)}_{\mu_k} J^* = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t T^{t+1} J^* = J^*.
\]
[cf. the induction step of Eq. (4.21)]. By combining the preceding two
relations, we obtain Eq. (4.22), and the proof is completed by using the
argument following that equation. Q.E.D.

The $\lambda$-PI algorithm has a useful property, which involves the mapping
$W_k : \mathcal{R}(X) \mapsto \mathcal{R}(X)$ given by

$$W_k J = (1 - \lambda) T_{\mu^k} J_k + \lambda T_{\mu^k} J.$$  \hfill (4.27)

In particular $J_{k+1}$ is a fixed point of $W_k$. Indeed, using the definition
$J_{k+1} = T^{(\lambda)}_{\mu^k} J_k$ [cf. Eq. (4.24)], and the linearity assumption (4.26), we
have

$$W_k J_{k+1} = (1 - \lambda) T_{\mu^k} J_k + \lambda T_{\mu^k} \left( T^{(\lambda)}_{\mu^k} J_k \right)$$

$$= (1 - \lambda) T_{\mu^k} J_k + \lambda T_{\mu^k} \left( T^{(\lambda)}_{\mu^k} J_k \right)$$

$$= T_{\mu^k} J_k$$

$$= J_{k+1}.$$

Thus $J_{k+1}$ can be calculated as a fixed point of $W_k$.

Consider now the case where $T_{\mu^k}$ is nonexpansive with respect to
some norm. Then from Eq. (4.27), it is seen that $W_k$ is a contraction of
modulus $\lambda$ with respect to that norm, so $J_{k+1}$ is the unique fixed point of
$W_k$. Moreover, if the norm is a weighted sup-norm, $J_{k+1}$ can be found using
the methods of Chapter 2 for contractive models. The following example applies this idea to finite-state SSP. The interesting aspect of this example is
that it implements the policy evaluation portion of $\lambda$-PI through solution of
a system of linear equations, similar to the exact policy evaluation method
of classical PI.

**Example 4.3.4 (Stochastic Shortest Path Problems with Nonpositive Costs)**

Consider the SSP problem of Example 1.2.6 with states $1, \ldots, n$, plus the
termination state 0. For all $u \in U(x)$, the state following $x$ is $y$ with prob-
ability $p_{xy}(u)$ and the expected cost incurred is nonpositive. This problem
arises when we wish to maximize nonnegative rewards up to termination. It
includes a classical search problem where the aim, roughly speaking, is to
move through the state space looking for states with favorable termination
rewards.

We view the problem within our abstract framework with $\bar{J}(x) \equiv 0$ and

$$T_{\mu} J = g_{\mu} + P_{\mu} J,$$  \hfill (4.28)

with $g_{\mu} \in \mathbb{R}^n$ being the corresponding nonpositive one-stage cost vector, and
$P_{\mu}$ being an $n \times n$ substochastic matrix. The components of $P_{\mu}$ are the
probabilities $p_{xy}(\mu(x))$, $x, y = 1, \ldots, n$. Clearly Assumption D holds.
Consider the $\lambda$-PI method (4.24), with $J_{k+1}$ computed by solving the fixed point equation $J = W_k J$, cf. Eq. (4.27). This is a nonsingular $n$-dimensional system of linear equations, and can be solved by matrix inversion, just like in exact PI for discounted $n$-state MDP. In particular, using Eqs. (4.27) and (4.28), we have

$$J_{k+1} = (I - \lambda P_{\mu k})^{-1}(g_{\mu k} + (1 - \lambda)P_{\mu k} J_k).$$

(4.29)

For a small number of states $n$, this matrix inversion-based policy evaluation may be simpler than the optimistic PI policy evaluation equation

$$J_{k+1} = T_{\mu k}^n J_k$$

[cf. Eq. (4.19)], which points to an advantage of $\lambda$-PI.

Note that based on the relation between the multistep mapping $T_{\mu}^{(\lambda)}$ and the proximal mapping, discussed in Section 1.2.5 and Exercise 1.2, the policy evaluation Eq. (4.29) may be viewed as an extrapolated proximal iteration. Note also that as $\lambda \to 1$, the policy evaluation Eq. (4.29) resembles the policy evaluation equation

$$J_{\mu k} = (I - \lambda P_{\mu k})^{-1}g_{\mu k}$$

for $\lambda$-discounted $n$-state MDP. An important difference, however, is that for a discounted finite-state MDP, exact PI will find an optimal policy in a finite number of iterations, while this is not guaranteed for $\lambda$-PI. Indeed $\lambda$-PI does not require that there exists an optimal policy or even that $J^*(x)$ is finite for all $x$.

**Policy Iteration Under I**

Contrary to the case of Assumption D, the important cost improvement property of PI holds under Assumption I. Thus, if $\mu$ is a policy and $\bar{\mu}$ satisfies the policy improvement equation $T_{\bar{\mu}} J_{\mu} = T J_{\mu}$, we have

$$J_{\mu} = T_{\mu} J_{\mu} \geq T J_{\mu} = T_{\bar{\mu}} J_{\mu},$$

from which we obtain

$$J_{\mu} \geq \lim_{k \to \infty} T_{\bar{\mu}}^k J_{\mu}.$$

Since $J_{\mu} \geq \bar{J}$ and $J_{\bar{\mu}} = \lim_{k \to \infty} T_{\bar{\mu}}^k \bar{J}$, it follows that

$$J_{\mu} \geq T J_{\mu} \geq J_{\bar{\mu}}.$$

(4.30)

However, this cost improvement property is not by itself sufficient for the validity of PI under Assumption I (see the deterministic shortest path example of Section 3.1.1). Thus additional conditions are needed to guarantee convergence. To this end we may use the semicontractive framework of Chapter 3, and take advantage of the fact that under Assumption I, $J^*$ is known to be a fixed point of $T$. 
In particular, suppose that we have a set \( S \subset E(X) \) such that \( J^*_S = J^* \). Then \( J^*_S \) is a fixed point of \( T \) and the theory of Section 3.2 comes into play. Thus, by Prop. 3.2.1 the following hold:

(a) We have \( T^kJ \to J^* \) for every \( J \in \mathcal{E}(X) \) such that \( J^* \leq J \leq \bar{J} \) for some \( \bar{J} \in S \).

(b) \( J^* \) is the only fixed point of \( T \) within the set of all \( J \in \mathcal{E}(X) \) such that \( J^* \leq J \leq \tilde{J} \) for some \( \tilde{J} \in S \).

Moreover, by Prop. 3.2.4, if \( S \) has the weak PI property and for each sequence \( \{J_m\} \subset \mathcal{E}(X) \) with \( J_m \downarrow J \) for some \( J \in \mathcal{E}(X) \), we have

\[
H(x,u,J) = \lim_{m \to \infty} H(x,u,J_m),
\]

then every sequence of \( S \)-regular policies \( \{\mu^k\} \) that can be generated by PI satisfies \( J_{\mu^k} \downarrow J^* \). If in addition the set of \( S \)-regular policies is finite, there exists \( k \geq 0 \) such that \( \mu^k \) is optimal.

For these properties to hold, it is of course critical that \( J^*_S = J^* \). If this is not so, but \( J^*_S \) is still a fixed point of \( T \), the VI and PI algorithms may converge to \( J^*_S \) rather than to \( J^* \) (cf. the linear quadratic problem of Section 3.5.4).

### 4.4 Regularity and Nonstationary Policies

In this section, we will extend the notion of regularity of Section 3.2 so that it applies more broadly, including situations where nonstationary policies need to be considered. We will then discuss some applications involving mostly monotone increasing models. We continue to focus on the infinite horizon case of the problem of Section 4.1, but we do not impose for the moment any additional assumptions, such as Assumption I or D.

We begin with the following extension of the definition of \( S \)-regularity, which we will use to prove a general result regarding the convergence properties of VI in the following Prop. 4.4.1. We will apply this result in the context of various applications in Sections 4.4.2-4.4.4.

**Definition 4.4.1:** For a nonempty set of functions \( S \subset \mathcal{E}(X) \), we say that a nonempty collection \( \mathcal{C} \) of policy-state pairs \((\pi, x)\), with \( \pi \in \Pi \) and \( x \in X \), is \( S \)-regular if

\[
J_\pi(x) = \limsup_{k \to \infty} (T_{\mu_0} \cdots T_{\mu_k} J)(x), \quad \forall (\pi, x) \in \mathcal{C}, \ J \in S.
\]

Thus for an \( S \)-regular collection of pairs \((\pi, x)\), the value of \( J_\pi(x) \) is not affected if the starting function is changed from \( \bar{J} \) to any \( J \in S \).
Note that this definition of regularity bears similarity to the one we used in Chapter 3 for stationary policies. In particular, for a set of functions $S \subset \mathcal{E}(X)$, a stationary policy $\mu$ is $S$-regular (in the sense of Section 3.2) if and only if the collection

$$C = \{(\mu, x) \mid x \in X\}$$

is $S$-regular (in the sense of Def. 4.4.1), and in addition $J_\mu \in S$ and $J_\mu = T_\mu J_\mu$. Thus, we may view the definition of regularity of this section as a generalization to nonstationary policies of the definition of Chapter 3. Using nonstationary policies may be important because in noncontractive models it is not generally true that the optimal cost over stationary policies is the same as the optimal cost over nonstationary policies (cf. the $\epsilon$-optimality result of Prop. 4.3.2, and the example given following the proof of that proposition).

For a given set $C$ of policy-state pairs $(\pi, x)$, let us consider the function $J_C^* \in \mathcal{E}(X)$, given by

$$J_C^*(x) = \inf_{\pi \mid (\pi, x) \in C} J_\pi(x), \quad x \in X. \quad (4.31)$$

Note that $J_C^*(x) \geq J_\pi^*(x)$ for all $x \in X$ [for those $x \in X$ for which the set of policies $\{\pi \mid (\pi, x) \in C\}$ is empty, we have $J_C^*(x) = \infty$]. In Sections 3.2-3.4 we saw that when $J_S^*$ is a fixed point of $T$, then favorable analytical and computational results follow. Similarly, in this section we will see that for an $S$-regular collection $C$, when $J_C^*$ is a fixed point of $T$, favorable results follow. The following is a key proposition, which may be compared with Prop. 3.2.1.

**Proposition 4.4.1:** Given a nonempty set $S \subset \mathcal{E}(X)$, let $C$ be a nonempty collection of policy-state pairs $(\pi, x)$ that is $S$-regular. Then:

(a) For all $J \in S$, we have

$$\liminf_{k \to \infty} T^k J \leq \limsup_{k \to \infty} T^k J \leq J_C^*.$$

(b) For all $J' \in \mathcal{E}(X)$ with $J' \leq T J'$, and all $J \in \mathcal{E}(X)$ such that $J' \leq J \leq \tilde{J}$ for some $\tilde{J} \in S$, we have

$$J' \leq \liminf_{k \to \infty} T^k J \leq \limsup_{k \to \infty} T^k J \leq J_C^*.$$

**Proof:** (a) Using the generic relation $TJ \leq T_\mu J$, $\mu \in \mathcal{M}$, and the monotonicity of $T$ and $T_\mu$, we have for all $k$

$$(T^k J)(x) \leq (T_{\mu_0} \cdots T_{\mu_{k-1}} J)(x), \quad \forall (\pi, x) \in C, \; J \in S.$$
By letting $k \to \infty$ and by using the definition of $S$-regularity, it follows that for all $(\pi, x) \in C$, and $J \in S$,
\[
\liminf_{k \to \infty} (T^k J)(x) \leq \limsup_{k \to \infty} (T^k J)(x) \leq \limsup_{k \to \infty} (T_{\mu_0} \cdots T_{\mu_{k-1}} J)(x) = J_\pi(x),
\]
and by taking infimum of the right side over $\{\pi \mid (\pi, x) \in C\}$, we obtain the result.

(b) Using the hypotheses $J' \leq T J'$, and $J' \leq J \leq \tilde{J}$ for some $\tilde{J} \in S$, and the monotonicity of $T$, we have
\[
J'(x) \leq (T J')(x) \leq \cdots \leq (T^k J')(x) \leq (T^k J)(x) \leq (T^k \tilde{J})(x).
\]
Letting $k \to \infty$ and using part (a), we obtain the result. \textbf{Q.E.D.}

Let us discuss some interesting implications of part (b) of the proposition. Suppose we are given a set $S \subset E(X)$, and a set $C \subset \Pi \times X$ that is $S$-regular. Then:

1. $J^*_C$ is an upper bound to every fixed point $J'$ of $T$ that lies below some $\tilde{J} \in S$ (i.e., $J' \leq \tilde{J}$). Moreover, for such a fixed point $J'$, the convergence of VI is characterized by the valid start region
\[
\{J \in E(X) \mid J^*_C \leq J \leq \tilde{J} \text{ for some } \tilde{J} \in S\},
\]
and the limit region
\[
\{J \in E(X) \mid J' \leq J \leq J^*_C\}.
\]
The VI algorithm, starting from the former, ends up asymptotically within the latter; cf. Fig. 4.4.1.

2. If $J^*_C$ is a fixed point of $T$ (a common case in our subsequent analysis), then VI converges to $J^*_C$ starting from any $J$ in the set
\[
W_{S,C} = \{J \in E(X) \mid J^*_C \leq J \leq \tilde{J} \text{ for some } \tilde{J} \in S\},
\]
which we refer to as the well-behaved region (cf. the corresponding definition in Section 3.2). For future reference, we state this observation as a proposition, which should be compared to Prop. 3.2.1, the stationary special case where $C$ is defined by the set of $S$-regular stationary policies, i.e., $C = \{(\mu, x) \mid \mu \in \mathcal{M}_S, x \in X\}$.

\textbf{Proposition 4.4.2:} Given a set $S \subset E(X)$, let $C$ be a collection of policy-state pairs $(\pi, x)$ that is $S$-regular, and assume that $J^*_C$ is a fixed point of $T$. Then:

(a) $J^*_C$ is the only fixed point of $T$ within $W_{S,C}$.

(b) We have $T^k J \to J^*_C$ for all $J \in W_{S,C}$. 

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Figure 4.4.1. Illustration of Prop. 4.4.1. Neither $J_C^*$ nor $J^*$ need to be fixed points of $T$, but if $C$ is $S$-regular, and there exists $\tilde{J} \in S$ with $J_C^* \leq \tilde{J}$, then $J_C^*$ demarcates from above the range of fixed points of $T$ that lie below $\tilde{J}$.

Proof: Let $J \in \mathcal{E}(X)$ and $\tilde{J} \in S$ be such that $J_C^* \leq J \leq \tilde{J}$. Using the fixed point property of $J_C^*$ and the monotonicity of $T$, we have

$$J_C^* = T^kJ_C^* \leq T^kJ \leq T^k\tilde{J}, \quad k = 0, 1, \ldots.$$  

From Prop. 4.4.1(b), with $J' = J_C^*$, it follows that $T^k\tilde{J} \to J_C^*$, so taking limit in the above relation as $k \to \infty$, we obtain $T^kJ \to J_C^*$. Q.E.D.

Examples and counterexamples illustrating the preceding proposition are provided by the problems of Section 3.1 for the stationary case where $C = \{(\mu, x) \mid \mu \in \mathcal{M}_S, x \in X\}$. Similar to the analysis of Chapter 3, the preceding proposition takes special significance when $C$ is rich enough so that $J_C^* = J^*$, as for example in the case where $C$ is the set $\Pi \times X$ of all $(\pi, x)$, or other choices to be discussed later. It then follows that VI converges to $J^*$ starting from any $J \in \mathcal{E}(X)$ such that $J^* \leq J \leq \tilde{J}$ for some $\tilde{J} \in S$.

Note that Prop. 4.4.2 does not say anything about fixed points of $T$ that lie below $J_C^*$, and does not give conditions under which $J_C^*$ is a fixed point. In particular, it does not address the question whether $J^*$ is a fixed point of $T$, or whether VI converges to $J^*$ starting from $\tilde{J}$ or from below $J^*$. Generally, it can happen that both, only one, or none of the two functions $J_C^*$ and $J^*$ is a fixed point of $T$, as we have seen in the examples of Section 3.1.

The Case Where $J_C^* \leq \tilde{J}$

We have seen in Section 4.3 that the results for monotone increasing and monotone decreasing models are markedly different. In the context of $S$-regularity of a collection $C$, it turns out that there are analogous significant differences between the cases $J_C^* \geq \tilde{J}$ and $J_C^* \leq \tilde{J}$. The following proposition establishes some favorable aspects of the condition $J_C^* \leq \tilde{J}$ in the
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context of VI. These can be attributed to the fact that \( \bar{J} \) can always be added to \( S \) without affecting the \( S \)-regularity of \( C \), so \( \bar{J} \) can serve as the element \( \tilde{J} \) of \( S \) with \( J^* \leq \tilde{J} \) in Props. 4.4.1 and 4.4.2 (see the subsequent proof). The following proposition may also be compared with the result on convergence of VI under Assumption D (cf. Prop. 4.3.13).

**Proposition 4.4.3:** Given a set \( S \subset \mathcal{E}(X) \), let \( C \) be a collection of policy-state pairs \((\pi, x)\) that is \( S \)-regular, and assume that \( J^* \leq \bar{J} \). Then:

(a) For all \( J' \in \mathcal{E}(X) \) with \( J' \leq T^*J' \), we have

\[
J' \leq \liminf_{k \to \infty} T^k\bar{J} \leq \limsup_{k \to \infty} T^k\bar{J} \leq J^*_C.
\]

(b) If \( J^*_C \) is a fixed point of \( T \), then \( J^* = J^*_C \) and we have \( T^k\bar{J} \to J^* \) as well as \( T^k\bar{J} \to J^* \) for every \( J \in \mathcal{E}(X) \) such that \( J^* \leq J \leq \bar{J} \) for some \( \tilde{J} \in S \).

**Proof:** (a) If \( S \) does not contain \( \bar{J} \), we can replace \( S \) with \( \bar{S} = S \cup \{ \bar{J} \} \), and \( C \) will still be \( \bar{S} \)-regular. By applying Prop. 4.4.1(b) with \( \bar{S} \) replaced by \( \bar{S} \) and \( \tilde{J} = \bar{J} \), the result follows.

(b) Assume without loss of generality that \( \bar{J} \in S \) [cf. the proof of part (a)]. By using Prop. 4.4.2 with \( \tilde{J} = \bar{J} \), we have \( J^*_C = \lim_{k \to \infty} T^k\bar{J} \). This relation yields for any policy \( \pi = \{ \mu_0, \mu_1, \ldots \} \in \Pi \),

\[
J^*_C = \lim_{k \to \infty} T^k\bar{J} \leq \limsup_{k \to \infty} T^k_\mu_0 \cdots T^k_\mu_{k-1} \bar{J} = J^*_\pi,
\]

so by taking the infimum over \( \pi \in \Pi \), we obtain \( J^*_C \leq J^* \). Since generically we have \( J^*_C \geq J^* \), it follows that \( J^*_C = J^* \). Finally, from Prop. 4.4.2, we obtain \( T^k\bar{J} \to J^* \) for all \( J \in \mathcal{E}(X) \) such that \( J^* \leq J \leq \bar{J} \) for some \( \tilde{J} \in S \).

Q.E.D.

As a special case of the preceding proposition, we have that if \( J^* \leq \bar{J} \) and \( J^* \) is a fixed point of \( T \), then \( J^* = \lim_{k \to \infty} T^k\bar{J} \), and for every other fixed point \( J' \) of \( T \) we have \( J' \leq J^* \) (apply the proposition with \( C = \Pi \times X \) and \( S = \{ \bar{J} \} \), in which case \( J^*_C = J^* \leq \bar{J} \)). This is relevant, among others, to the monotone decreasing models, where \( T^k_\mu \bar{J} \leq \bar{J} \) for all \( \mu \in \mathcal{M} \). A special case is the convergence of VI under Assumption D (cf. Prop. 4.3.5). The proposition also applies to a classical type of search problem with both positive and negative costs per stage. This is the SSP problem, where at each \( x \in X \) we have cost \( E\{g(x, u, w)\} \geq 0 \) for all \( u \) except one that leads to a termination state with probability 1 and nonpositive cost; here we
have $\bar{J}(x) = 0$ and $J^*_c(x) \leq 0$ for all $x \in X$, but Assumption D need not hold.

### 4.4.1 Regularity and Monotone Increasing Models

We will now return to the monotone increasing model, cf. Assumption I. For this model, we know from Section 4.3 that $J^*$ is the smallest fixed point of $T$ within the class of functions $J \geq \bar{J}$, under certain relatively mild assumptions. However, VI may not converge to $J^*$ starting from below $J^*$ (e.g., starting from $\bar{J}$), and also starting from above $J^*$. In this section we will address the question of convergence of VI from above $J^*$ by using regularity ideas, and in Section 4.5 we will consider the characterization of the largest fixed point of $T$ in the context of deterministic optimal control and infinite-space shortest path problems. We summarize the results of Section 4.3 that are relevant to our development in the following proposition (cf., Props. 4.3.2, 4.3.3, 4.3.9, and 4.3.14).

**Proposition 4.4.4:** Let Assumption I hold. Then:

(a) $J^* = TJ^*$, and if $J' \in E(X)$ is such that $J' \geq \bar{J}$ and $J' \geq TJ'$, then $J' \geq J^*$.
(b) For all $\mu \in \mathcal{M}$ we have $J_\mu = T_\mu J_\mu$, and if $J' \in E(X)$ is such that $J' \geq \bar{J}$ and $J' \geq T_\mu J'$, then $J' \geq J_\mu$.
(c) $\mu^* \in \mathcal{M}$ is optimal if and only if $T_{\mu^*} J^* = TJ^*$.
(d) If $U$ is a metric space and the sets

$$U_k(x, \lambda) = \{u \in U(x) \mid H(x, u, T_k \bar{J}) \leq \lambda\} \quad (4.33)$$

are compact for all $x \in X$, $\lambda \in \mathbb{R}$, and $k$, then there exists at least one optimal stationary policy, and we have $T_k J \rightarrow J^*$ for all $J \in E(X)$ with $J \leq J^*$.
(e) Given any $\epsilon > 0$, there exists a policy $\pi_\epsilon \in \Pi$ such that

$$J^* \leq J_{\pi_\epsilon} \leq J^* + \epsilon.$$ 

Furthermore, if the scalar $\alpha$ in part (c) of Assumption I satisfies $\alpha < 1$, the policy $\pi_\epsilon$ can be taken to be stationary.

We have noted that under Assumption I there may exist fixed points $J'$ of $T$ with $J^* \leq J'$, implying that VI may not converge to $J^*$ starting from above $J^*$. However, convergence of VI to $J^*$ from above, if it occurs, is often much faster than convergence from below, so starting points $J \geq$
$J^*$ may be desirable. One well-known such case is deterministic finite-state shortest path problems where major algorithms, such as the Bellman-Ford method or other label correcting methods have polynomial complexity, when started from $J$ above $J^*$, but only pseudopolynomial complexity when started from $J = 0$.

In the next two subsections, we will consider discounted and undiscounted optimal control problems with nonnegative cost per stage, and we will establish conditions that guarantee the uniqueness of $J^*$ as a fixed point of $T$, and the convergence of VI. Our analysis will proceed as follows:

(a) Define a collection $C$ such that $J_C^* = J^*$.

(b) Define a set $S \subseteq E^+(X)$ such that $J^* \in S$ and $C$ is $S$-regular.

(c) Use Prop. 4.4.2 (which shows that $J_C^*$ is the largest fixed point of $T$ within $S$) in conjunction with Prop. 4.4.4(a) (which shows that $J^*$ is the smallest fixed point of $T$ within $S$) to show that $J^*$ is the unique fixed point of $T$ within $S$. Use also Prop. 4.4.2 to show that the VI algorithm converges to $J^*$ starting from $J \in S$ such that $J \geq J^*$.

(d) Use the compactness condition of Prop. 4.4.4(d), to enlarge the set of functions starting from which VI converges to $J^*$.

### 4.4.2 Nonnegative Cost Stochastic Optimal Control

Let us consider the undiscounted stochastic optimal control problem that involves the mapping

$$H(x, u, J) = \mathbb{E}\{g(x, u, w) + J(f(x, u, w))\}, \quad (4.34)$$

where $g$ is the one-stage cost function and $f$ is the system function. The expected value is taken with respect to the distribution of the random variable $w$ (which takes values in a countable set $W$). We assume that

$$0 \leq g(x, u, w) \leq \infty, \quad \forall \ x \in X, \ u \in U(x), \ w \in W.$$ 

We consider the abstract DP model with $H$ as above, and with $\bar{J}(x) \equiv 0$. We will apply the analysis of this section with

$$C = \{(\pi, x) \ | \ J_\pi(x) < \infty\}, \quad (4.35)$$

for which $J_C^* = J^*$. We assume that $C$ is nonempty, which is true if and only if $J^*$ is not identically $\infty$, i.e., $J^*(x) < \infty$ for some $x \in X$.

Let us denote by $E_{\pi x_0}\{\cdot\}$ the expected value with respect to the probability distribution induced by $\pi \in \Pi$ under initial state $x_0$, and consider the set

$$S = \{J \in E^+(X) \mid E_{\pi x_0}^+\{J(x_k)\} \to 0, \ \forall \ (\pi, x_0) \in C\}. \quad (4.36)$$
We will show that \( J^* \in S \) and that \( C \) is \( S \)-regular. Once this is done, it will follow from Prop. 4.4.2 and the fixed point property of \( J^* \) (cf. Prop. 4.4.4) that \( T^kJ \to J^* \) for all \( J \in S \) that satisfy \( J \geq J^* \). If the sets \( U_k(x, \lambda) \) of Eq. (4.33) are compact, the convergence of VI starting from below \( J^* \) will also be guaranteed. We have the following proposition.

**Proposition 4.4.5: (Convergence of VI)** Consider the problem corresponding to the mapping (4.34) with \( g \geq 0 \), and assume that \( J^* \) is not identically \( \infty \). Then:

(a) \( J^* \) is the unique fixed point of \( T \) within \( S \), and we have \( T^kJ \to J^* \) for all \( J \geq J^* \) with \( J \in S \).

(b) If \( U \) is a metric space, and the sets \( U_k(x, \lambda) \) of Eq. (4.33) are compact for all \( x \in X \), \( \lambda \in \mathbb{R} \), and \( k \), we have \( T^kJ \to J^* \) for all \( J \in S \), and an optimal stationary policy is guaranteed to exist.

**Proof:** (a) We have for all \( J \in \mathcal{E}(X) \), \((\pi, x_0) \in C\), and \( k \),

\[
(T_{\mu_0} \cdots T_{\mu_{k-1}}J)(x_0) = E_{x_0}^{\pi} \left\{ J(x_k) \right\} + E_{x_0}^{\pi} \left\{ \sum_{m=0}^{k-1} g(x_m, \mu_m(x_m), w_m) \right\},
\]

where \( \mu_m, m = 0, 1, \ldots \), denote generically the components of \( \pi \). The rightmost term above converges to \( J_{\pi}(x_0) \) as \( k \to \infty \), so by taking upper limit, we obtain

\[
\limsup_{k \to \infty} (T_{\mu_0} \cdots T_{\mu_{k-1}}J)(x_0) = \limsup_{k \to \infty} E_{x_0}^{\pi} \left\{ J(x_k) \right\} + J_{\pi}(x_0).
\]

Thus in view of the definition of \( S \), we see that for all \((\pi, x_0) \in C\) and \( J \in S \), we have

\[
\limsup_{k \to \infty} (T_{\mu_0} \cdots T_{\mu_{k-1}}J)(x_0) = J_{\pi}(x_0),
\]

so \( C \) is \( S \)-regular.

We next show that \( J^* \in S \). Given a policy \( \pi = \{\mu_0, \mu_1, \ldots\} \), we denote by \( \pi_k \) the policy

\[
\pi_k = \{\mu_k, \mu_{k+1}, \ldots\}.
\]

We have for all \((\pi, x_0) \in C\)

\[
J_\pi(x_0) = E_{x_0}^{\pi} \left\{ g(x_0, \mu_0(x_0), w_0) \right\} + E_{x_0}^{\pi} \left\{ J_{\pi_1}(x_1) \right\}.
\]
and more generally,
\[
E_\pi x_0 \{ J_{\pi_m}(x_m) \} = E_\pi x_0 \{ g(x_m, \mu_m(x_m), w_m) \} + E_\pi x_0 \{ J_{\pi_{m+1}}(x_{m+1}) \},
\]
for all \( m = 0, 1, \ldots \), where \( \{ x_m \} \) is the sequence generated starting from \( x_0 \) and using \( \pi \). Using the defining property \( J_\pi(x_0) < \infty \) of \( C \), it follows that all the terms in the above relations are finite, and in particular
\[
E_\pi x_0 \{ J_{\pi_m}(x_m) \} < \infty, \quad \forall (\pi, x_0) \in C, \ m = 0, 1, \ldots.
\]

By adding Eq. (4.38) for \( m = 0, \ldots, k - 1 \), and canceling the finite terms \( E_\pi x_0 \{ J_{\pi_m}(x_m) \} \) for \( m = 1, \ldots, k - 1 \), we obtain for all \( k = 1, 2, \ldots \),
\[
J_\pi(x_0) = E_\pi x_0 \{ J_{\pi_k}(x_k) \} + \sum_{m=0}^{k-1} E_\pi x_0 \{ g(x_m, \mu_m(x_m), w_m) \}, \quad \forall (\pi, x_0) \in C.
\]
The rightmost term above tends to \( J_\pi(x_0) \) as \( k \to \infty \), so we obtain
\[
E_\pi x_0 \{ J_{\pi_k}(x_k) \} \to 0, \quad \forall (\pi, x_0) \in C.
\]
Since \( 0 \leq J^* \leq J_{\pi_k} \), it follows that
\[
E_\pi x_0 \{ J^*(x_k) \} \to 0, \quad \forall x_0 \text{ with } J^*(x_0) < \infty.
\]
Thus \( J^* \in S \), while by Prop. 4.1.1, \( J^* \) (which is equal to \( J_\pi^* \)) is a fixed point of \( T \). Hence, by Prop. 4.4.2, \( J^* \) is the unique fixed point of \( T \) within the set \( \{ J \in S \mid J \geq J^* \} \). Similarly, by Prop. 4.4.2, we have \( T^k J \to J^* \) for all \( J \in S \).

(b) This part follows from part (a) and Prop. 4.4.4(d). \textbf{Q.E.D.}

A consequence of the preceding proposition is the following condition for VI convergence from above, first proved in the paper [YuB15], which also addressed universal measurability issues.

**Proposition 4.4.6:** Under the conditions of Prop. 4.4.5, if a function \( J \in \mathcal{F}^+(X) \) satisfies
\[
J^* \leq J \leq h(J^*), \quad (4.39)
\]
for a function \( h : \mathcal{F}^+(X) \to \mathcal{F}^+(X) \) such that \( \lim_{J \to 0} h(J) = 0 \), we have \( T^k J \to J^* \).

**Proof:** Since \( J^* \in S \) as shown in Prop. 4.4.5, any \( J \) satisfying Eq. (4.39), also belongs to the set \( S \) of Eq. (4.36), and the result follows from Prop. 4.4.5. \textbf{Q.E.D.}
An example of a function $h$ satisfying Eq. (4.39) is $h(J) = cJ$, where $c > 1$. Note that under the conditions of Prop. 4.4.5, either $T$ has a unique fixed point within $E^+(X)$ (namely $J^*$), or else all the additional fixed points of $T$ lie outside $S$. To illustrate the limitations of this result, consider the shortest path problem of Section 3.1.1 for the case where the choice at state 1 is either to stay at 1 at cost 0, or move to the destination at cost $b > 0$. Then Bellman’s equation at state 1 is $J(1) = \min \{b, J(1)\}$, and its set of solutions is the interval $[0, b]$, while we have $J^* = 0$. The set $S$ of Eq. (4.36) here consists of just $J^*$, Prop. 4.4.5 holds, but is not very useful. Similarly, in the linear-quadratic example of Section 3.1.4, where $T$ has the two fixed points $J^*(x) = 0$ and $\tilde{J}(x) = (\gamma^2 - 1)x^2$, the set $S$ of Eq. (4.36) consists of just $J^*$. Thus the regularity framework of this section cannot be used to differentiate between multiple fixed points of $T$, and to explain the unusual behavior in the preceding examples. In Section 4.5, we address this issue within the more restricted context of deterministic optimal control.

### 4.4.3 Discounted Stochastic Optimal Control

We will now consider a discounted version of the stochastic optimal control problem of the preceding section. For a policy $\pi = \{\mu_0, \mu_1, \ldots\}$ we have

$$J_\pi(x_0) = \lim_{k \to \infty} E^\pi_{x_0} \left\{ \sum_{m=0}^{k-1} \alpha^m g(x_m, \mu_m(x_m), w_m) \right\},$$

where $\alpha \in (0, 1)$ is the discount factor, and as earlier $E^\pi_{x_0}\{\cdot\}$ denotes expected value with respect to the probability measure induced by $\pi \in \Pi$ under initial state $x_0$. We can view this problem within the abstract DP framework by defining the mapping $H$ as

$$H(x, u, J) = E\{g(x, u, w) + \alpha J(f(x, u, w))\},$$

and $\tilde{J}(x) \equiv 0$. We assume that the one-stage cost is nonnegative,

$$0 \leq g(x, u, w) \leq \infty, \quad \forall \ x \in X, \ u \in U(x), \ w \in W.$$  

We also assume that $X$ is a normed space. Note that because of the discount factor, the existence of a terminal set of states is not essential for the optimal costs to be finite. Moreover, the nonnegativity of $g$ is not essential for our analysis. Any problem where $g$ can take both positive and negative values, but is bounded below, can be converted to an equivalent problem where $g$ is nonnegative, by adding a suitable constant $c$ to $g$. Then the cost of all policies will simply change by the constant

$$\sum_{k=0}^{\infty} \alpha^k c = \frac{c}{1 - \alpha}.$$
Sec. 4.4 Regularity and Nonstationary Policies

The line of analysis of this section makes a connection between the regularity concepts of this section and a notion of stability that is common in feedback control theory. This connection will be explored further in Section 4.5. We introduce the set

\[ X_f = \{ x \in X \mid J^*(x) < \infty \}, \]

which we assume to be nonempty. Given a state \( x \in X_f \), we say that a policy \( \pi \) is stable from \( x \) if there exists a bounded subset of \( X_f \) (that depends on \((\pi, x)\)) such that the (random) sequence \( \{x_k\} \) generated starting from \( x \) and using \( \pi \) lies with probability 1 within that subset. We consider the set of policy-state pairs

\[ C = \{ (\pi, x) \mid x \in X_f, \pi \text{ is stable from } x \}, \]

and we assume that \( C \) is nonempty.

Let us say that a function \( J \in \mathcal{E}(X) \) is bounded on bounded subsets of \( X_f \) if for every bounded subset \( \tilde{X} \subset X_f \) there is a scalar \( b \) such that \( J_x \leq b \) for all \( x \in \tilde{X} \). Let us also introduce the set

\[ S = \{ J \in \mathcal{E}(X) \mid J \text{ is bounded on bounded subsets of } X_f \}. \]

We assume that \( C \) is nonempty, \( J^* \in S \), and for every \( x \in X_f \) and \( \epsilon > 0 \), there exists a policy \( \pi \) that is stable from \( x \) and satisfies \( J_\pi(x) \leq J^*(x) + \epsilon \). We have the following proposition.

**Proposition 4.4.7:** Under the preceding assumptions, \( J^* \) is the unique fixed point of \( T \) within \( S \), and we have \( T^k J \to J^* \) for all \( J \in S \) with \( J^* \leq J \). If in addition \( U \) is a metric space, and the sets \( U_k(x, \lambda) \) of Eq. (4.33) are compact for all \( x \in X, \lambda \in \mathbb{R} \), and \( k \), we have \( T^k J \to J^* \) for all \( J \in S \), and an optimal stationary policy is guaranteed to exist.

**Proof:** We have for all \( J \in \mathcal{E}(X), (\pi, x_0) \in C \), and \( k \),

\[
(T_{\mu_0} \cdots T_{\mu_{k-1}} J)(x_0) = \alpha^k E_{x_0}^\pi \left\{ J(x_k) \right\} + E_{x_0}^\pi \left\{ \sum_{m=0}^{k-1} \alpha^m g(x_m, \mu_m(x_m), w_m) \right\}
\]

[cf. Eq. (4.37)]. The fact \((\pi, x_0) \in C\) implies that there is a bounded subset of \( X_f \) such that \( \{x_k\} \) belongs to that subset with probability 1, so if \( J \in S \) it follows that \( \alpha^k E_{x_0}^\pi \{J(x_k)\} \to 0 \). Thus by taking for all \((\pi, x_0) \in C\) and \( J \in S \),

\[
\lim_{k \to \infty} (T_{\mu_0} \cdots T_{\mu_{k-1}} J)(x_0) = \lim_{k \to \infty} E_{x_0}^\pi \left\{ \sum_{m=0}^{k-1} \alpha^m g(x_m, \mu_m(x_m), w_m) \right\} = J_\pi(x_0),
\]
so $C$ is $S$-regular. Since $J^*_c$ is equal to $J^*$ which is a fixed point of $T$ (by Prop. 4.4.4), it follows from Prop. 4.4.2 that $T^kJ \to J^*$ for all $J \in S$. The last conclusion follows from Prop. 4.4.4. Q.E.D.

### 4.4.4 Convergent Models

In this section we consider a case of an abstract DP model that generalizes both the monotone increasing and the monotone decreasing models. The model is patterned after the stochastic optimal control problem of Example 1.2.1, where the cost per stage function $g$ can take negative as well as positive values. Our main assumption is that the cost functions of all policies are defined as limits (rather than upper limits), and that $-\infty < \bar{J}(x) \leq J^*(x)$, $\forall \ x \in X$.

The preceding condition is somewhat restrictive and makes the model more similar to the monotone increasing than to the monotone decreasing model, but is essential for the results of this section to hold (for a discussion of the pathological behaviors that can occur without the condition $J^* \geq \bar{J}$, see the paper [Yu15]). We will show that $J^*$ is a fixed point of $T$, and that there exists an $\epsilon$-optimal policy for every $\epsilon > 0$. This will bring to bear the regularity ideas and results of Prop. 4.4.2, and will provide a convergence result for the VI algorithm.

In particular, we denote $E_b(X) = \{ J \in E(X) \mid J(x) > -\infty, \forall \ x \in X \}$, and we will assume the following.

**Assumption 4.4.1:**

(a) For all $\pi = \{\mu_0, \mu_1, \ldots\} \in \Pi$, $J_\pi$ can be defined as a limit:

$$J_\pi(x) = \lim_{k \to \infty} (T_{\mu_1} \cdots T_{\mu_k} \bar{J})(x), \quad \forall \ x \in X. \quad (4.40)$$

Furthermore, we have $\bar{J} \in E_b(X)$ and $\bar{J} \leq J^*$.

(b) For each sequence $\{J_m\} \subset E_b(X)$ with $J_m \to J \in E_b(X)$, we have

$$\lim_{m \to \infty} H(x, u, J_m) = H(x, u, J), \quad \forall \ x \in X, \ u \in U(x).$$

(c) There exists $\alpha > 0$ such that for all $J \in E_b(X)$ and $r \in \mathbb{R}$,

$$H(x, u, J + re) \leq H(x, u, J) + \alpha r, \quad \forall \ x \in X, \ u \in U(x),$$

where $e$ is the unit function, $e(x) \equiv 1$. 
For an example of a type of problem where the convergence condition (4.40) is satisfied, consider the stochastic optimal control problem of Example 1.2.1, assuming that the state space consists of two regions: $X_1$ where the cost per stage is nonnegative under all controls, and $X_2$ where the cost per stage is nonpositive. Assuming that once the system enters $X_1$ it can never return to $X_2$, the convergence condition (4.40) is satisfied for all $\pi$. The same is true for the reverse situation, where once the system enters $X_2$ it can never return to $X_1$. Optimal stopping problems and stochastic shortest path problems are often of this type.

We first prove the existence of $\epsilon$-optimal policies and then use it to establish that $J^*$ is a fixed point of $T$. The proofs are patterned after the ones under Assumption I (cf. Props. 4.3.2 and 4.3.3).

**Proposition 4.4.8:** Let Assumption 4.4.1 hold. Given any $\epsilon > 0$, there exists a policy $\pi_{\epsilon} \in \Pi$ such that

$$J^* \leq J_{\pi_{\epsilon}} \leq J^* + \epsilon \alpha.$$

**Proof:** Let $\{\epsilon_k\}$ be a sequence such that $\epsilon_k > 0$ for all $k$ and

$$\sum_{k=0}^{\infty} \alpha^k \epsilon_k = \epsilon,$$

(4.41)

where $\alpha$ is the scalar of Assumption 4.4.1(c). For each $x \in X$, consider a sequence of policies $\{\pi_k[x]\} \subset \Pi$, with components of $\pi_k[x]$ (to emphasize their dependence on $x$) denoted by $\mu_k^m[x]$, $m = 0, 1, \ldots$,

$$\pi_k[x] = \{\mu_k^m[x], \mu_k^1[x], \ldots\},$$

(4.42)

such that for $k = 0, 1, \ldots$,

$$J_{\pi_k[x]}(x) \leq J^*(x) + \epsilon_k.$$

(4.43)

Such a sequence exists since $J^* \in E_0(X)$.

Consider the functions $\overline{\pi}_k$ defined by

$$\overline{\pi}_k(x) = \mu_k^0[x](x), \quad \forall x \in X,$$

(4.44)

and the functions $\overline{J}_k$ defined by

$$\overline{J}_k(x) = H\left(x, \overline{\pi}_k(x), \lim_{m \to \infty} T_{\mu_k^m[x]} \cdots T_{\mu_k^0[x]} \overline{J}\right), \quad \forall x \in X, \ k = 0, 1, \ldots.$$  

(4.45)
By using Eqs. (4.43)-(4.45), and the continuity property of Assumption 4.4.1(b), we obtain for all \( x \in X \) and \( k = 0, 1, \ldots \)

\[
\tilde{J}_k(x) = H \left( x, \mu^*_0(x), \lim_{m \to \infty} T_{\mu^*_m}[x] \tilde{J} \right) \\
= \lim_{m \to \infty} H \left( x, \mu^*_0(x), T_{\mu^*_m}[x] \tilde{J} \right) \\
= \lim_{m \to \infty} \left( T_{\mu^*_0}[x] \cdots T_{\mu^*_m}[x] \tilde{J} \right)(x) \\
= J_{\pi_k[x]}(x) \\
\leq J^*(x) + \epsilon_k.
\]

From Eqs. (4.45), (4.46), and Assumption 4.4.1(c), we have for all \( x \in X \) and \( k = 1, 2, \ldots \),

\[
(T_{\pi_{k-1}} \tilde{J}_k)(x) = H \left( x, \pi_{k-1}(x), \tilde{J}_k \right) \\
\leq H \left( x, \pi_{k-1}(x), J^* + \epsilon_k e \right) \\
\leq H \left( x, \pi_{k-1}(x), J^* + \alpha e \right) \\
\leq H \left( x, \pi_{k-1}(x), \lim_{m \to \infty} T_{\mu^*_m}[x] \cdots T_{\mu^*_{k-1}}[x] \tilde{J} \right) + \alpha e_k \\
= \tilde{J}_{k-1}(x) + \alpha e_k,
\]

and finally

\[
T_{\pi_{k-1}} \tilde{J}_k \leq \tilde{J}_{k-1} + \alpha e_k e, \quad k = 1, 2, \ldots
\]

Using this inequality and Assumption 4.4.1(c), we obtain

\[
T_{\pi_{k-2}} T_{\pi_{k-1}} \tilde{J}_k \leq T_{\pi_{k-2}} (\tilde{J}_{k-1} + \alpha e_k e) \\
\leq T_{\pi_{k-2}} \tilde{J}_{k-1} + \alpha^2 e_k e \\
\leq \tilde{J}_{k-2} + (\alpha e_{k-1} + \alpha^2 e_k) e.
\]

Continuing in the same manner, we have for \( k = 1, 2, \ldots \),

\[
T_{\pi_0} \cdots T_{\pi_{k-1}} \tilde{J}_k \leq \tilde{J}_0 + (\alpha e_1 + \cdots + \alpha^k e_k) e \leq J^* + \left( \sum_{i=0}^{k} \alpha^i e_i \right) e.
\]

Since by Assumption 4.4.1(c), we have \( \tilde{J} \leq J^* \leq \tilde{J}_k \), it follows that

\[
T_{\pi_0} \cdots T_{\pi_{k-1}} \tilde{J} \leq J^* + \left( \sum_{i=0}^{k} \alpha^i e_i \right) e.
\]

Denote \( \pi_e = \{\pi_0, \pi_1, \ldots\} \). Then by taking the limit in the preceding inequality and using Eq. (4.41), we obtain

\[
J_{\pi_e} \leq J^* + \epsilon e.
\]
Q.E.D.

By using Prop. 4.4.8 we can prove the following.

**Proposition 4.4.9:** Let Assumption 4.4.1 hold. Then $J^*$ is a fixed point of $T$.

**Proof:** For every $\pi = \{\mu_0, \mu_1, \ldots\} \in \Pi$ and $x \in X$, we have using the continuity property of Assumption 4.4.1(b) and the monotonicity of $H$,

$$J_{\pi}(x) = \lim_{k \to \infty} (T_{\mu_0} T_{\mu_1} \cdots T_{\mu_k} \bar{J})(x)$$

$$= T_{\mu_0} \left( \lim_{k \to \infty} T_{\mu_1} \cdots T_{\mu_k} \bar{J} \right)(x)$$

$$\geq (T_{\mu_0} J^*)(x)$$

$$\geq (T J^*)(x).$$

By taking the infimum of the left-hand side over $\pi \in \Pi$, we obtain

$$J^* \geq T J^*.$$

To prove the reverse inequality, let $\epsilon_1$ and $\epsilon_2$ be any positive scalars, and let $\pi = \{\mu_0, \mu_1, \ldots\}$ be such that

$$T_{\mu_0} J^* \leq T J^* + \epsilon_1 e, \quad J_{\pi_1} \leq J^* + \epsilon_2 e,$$

where $\pi_1 = \{\mu_1, \mu_2, \ldots\}$ (such a policy exists by Prop. 4.4.8). By using the preceding relations and Assumption 4.4.1(c), we have

$$J^* \leq J_{\pi}$$

$$= \lim_{k \to \infty} T_{\pi_0} T_{\pi_1} \cdots T_{\pi_k} \bar{J}$$

$$= T_{\pi_0} \left( \lim_{k \to \infty} T_{\pi_1} \cdots T_{\pi_k} \bar{J} \right)$$

$$= T_{\pi_0} J_{\pi_1}$$

$$\leq T_{\pi_0} (J^* + \epsilon_2 e)$$

$$\leq T_{\pi_0} J^* + \alpha \epsilon_2 e$$

$$\leq T J^* + (\epsilon_1 + \alpha \epsilon_2) e.$$  

Since $\epsilon_1$ and $\epsilon_2$ can be taken arbitrarily small, it follows that

$$J^* \leq T J^*.$$  

Hence $J^* = T J^*$.

Q.E.D.
It is known that $J^*$ may not be a fixed point of $T$ if the convergence condition (a) of Assumption 4.4.1 is violated (see the example of Section 3.1.2). Moreover, $J^*$ may not be a fixed point of $T$ if either part (b) or part (c) of Assumption 4.4.1 is violated, even when the monotone increase condition $\bar{J} \leq TJ$ [and hence also the convergence condition of part (a)] is satisfied (see Examples 4.3.1 and 4.3.2). By applying Prop. 4.4.2, we have the following proposition, which supplements the results given in Section 3.2 for semicontractive models.

**Proposition 4.4.10:** Let Assumption 4.4.1 hold, and let $\mathcal{C}$ be a set of policy-state pairs such that $J^*_C = J^*$, and let $S$ be any subset of $\mathcal{E}(X)$ such that $\mathcal{C}$ is $S$-regular. Then:

(a) $J^*$ is the unique fixed point of $T$ within the set \{$J \in S \mid J \geq J^*$\}.

(b) We have $T^kJ \to J^*$ for every $J \in \mathcal{E}(X)$ such that $J^* \leq J \leq J'$ for some $J' \in S$.

**Proof:** By Prop. 4.4.9, $J^*$ is a fixed point of $T$. The result follows from Prop. 4.4.2. Q.E.D.

### 4.5 STABLE POLICIES AND DETERMINISTIC OPTIMAL CONTROL

In this section, we will consider the use of the regularity ideas of the preceding section in conjunction with a particularly favorable class of monotone increasing models. These are the discrete-time infinite horizon deterministic optimal control problems with nonnegative cost, and a destination that is cost-free and absorbing. The classical linear-quadratic regulator problem of Section 3.5.4 is a special case. Our assumptions are very general, and allow the possibility that the optimal policy may not be stabilizing the system, e.g., may not reach the destination either asymptotically or in a finite number of steps. This situation is illustrated by the one-dimensional linear-quadratic example of Section 3.1.4, where we saw that the Riccati equation may have multiple nonnegative solutions, with the largest solution corresponding to the restricted optimal cost over just the stable policies.

Our approach is similar to the one of the preceding section, and involves classes of regular policy-state pairs. We use forcing functions $p$ and a perturbation line of analysis like the one of Section 3.4. We introduce a new unifying notion of $p$-stability, which in addition to implying convergence of the generated states to the destination, quantifies the speed of convergence. We consider the properties of several distinct cost functions: $J^*$, the overall optimal, and $\hat{J}_p$, the restricted optimal over just the $p$-stable
Sec. 4.5 Stable Policies and Deterministic Optimal Control

System

\[ u_k = \mu_k(x_k) \]

\[ x_{k+1} = f(x_k, u_k) \]

Cost: \( g(x_k, u_k) \geq 0 \)

“Destination” \( t \)
(cost-free and absorbing)

Figure 4.5.1 A deterministic optimal control problem with nonnegative cost per stage, and a cost-free and absorbing destination \( t \).

policies. Different choices of \( p \) may yield different classes of \( p \)-stable policies, with different speeds of convergence. We will show that for any \( p \) and associated class of \( p \)-stable policies, \( \hat{J}_p \) is a solution of Bellman’s equation, and we will characterize the smallest and the largest solutions: they are \( J^- \) and \( J^+ \), the restricted optimal cost function over the class of (finitely) terminating policies. We will also discuss various modified versions of the VI and PI algorithms, as substitutes for the standard algorithms, which may not work in general.

Consider a deterministic discrete-time infinite horizon optimal control problem involving the system

\[ x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \ldots, \quad (4.47) \]

where \( x_k \) and \( u_k \) are the state and control at stage \( k \), which belong to sets \( X \) and \( U \), referred to as the state and control spaces, respectively, and \( f : X \times U \mapsto X \) is a given function. The control \( u_k \) must be chosen from a constraint set \( U(x_k) \subset U \) that may depend on the current state \( x_k \). The cost for the \( k \)th stage, \( g(x_k, u_k) \), is assumed nonnegative and possibly extended real-valued:

\[ 0 \leq g(x_k, u_k) \leq \infty, \quad \forall \ x_k \in X, \ u_k \in U(x_k), \ k = 0, 1, \ldots \quad (4.48) \]

We assume that \( X \) contains a special state, denoted \( t \), which is referred to as the destination, and is cost-free and absorbing:

\[ f(t, u) = t, \quad g(t, u) = 0, \quad \forall \ u \in U(t). \quad (4.49) \]

Except for the cost nonnegativity assumption (4.48), this problem is similar to the one of Section 3.5.5; see Fig. 4.5.1. It arises in many classical control applications involving regulation around a set point, and in infinite-state versions of shortest path applications; see Fig. 4.5.1.

As earlier, we denote policies by \( \pi \) and stationary policies by \( \mu \). Given an initial state \( x_0 \), a policy \( \pi = \{\mu_0, \mu_1, \ldots\} \) when applied to the system
(4.47), generates a unique sequence of state-control pairs \((x_k, \mu_k(x_k))\), \(k = 0, 1, \ldots\). The cost of \(\pi\) starting from \(x\) is

\[
J_{\pi}(x_0) = \sum_{k=0}^{\infty} g(x_k, \mu_k(x_k)), \quad x_0 \in X,
\]

[the series converges to some number in \([0, \infty]\) thanks to the nonnegativity assumption (4.48)]. The optimal cost function over the set of all policies \(\Pi\) is

\[
J^*(x) = \inf_{\pi \in \Pi} J_{\pi}(x), \quad x \in X.
\]

We denote by \(\mathcal{E}^+(X)\) the set of functions \(J : X \mapsto [0, \infty]\). In our analysis, we will use the set of functions

\[
\mathcal{J} = \{ J \in \mathcal{E}^+(X) \mid J(t) = 0 \}.
\]

Since \(t\) is cost-free and absorbing, this set contains the cost function \(J_\pi\) of every \(\pi \in \Pi\), as well as \(J^*\).

Under the cost nonnegativity assumption (4.48), the problem can be cast as a special case of the monotone increasing model with

\[
H(x, u, J) = g(x, u) + J(f(x, u)),
\]

and the initial function \(\bar{J}\) being identically zero. Thus Prop. 4.4.4 applies and in particular \(J^*\) satisfies Bellman’s equation:

\[
J^*(x) = \inf_{u \in U(x)} \left\{ g(x, u) + J^*(f(x, u)) \right\}, \quad x \in X.
\]

Moreover, an optimal stationary policy (if it exists) may be obtained through the minimization in the right side of this equation, cf. Prop. 4.4.4(c).

The VI method starts from some function \(J_0 \in \mathcal{J}\), and generates a sequence of functions \(\{J_k\} \subset \mathcal{J}\) according to

\[
J_{k+1} = \inf_{u \in U(x)} \left\{ g(x, u) + J_k(f(x, u)) \right\}, \quad x \in X, \quad k = 0, 1, \ldots
\]

By specializing Prop. 4.4.6, we have that the VI sequence \(\{J_k\}\) converges to \(J^*\) starting from any function \(J_0 \in \mathcal{E}^+(X)\) that satisfies

\[
J^* \leq J_0 \leq h(J^*),
\]

for some function \(h : \mathcal{E}^+(X) \mapsto \mathcal{E}^+(X)\) such that \(\lim_{t \to 0} h(J) = 0\); for example it converges to \(J^*\) starting from any \(J_0\) with

\[
J^* \leq J_0 \leq cJ^*,
\]
where \( c \) is a positive scalar. We also have that VI converges to \( J^* \) starting from any \( J_0 \) with \( 0 \leq J_0 \leq J^* \) under the compactness condition of Prop. 4.4.4(d). However, \( \{ J_k \} \) may not always converge to \( J^* \) because, among other reasons, Bellman’s equation may have multiple solutions within \( \mathcal{J} \). In this section we will provide assumptions that guarantee convergence to various solutions of Bellman’s equation starting from additional initial \( J_0 \).

The PI method starts from a stationary policy \( \mu^0 \), and generates a sequence of stationary policies \( \{ \mu^k \} \) via a sequence of policy evaluations to obtain \( J_{\mu^k} \) from the equation
\[
J_{\mu^k}(x) = g(x, \mu^k(x)) + J_{\mu^k}(f(x, \mu^k(x))), \quad x \in X, \tag{4.51}
\]
interleaved with policy improvements to obtain \( \mu^{k+1} \) from \( J_{\mu^k} \) according to
\[
\mu^{k+1}(x) \in \arg\min_{u \in U(x)} \{ g(x, u) + J_{\mu^k}(f(x, u)) \}, \quad x \in X. \tag{4.52}
\]
Here, we implicitly assume that the minimum in Eq. (4.52) is attained for each \( x \in X \), which is true under some compactness condition on either \( U(x) \) or the level sets of the function \( g(x, \cdot) + J_k(f(x, \cdot)) \), or both. However, as noted in Section 4.3.3, PI may not produce a strict improvement of the cost function of a nonoptimal policy, a fact that was demonstrated with the simple deterministic shortest path example of Section 3.1.1.

The uniqueness of solution of Bellman’s equation within \( \mathcal{J} \), and the convergence of VI to \( J^* \) and of PI to an optimal policy have been investigated as part of the analysis of Section 3.5.5. There we introduced conditions guaranteeing that \( J^* \) is the unique solution of Bellman’s equation within \( \mathcal{J} \) [the near-optimal termination Assumption 3.5.10, but not the cost nonnegativity assumption (4.48)]. Our approach here will make use of the cost nonnegativity but will address the problem under otherwise weaker conditions. In particular, under our assumptions, Bellman’s equation need not have a unique solution within \( \mathcal{J} \), while the VI and PI algorithms may be unreliable.

Our analytical approach will also be different than the approach of Section 3.5.5, in that it will implicitly rely on the regularity ideas for nonstationary policies that we introduced in Section 4.4, and will make a connection with traditional notions of feedback control system stability. Using nonstationary policies may be important in undiscounted optimal control problems with nonnegative cost per stage because it is not generally true that the optimal cost over stationary policies is the same as the optimal cost over nonstationary policies [cf. the \( \epsilon \)-optimality result of Prop. 4.4.4(e)].

### 4.5.1 Forcing Functions and \( p \)-Stable Policies

We will introduce a notion of stability that involves a function \( p : X \mapsto [0, \infty) \) such that
\[
p(t) = 0, \quad p(x) > 0, \quad \forall \ x \neq t,
\]
As in Section 3.4, we refer to \( p \) as the forcing function, and we associate with it the \( p\)-\( \delta \)-perturbed optimal control problem, where \( \delta > 0 \) is a given scalar. This is the same problem as the original, except that the cost per stage is changed to
\[
g(x, u) + \delta p(x).
\]
We denote by \( J_{\pi,p,\delta} \) the cost function of a policy \( \pi \in \Pi \) in the \( p\)-\( \delta \)-perturbed problem:
\[
J_{\pi,p,\delta}(x_0) = J_\pi(x_0) + \delta \sum_{k=0}^{\infty} p(x_k),
\]
where \( \{x_k\} \) is the sequence generated starting from \( x_0 \) and using \( \pi \). We also denote by \( \hat{J}_{p,\delta} \), the corresponding optimal cost function, \( \hat{J}_{p,\delta} = \inf_{\pi \in \Pi} J_{\pi,p,\delta} \).

**Definition 4.5.1:** Let \( p \) be a given forcing function. For a state \( x \in X \), we say that a policy \( \pi \) is \( p \)-stable from \( x \) if for all \( \delta > 0 \) we have
\[
J_{\pi,p,\delta}(x) < \infty.
\]
The set of all such policies is denoted by \( \Pi_{p,x} \). We define the restricted optimal cost function over \( \Pi_{p,x} \) by
\[
\hat{J}_p(x) = \inf_{\pi \in \Pi_{p,x}} J_\pi(x), \quad x \in X.
\]
We say that \( \pi \) is \( p \)-stable if \( \pi \in \Pi_{p,x} \) simultaneously from all \( x \in X \) such that \( \Pi_{p,x} \neq \emptyset \). The set of all \( p \)-stable policies is denoted by \( \Pi_{p} \).

Note that the set \( \Pi_{p,x} \) depends on the forcing function \( p \). As an example, let \( X = \mathbb{R}^n \) and
\[
p(x) = \|x\|^\rho,
\]
where \( \rho > 0 \) is a scalar. Then roughly speaking, \( \rho \) quantifies the rate at which the destination is approached using the \( p \)-stable policies. In particular, the policies \( \pi \in \Pi_{p,x_0} \) are the ones that force \( x_k \) towards 0 at a rate faster than \( O(1/k^\rho) \), so slower policies would be excluded from \( \Pi_{p,x_0} \).

Let us make some observations regarding \( p \)-stability:

(a) **Equivalent definition of \( p \)-stability:** Given any policy \( \pi \) and state \( x_0 \in X \), from Eq. (4.53) it follows that
\[
\pi \in \Pi_{p,x_0} \quad \text{if and only if} \quad J_\pi(x_0) < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} p(x_k) < \infty,
\]
where \( \{x_k\} \) is the sequence generated starting from \( x_0 \) and using \( \pi \). Since the right-hand side of the preceding relation does not depend

\[
\text{on the forcing function } p \text{.}
\]
on \( \delta \), it also follows that an equivalent definition of a policy \( \pi \) that is \( p \)-stable from \( x \) is that \( J_{\pi,p,\delta}(x) < \infty \) for some \( \delta > 0 \) (rather than all \( \delta > 0 \)).

(b) \textit{Approximation property of} \( J_{\pi,p,\delta}(x) \): Consider a pair \((\pi, x_0)\) with \( \pi \in \Pi_{p,x_0} \). By taking the limit as \( k \to \infty \) in the expression

\[
J_{\pi,p,\delta}(x_0) = J_\pi(x_0) + \delta \sum_{k=0}^{\infty} p(x_k),
\]

[cf. Eq. (4.53)] and by using Eq. (4.55), it follows that

\[
\lim_{\delta \to 0} J_{\pi,p,\delta}(x_0) = J_\pi(x_0), \quad \forall \text{ pairs } (\pi, x_0) \text{ with } \pi \in \Pi_{p,x_0}. \quad (4.56)
\]

From this equation, we have that if \( \pi \in \Pi_{p,x} \), then \( J_{\pi,p,\delta}(x) \) is finite and differs from \( J_\pi(x) \) by \( O(\delta) \). By contrast, if \( \pi \notin \Pi_{p,x} \), then \( J_{\pi,p,\delta}(x) = \infty \) by the definition of \( p \)-stability.

(c) \textit{Limiting property of} \( \hat{J}_p(x_k) \): Consider a pair \((\pi, x_0)\) with \( \pi \in \Pi_{p,x_0} \). By breaking down \( J_{\pi,p,\delta}(x_0) \) into the sum of the costs of the first \( k \) stages and the remaining stages, we have for all \( \delta > 0 \) and \( k > 0 \),

\[
J_{\pi,p,\delta}(x_0) = \sum_{m=0}^{k-1} g(x_m, \mu_m(x_m)) + \delta \sum_{m=0}^{k-1} p(x_m) + J_{\pi_k,p,\delta}(x_k), \quad (4.57)
\]

where \( \{x_k\} \) is the sequence generated starting from \( x_0 \) and using \( \pi \), and \( \pi_k \) is the policy \( \{\mu_k, \mu_{k+1}, \ldots\} \). By taking the limit as \( k \to \infty \) and using Eq. (4.53), it follows that

\[
\lim_{k \to \infty} J_{\pi_k,p,\delta}(x_k) = 0, \quad \forall \text{ pairs } (\pi, x_0) \text{ with } \pi \in \Pi_{p,x_0}, \delta > 0. \quad (4.58)
\]

Also, since \( \hat{J}_p(x_k) \leq \hat{J}_p(x_k) \leq J_{\pi_k,p,\delta}(x_k) \), it follows that

\[
\lim_{k \to \infty} J_{p,\delta}(x_k) = 0, \quad \forall (\pi, x_0) \text{ with } x_0 \in X \text{ and } \pi \in \Pi_{p,x_0}, \delta > 0, \quad (4.59)
\]

\[
\lim_{k \to \infty} \hat{J}_p(x_k) = 0, \quad \forall (\pi, x_0) \text{ with } x_0 \in X \text{ and } \pi \in \Pi_{p,x_0}. \quad (4.60)
\]

\textbf{Terminating Policies and Controllability}

An important special case is when \( p \) is equal to the function

\[
p^+(x) = \begin{cases} 
0 & \text{if } x = t, \\
1 & \text{if } x \neq t.
\end{cases} \quad (4.61)
\]
For $p = p^+$, a policy $\pi$ is $p^+$-stable from $x$ if and only if it is terminating from $x$, i.e., reaches $t$ in a finite number of steps starting from $x$ [cf. Eq. (4.55)]. The set of terminating policies from $x$ is denoted by $\Pi^+_x$ and it is contained within every other set of $p$-stable policies $\Pi_{p,x}$, as can be seen from Eq. (4.55). As a result, the restricted optimal cost function over $\Pi^+_x$,

$$J^+(x) = \inf_{\pi \in \Pi^+_x} J_\pi(x), \quad x \in X,$$

satisfies $J^*(x) \leq \hat{J}_p(x) \leq J^+(x)$ for all $x \in X$. A policy $\pi$ is said to be terminating if it is simultaneously terminating from all $x \in X$ such that $\Pi^+_x \neq \emptyset$. The set of all terminating policies is denoted by $\Pi^+$. Note that if the state space $X$ is finite, we have for every forcing function $p$

$$\beta p^+(x) \leq p(x) \leq \bar{\beta} p^+(x), \quad \forall \ x \in X,$$

for some scalars $\beta, \bar{\beta} > 0$. As a result it can be seen that $\Pi_{p,x} = \Pi^+_x$ and $\hat{J}_p = J^+$, so in effect the case where $p = p^+$ is the only case of interest for finite-state problems.

The notion of a terminating policy is related to the notion of controllability. In classical control theory terms, the system $x_{k+1} = f(x_k, u_k)$ is said to be completely controllable if for every $x_0 \in X$, there exists a policy that drives the state $x_k$ to the destination in a finite number of steps. This notion of controllability is equivalent to the existence of a terminating policy from each $x \in X$.

One of our main results, to be shown shortly, is that $J^*$, $\hat{J}_p$, and $J^+$ are solutions of Bellman’s equation, with $J^*$ being the “smallest” solution and $J^+$ being the “largest” solution within $\mathcal{J}$. The most favorable situation arises when $J^* = J^+$, in which case $J^*$ is the unique solution of Bellman’s equation within $\mathcal{J}$. Moreover, in this case it will be shown that the VI algorithm converges to $J^*$ starting with any $J_0 \in \mathcal{J}$ with $J_0 \geq J^*$, and the PI algorithm converges to $J^*$ as well (see Section 4.1).

### 4.5.2 Restricted Optimization over Stable Policies

For a given forcing function $p$, we denote by $\hat{X}_p$ the effective domain of $\hat{J}_p$, the set of all $x$ where $\hat{J}_p$ is finite,

$$\hat{X}_p = \{ x \in X \mid \hat{J}_p(x) < \infty \}.$$

Since $\hat{J}_p(x) < \infty$ if and only if $\Pi_{p,x} \neq \emptyset$ [cf. Eqs. (4.54) and (4.55)], or equivalently $J_{x,\delta}(x) < \infty$ for some $\pi$ and all $\delta > 0$, it follows that $\hat{X}_p$ is also the effective domain of $\hat{J}_{p,\delta}$,

$$\hat{X}_p = \{ x \in X \mid \Pi_{p,x} \neq \emptyset \} = \{ x \in X \mid \hat{J}_{p,\delta}(x) < \infty \}, \quad \forall \ \delta > 0.$$
Note that $\hat{X}_p$ may depend on $p$ and may be a strict subset of the effective domain of $J^*$, which is denoted by

$$X^* = \{x \in X \mid J^*(x) < \infty\}.$$ 

The reason is that there may exist a policy $\pi$ such that $J_{\pi}(x) < \infty$, even when there is no $p$-stable policy from $x$.

Our first objective is to show that as $\delta \downarrow 0$, the $p$-$\delta$-perturbed optimal cost function $\hat{J}_{p,\delta}$ converges to the restricted optimal cost function $\hat{J}_p$.

**Proposition 4.5.1 (Approximation Property of $\hat{J}_{p,\delta}$):** Let $p$ be a given forcing function and $\delta > 0$.

(a) We have

$$J_{\pi_{p,\delta}}(x) = J_{\pi}(x) + w_{\pi,p,\delta}(x), \quad \forall \ x \in X, \ \pi \in \Pi_{p,x}, \quad (4.62)$$

where $w_{\pi,p,\delta}$ is a function such that $\lim_{\delta \downarrow 0} w_{\pi,p,\delta}(x) = 0$ for all $x \in X$.

(b) We have

$$\lim_{\delta \downarrow 0} \hat{J}_{p,\delta}(x) = \hat{J}_p(x), \quad \forall \ x \in X.$$ 

**Proof:** (a) Follows by using Eq. (4.56) for $x \in \hat{X}_p$, and by taking $w_{p,\delta}(x) = 0$ for $x \notin \hat{X}_p$.

(b) By Prop. 4.4.4(e), there exists an $\epsilon$-optimal policy $\pi_{\epsilon}$ for the $p$-$\delta$-perturbed problem, i.e., $J_{\pi_{\epsilon},p,\delta}(x) \leq J_{p,\delta}(x) + \epsilon$ for all $x \in X$. Moreover, for $x \in \hat{X}_p$ we have $\hat{J}_{p,\delta}(x) < \infty$, so $J_{\pi_{\epsilon},p,\delta}(x) < \infty$. Hence $\pi_{\epsilon}$ is $p$-stable from all $x \in \hat{X}_p$, and we have $\hat{J}_{p} \leq J_{\pi_{\epsilon}}$. Using also Eq. (4.62), we have for all $\delta > 0$, $\epsilon > 0$, $x \in X$, and $\pi \in \Pi_{p,x}$,

$$\hat{J}_p(x) - \epsilon \leq J_{\pi_{\epsilon}}(x) - \epsilon \leq J_{\pi_{\epsilon},p,\delta}(x) - \epsilon \leq \hat{J}_{p,\delta}(x) \leq J_{\pi_{\epsilon},p,\delta}(x) = J_{\pi}(x) + w_{\pi,p,\delta}(x),$$

where $\lim_{\delta \downarrow 0} w_{\pi,p,\delta}(x) = 0$ for all $x \in X$. By taking the limit as $\epsilon \downarrow 0$, we obtain for all $\delta > 0$ and $\pi \in \Pi_{p,x}$,

$$\hat{J}_p(x) \leq \hat{J}_{p,\delta}(x) \leq J_{\pi}(x) + w_{\pi,p,\delta}(x), \quad \forall \ x \in X.$$ 

By taking the limit as $\delta \downarrow 0$ and then the infimum over all $\pi \in \Pi_{p,x}$, we have

$$\hat{J}_p(x) \leq \lim_{\delta \downarrow 0} \hat{J}_{p,\delta}(x) \leq \inf_{\pi \in \Pi_{p,x}} J_{\pi}(x) = \hat{J}_p(x), \quad \forall \ x \in X.$$
from which the result follows. Q.E.D.

We now consider approximately optimal policies. Given any \( \epsilon > 0 \), by Prop. 4.4.4(e), there exists an \( \epsilon \)-optimal policy for the \( p\)-\( \delta \)-perturbed problem, i.e., a policy \( \pi \) such that \( J_\pi(x) \leq \hat{J}_{p,\delta}(x) + \epsilon \) for all \( x \in X \). We address the question whether there exists a \( p \)-stable policy \( \pi \) that is \( \epsilon \)-optimal for the restricted optimization over \( p \)-stable policies, i.e., a policy \( \pi \) that is \( p \)-stable simultaneously from all \( x \in X_p \), (i.e., \( \pi \in \Pi_p \)) and satisfies

\[
J_\pi(x) \leq \hat{J}_p(x) + \epsilon, \quad \forall \ x \in X.
\]

We refer to such a policy as a \( p \)-\( \epsilon \)-optimal policy.

\[\text{Proposition 4.5.2 (Existence of } p \text{-}\epsilon \text{-Optimal Policy): Let } p \text{ be a given forcing function and } \delta > 0. \text{ For every } \epsilon > 0, \text{ a policy } \pi \text{ that is } \epsilon \text{-optimal for the } p\text{-}\delta\text{-perturbed problem is } p\text{-}\epsilon\text{-optimal, and hence belongs to } \Pi_p.\]

\[\text{Proof: For any } \epsilon \text{-optimal policy } \pi_\epsilon \text{ for the } p\text{-}\delta\text{-perturbed problem, we have}
J_{\pi_\epsilon,p,\delta}(x) \leq \hat{J}_{p,\delta}(x) + \epsilon < \infty, \quad \forall \ x \in \hat{X}_p.
\]

This implies that \( \pi_\epsilon \in \Pi_p \). Moreover, for all sequences \( \{x_k\} \) generated from initial state-policy pairs \((\pi, x_0)\) with \( x_0 \in \hat{X}_p \) and \( \pi \in \Pi_p, x_0 \), we have

\[
J_{\pi_\epsilon}(x_0) \leq J_{\pi_\epsilon,p,\delta}(x_0) \leq \hat{J}_{p,\delta}(x_0) + \epsilon \leq J_\pi(x_0) + \delta \sum_{k=0}^{\infty} p(x_k) + \epsilon.
\]

Taking the limit as \( \delta \downarrow 0 \) and using the fact \( \sum_{k=0}^{\infty} p(x_k) < \infty \) (since \( \pi \in \Pi_p, x_0 \)), we obtain

\[
J_{\pi_\epsilon}(x_0) \leq J_\pi(x_0) + \epsilon, \quad \forall \ x_0 \in \hat{X}_p, \ \pi \in \Pi_p, x_0.
\]

By taking infimum over \( \pi \in \Pi_p, x_0 \), it follows that

\[
J_{\pi_\epsilon}(x_0) \leq \hat{J}_p(x_0) + \epsilon, \quad \forall \ x_0 \in \hat{X}_p,
\]

which in view of the fact \( J_{\pi_\epsilon}(x_0) = \hat{J}_p(x_0) = \infty \) for \( x_0 \notin \hat{X}_p \), implies that \( \pi_\epsilon \) is \( p \)-\( \epsilon \)-optimal. Q.E.D.

Note that the preceding proposition implies that

\[
\hat{J}_p(x) = \inf_{\pi \in \Pi_p} J_\pi(x), \quad \forall \ x \in X,
\]  

(4.63)
which is a stronger statement than the definition $\hat{J}_p(x) = \inf_{\pi \in \Pi_{p,x}} J_\pi(x)$ for all $x \in X$. However, it can be shown through examples that there may not exist a restricted-optimal $p$-stable policy, i.e., a $\pi \in \Pi_p$ such that $J_\pi = \hat{J}_p$, even if there exists an optimal policy for the original problem. One such example is the one-dimensional linear-quadratic problem of Section 3.1.4 for the case where $p = p^+$. Then, there exists a unique linear stable policy that attains the restricted optimal cost $J^+(x)$ for all $x$, but this policy is not terminating. Note also that there may not exist a stationary $p$-optimal policy, since generally in undiscounted nonnegative optimal control problems there may not exist a stationary $\epsilon$-optimal policy (an example is given following Prop. 4.4.8).

Our next proposition is preliminary for our main result. It involves the set of functions $S_p$ given by

$$S_p = \left\{ J \in \mathcal{J} \mid J(x_k) \to 0 \text{ for all sequences } \{x_k\} \text{ generated from initial state-policy pairs } (\pi, x_0) \text{ with } x_0 \in X \text{ and } \pi \in \Pi_{p,x_0} \right\}.$$  

(4.64)

In words, $S_p$ is the set of functions in $\mathcal{J}$ whose value is asymptotically driven to 0 by all the policies that are $p$-stable starting from some $x_0 \in X$ (and thus have the character of Lyapounov functions for these policies).

Note that $S_p$ contains $\hat{J}_p$ and $\hat{J}_{p,\delta}$ for all $\delta > 0$ [cf. Eq. (4.59)]. Moreover, $S_p$ contains all functions $J$ such that

$$0 \leq J \leq h(\hat{J}_{p,\delta})$$

for some $\delta > 0$ and function $h : X \to X$ such that $h(J) \to 0$ as $J \to 0$. For example $S_p$ contains all $J$ such that $0 \leq J \leq c \hat{J}_{p,\delta}$ for some $c > 0$ and $\delta > 0$.

We summarize the preceding discussion in the following proposition, which also shows uniqueness of solution (within $S_p$) of Bellman’s equation for the $p$-$\delta$-perturbed problem. The significance of this is that the $p$-$\delta$-perturbed problem can be solved more reliably than the original problem (including by VI methods), and yield a close approximation to $\hat{J}_p$ [cf. Prop. 4.5.1(b)].

**Proposition 4.5.3:** Let $p$ be a forcing function and $\delta > 0$. The function $\hat{J}_{p,\delta}$ belongs to the set $S_p$, and is the unique solution within $S_p$ of Bellman’s equation for the $p$-$\delta$-perturbed problem,

$$\hat{J}_{p,\delta}(x) = \inf_{u \in U(x)} \left\{ g(x, u) + \delta p(x) + \hat{J}_{p,\delta}(f(x, u)) \right\}, \quad x \in X.$$  

(4.65)
Moreover, \( S_p \) contains \( \hat{J}_p \) and all functions \( J \) satisfying

\[
0 \leq J \leq h(\hat{J}_p,\delta)
\]

for some \( h : X \mapsto X \) with \( h(J) \to 0 \) as \( J \to 0 \).

**Proof:** We have \( \hat{J}_{p,\delta} \in S_p \) and \( \hat{J}_p \in S_p \) by Eq. (4.59), as noted earlier. We also have that \( \hat{J}_{p,\delta} \) is a solution of Bellman’s equation (4.65) by Prop. 4.4.4(a). To show that \( \hat{J}_{p,\delta} \) is the unique solution within \( S_p \), let \( \tilde{J} \in S_p \) be another solution, so that using also Prop. 4.4.4(a), we have

\[
\hat{J}_{p,\delta}(x) \leq \hat{J}(x) \leq g(x,u) + \delta p(x) + \hat{J}(f(x,u)), \quad \forall \ x \in X, \ u \in U(x).
\]

(4.66)

Fix \( \epsilon > 0 \), and let \( \pi = \{\mu_0, \mu_1, \ldots\} \) be an \( \epsilon \)-optimal policy for the \( p,\delta \)-perturbed problem. By repeatedly applying the preceding relation, we have for any \( x_0 \in \hat{X}_p \),

\[
\hat{J}_{p,\delta}(x_0) \leq \tilde{J}(x_0) \leq \hat{J}(x_k) + \delta \sum_{m=0}^{k-1} p(x_m) + \sum_{m=0}^{k-1} g(x_m, \mu_m(x_m)), \quad \forall \ k \geq 1,
\]

(4.67)

where \( \{x_k\} \) is the state sequence generated starting from \( x_0 \) and using \( \pi \). We have \( \hat{J}(x_k) \to 0 \) (since \( \hat{J} \in S_p \) and \( \pi \in \Pi_p \) by Prop. 4.5.2), so that

\[
\lim_{k \to \infty} \left\{ \hat{J}(x_k) + \delta \sum_{m=0}^{k-1} p(x_m) + \sum_{m=0}^{k-1} g(x_m, \mu_m(x_m)) \right\} = J_{\pi,\delta}(x_0)
\]

\[
\leq \hat{J}_{p,\delta}(x_0) + \epsilon.
\]

(4.68)

By combining Eqs. (4.67) and (4.68), we obtain

\[
\hat{J}_{p,\delta}(x_0) \leq \tilde{J}(x_0) \leq \hat{J}_{p,\delta}(x_0) + \epsilon, \quad \forall \ x_0 \in \hat{X}_p.
\]

By letting \( \epsilon \to 0 \), it follows that \( \hat{J}_{p,\delta}(x_0) = \tilde{J}(x_0) \) for all \( x_0 \in \hat{X}_p \). Also for \( x_0 \notin \hat{X}_p \), we have \( \hat{J}_{p,\delta}(x_0) = \tilde{J}(x_0) = \infty \) [since \( \hat{J}_{p,\delta}(x_0) = \infty \) for \( x_0 \notin \hat{X}_p \) and \( \hat{J}_{p,\delta} \leq \hat{J} \), cf. Eq. (4.66)]. Thus \( \hat{J}_{p,\delta} = \hat{J} \), proving that \( \hat{J}_{p,\delta} \) is the unique solution of the Bellman Eq. (4.65) within \( S_p \). **Q.E.D.**

We next show that \( \hat{J}_p \) is the unique solution of Bellman’s equation within the set of functions

\[
\mathcal{W}_p = \{J \in S_p \mid \hat{J}_p \leq J\},
\]

(4.69)

and that the VI algorithm yields \( \hat{J}_p \) in the limit for any initial \( J_0 \in \mathcal{W}_p \).
Proposition 4.5.4: Let \( p \) be a given forcing function. Then:

(a) \( \hat{J}_p \) is the unique solution of Bellman’s equation

\[
J(x) = \inf_{u \in U(x)} \left\{ g(x, u) + J(f(x, u)) \right\}, \quad x \in X, \tag{4.70}
\]

within the set \( \mathcal{W}_p \) of Eq. (4.69).

(b) (VI Convergence) If \( \{J_k\} \) is the sequence generated by the VI algorithm (4.50) starting with some \( J_0 \in \mathcal{W}_p \), then \( J_k \to \hat{J}_p \).

(c) (Optimality Condition) If \( \hat{\mu} \) is a \( p \)-stable stationary policy and

\[
\hat{\mu}(x) \in \arg \min_{u \in U(x)} \left\{ g(x, u) + \hat{J}_p(f(x, u)) \right\}, \quad \forall x \in X, \tag{4.71}
\]

then \( \hat{\mu} \) is optimal over the set of \( p \)-stable policies. Conversely, if \( \hat{\mu} \) is optimal within the set of \( p \)-stable policies, then it satisfies the preceding condition (4.71).

Proof: (a), (b) We first show that \( \hat{J}_p \) is a solution of Bellman’s equation. Since \( \hat{J}_{p,\delta} \) is a solution [cf. Prop. 4.5.3] and \( \hat{J}_{p,\delta} \geq \hat{J}_p \) [cf. Prop. 4.5.1(b)], we have for all \( \delta > 0 \),

\[
\hat{J}_{p,\delta}(x) = \inf_{u \in U(x)} \left\{ g(x, u) + \delta p(x) + \hat{J}_{p,\delta}(f(x, u)) \right\} \\
\geq \inf_{u \in U(x)} \left\{ g(x, u) + \hat{J}_{p,\delta}(f(x, u)) \right\} \\
\geq \inf_{u \in U(x)} \left\{ g(x, u) + \hat{J}_p(f(x, u)) \right\}.
\]

By taking the limit as \( \delta \downarrow 0 \) and using the fact \( \lim_{\delta \downarrow 0} \hat{J}_{p,\delta} = \hat{J}_p \) [cf. Prop. 4.5.1(b)], we obtain

\[
\hat{J}_p(x) \geq \inf_{u \in U(x)} \left\{ g(x, u) + \hat{J}_p(f(x, u)) \right\}, \quad \forall x \in X. \tag{4.72}
\]

For the reverse inequality, let \( \{\delta_m\} \) be a sequence with \( \delta_m \downarrow 0 \). From Prop. 4.5.3, we have for all \( m, x \in X \), and \( u \in U(x) \),

\[
g(x, u) + \delta_m p(x) + \hat{J}_{p,\delta_m}(f(x, u)) \geq \inf_{v \in U(x)} \left\{ g(x, v) + \delta_m p(x) \\
+ \hat{J}_{p,\delta_m}(f(x, v)) \right\} \\
= \hat{J}_{p,\delta_m}(x).
\]
Taking the limit as \( m \to \infty \), and using the fact \( \lim_{\delta_m \downarrow 0} \hat{J}_{p, \delta_m} = \hat{J}_p \) [cf. Prop. 4.5.1(b)], we have
\[
 g(x, u) + \hat{J}_p(f(x, u)) \geq \hat{J}_p(x), \quad \forall \, x \in X, \, u \in U(x),
\]
so that
\[
 \inf_{u \in U(x)} \{ g(x, u) + \hat{J}_p(f(x, u)) \} \geq \hat{J}_p(x), \quad \forall \, x \in X. \tag{4.73}
\]

By combining Eqs. (4.72) and (4.73), we see that \( \hat{J}_p \) is a solution of Bellman’s equation. We also have \( \hat{J}_p \in S_p \) by Prop. 4.5.3, implying that \( \hat{J}_p \in W_p \) and proving part (a) except for the uniqueness assertion.

We will now prove part (b). Let \( \pi = \{ \mu_0, \mu_1, \ldots \} \in \Pi_p \) [which is nonempty by Prop. 4.5.1(c)], and for \( x_0 \in \hat{X}_p \), let \( \{ x_k \} \) be the generated sequence starting from \( x_0 \) and using \( \pi \). We have \( J_0(x_k) \to 0 \) since \( J_0 \in S_p \). Since from the definition of the VI sequence \( \{ J_k \} \), we have
\[
 J_k(x) \leq g(x, u) + J_{k-1}(f(x, u)), \quad \forall \, x \in X, \, u \in U(x), \, k = 1, 2, \ldots,
\]
it follows that
\[
 J_k(x_0) \leq J_0(x_k) + \sum_{m=0}^{k-1} g(x_m, \mu_m(x_m)).
\]

By taking limit as \( k \to \infty \) and using the fact \( J_0(x_k) \to 0 \), it follows that \( \limsup_{k \to \infty} J_k(x_0) \leq J_\pi(x_0) \). By taking the infimum over all \( \pi \in \Pi_p \), we obtain \( \limsup_{k \to \infty} J_k(x_0) \leq \hat{J}_p(x_0) \). Conversely, since \( \hat{J}_p \leq J_0 \) and \( \hat{J}_p \) is a solution of Bellman’s equation (as shown earlier), it follows by induction that \( \hat{J}_p \leq J_k \) for all \( k \). Thus \( \hat{J}_p(x_0) \leq \liminf_{k \to \infty} J_k(x_0) \), implying that \( J_k(x_0) \to \hat{J}_p(x_0) \) for all \( x_0 \in \hat{X}_p \). We also have \( \hat{J}_p \leq J_k \) for all \( k \), so that \( \hat{J}_p(x_0) = J_k(x_0) = \infty \) for all \( x_0 \notin \hat{X}_p \). This completes the proof of part (b). Finally, since \( \hat{J}_p \in W_p \) and \( \hat{J}_p \) is a solution of Bellman’s equation, part (b) implies the uniqueness assertion of part (a).

(c) If \( \mu \) is \( p \)-stable and Eq. (4.71) holds, then
\[
 \hat{J}_\mu(x) = g(x, \mu(x)) + \hat{J}_p(f(x, \mu(x))), \quad x \in X.
\]
By Prop. 4.4.4(b), this implies that \( J_\mu \leq \hat{J}_p \), so \( \mu \) is optimal over the set of \( p \)-stable policies. Conversely, assume that \( \mu \) is \( p \)-stable and \( J_\mu = \hat{J}_p \). Then by Prop. 4.4.4(b), we have
\[
 \hat{J}_p(x) = g(x, \mu(x)) + \hat{J}_p(f(x, \mu(x))), \quad x \in X,
\]
and since [by part (a)] \( \hat{J}_p \) is a solution of Bellman’s equation,

\[
J_p(x) = \inf_{u \in U(x)} \{ g(x, u) + \hat{J}_p(f(x, u)) \}, \quad x \in X.
\]

Combining the last two relations, we obtain Eq. (4.71). Q.E.D.

We now consider the special case where \( p \) is equal to the function
\( p(x) = 1 \) for all \( x \neq t \) [cf. Eq. (4.61)]. Then the set of \( p^+ \)-stable policies from \( x \) is \( \Pi^+_x \), the set of terminating policies from \( x \), and the corresponding restricted optimal cost is

\[
J^+(x) = \hat{J}_{p^+}(x) = \inf_{\pi \in \Pi^+_x} J_\pi(x) = \inf_{\pi \in \Pi^+} J_\pi(x), \quad x \in X,
\]

[the last equality follows from Eq. (4.63)]. In this case, the set \( S_{p^+} \) of Eq. (4.64) is the entire set \( J \), since for all \( J \in J \) and all sequences \( \{x_k\} \) generated from initial state-policy pairs \( (\pi, x_0) \) with \( x_0 \in X \) and \( \pi \) terminating from \( x_0 \), we have \( J(x_k) = 0 \) for \( k \) sufficiently large. Thus, the set \( W_{p^+} \) of Eq. (4.69) is

\[
W_{p^+} = \{ J \in J | J^+ \leq J \}.
\]

By specializing to the case \( p = p^+ \) the result of Prop. 4.5.4, we obtain the following proposition, which makes a stronger assertion than Prop. 4.5.4(a), namely that \( J^+ \) is the largest solution of Bellman’s equation within \( J \) (rather than the smallest solution within \( W_{p^+} \)).

\begin{proposition}
(a) \( J^+ \) is the largest solution of the Bellman equation (4.70) within \( J \), i.e., if \( \hat{J} \in J \) is a solution of Bellman’s equation, then \( \hat{J} \leq J^+ \).

(b) (VI Convergence) If \( \{J_k\} \) is the sequence generated by the VI algorithm (4.50) starting with some \( J_0 \in J \) with \( J_0 \geq J^+ \), then \( J_k \to J^+ \).

(c) (Optimality Condition) If \( \mu^+ \) is a terminating stationary policy and

\[
\mu^+(x) \in \arg \min_{u \in U(x)} \{ g(x, u) + J^+(f(x, u)) \}, \quad \forall x \in X,
\]

then \( \mu^+ \) is optimal over the set of terminating policies. Conversely, if \( \mu^+ \) is optimal within the set of terminating policies, then it satisfies the preceding condition (4.75).

\end{proposition}

**Proof:** In view of Prop. 4.5.4 and the expression (4.74) for \( W_{p^+} \), we only need to show that \( \hat{J} \leq J^+ \) for every solution \( \hat{J} \in J \) of Bellman’s equation.
Indeed, let \( \tilde{J} \) be such a solution. We have \( \tilde{J}(x_0) \leq J^+(x_0) \) for all \( x_0 \) with \( J^+(x_0) = \infty \), so in order to show that \( \tilde{J} \leq J^+ \), it will suffice to show that for every \( (\pi, x_0) \) with \( \pi \in \Pi_{x_0}^+ \), we have \( \tilde{J}(x_0) \leq J_{\pi}(x_0) \). Indeed, consider \( (\pi, x_0) \) with \( \pi \in \Pi_{x_0}^+ \), and let \( \{x_0, \ldots, x_k, t\} \) be the terminating state sequence generated starting from \( x_0 \) and using \( \pi \). Since \( \tilde{J} \) solves Bellman’s equation, we have

\[
\tilde{J}(x_m) \leq g(x_m, \mu_m(x_m)) + \tilde{J}(x_{m+1}), \quad m = 0, \ldots, k - 1,
\]

\[
\tilde{J}(x_k) \leq g(x_k, \mu_k(x_k)).
\]

By adding these relations, we obtain

\[
\tilde{J}(x_0) \leq \sum_{m=0}^{k} g(x_m, \mu_m(x_m)) = J_{\pi}(x_0), \quad \forall (\pi, x_0) \text{ with } \pi \in \Pi_{x_0}^+,
\]

and by taking the infimum of the right side over \( \pi \in \Pi_{x_0}^+ \), we obtain \( \tilde{J}(x_0) \leq J^+(x_0) \). Q.E.D.

We illustrate Props. 4.5.4 and 4.5.5 in Figs. 4.5.2 and 4.5.3. In particular, each forcing function \( p \) delineates the set of initial functions \( \mathcal{W}_p \) from which VI converges to \( \tilde{J}_p \). The function \( \tilde{J}_p \) is the minimal element of \( \mathcal{W}_p \). Moreover, we have \( \mathcal{W}_p \cap \mathcal{W}_{p'} = \emptyset \) if \( \tilde{J}_p \neq \tilde{J}_{p'} \), in view of the VI convergence result of Prop. 4.5.4(b).

Note that Prop. 4.5.5(b) implies that VI converges to \( J^+ \) starting from the particular initial condition

\[
J_0(x) = \begin{cases} 
0 & \text{if } x = t, \\
\infty & \text{if } x \neq t. 
\end{cases} \tag{4.76}
\]

For this choice of \( J_0 \), the value \( J_k(x) \) generated by VI is the optimal cost that can be achieved starting from \( x \) subject to the constraint that \( t \) is reached in \( k \) steps or less.
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VI converges to \( J^+ \) from within \( W \):

- Functions \( J \geq J_p \) with \( J(x_k) \to 0 \) for all \( p \)-stable \( \pi \)
- Functions \( J \geq J_{p'} \) with \( J(x_k) \to 0 \) for all \( p' \)-stable \( \pi \)

\[ W_p := \{ J | J \geq J_p, J(t) = 0 \} \]

\[ W_{p'} := \{ J | J \geq J_{p'}, J(t) = 0 \} \]

**Figure 4.5.3** Illustration of the VI convergence results of Prop. 4.5.4 and 4.5.5. Each \( p \) defines the set of initial functions \( W_p \) from which VI converges to \( J_p \) from above. For two forcing functions \( p \) and \( p' \), we have \( W_p \cap W_{p'} = \emptyset \) if \( J_p \neq J_{p'} \).

Suppose now that the set of terminating policies is sufficient in the sense that it can achieve the same optimal cost as the set of all policies, i.e., \( J^+ = J^* \). Then, from Prop. 4.5.5, it follows that \( J^* \) is the unique solution of Bellman’s equation within \( J \), and the VI algorithm converges to \( J^* \) from above, i.e., starting from any \( J_0 \in J \) with \( J_0 \geq J^* \). Under additional conditions, such as finiteness of \( U(x) \) for all \( x \in X \) [cf. Prop. 4.4.4(d)], VI converges to \( J^* \) starting from any \( J_0 \in E^+(X) \) with \( J_0(t) = 0 \).

Examples of problems where terminating policies are sufficient include linear-quadratic problems under the classical conditions of controllability and observability, and finite-node deterministic shortest path problems with all cycles having positive length. Note that in the former case, despite the fact \( J^+ = J^* \), there is no optimal terminating policy, since the only optimal policy is a linear policy that drives the system to the origin asymptotically, but not in finite time.

Let us illustrate the results of this section with two examples.

**Example 4.5.1 (Minimum Energy Stable Control of Linear Systems)**

Consider the linear-quadratic problem of Section 3.1.4. We assume that the pair \((A, B)\) is stabilizable. However, we are making no assumptions on the state weighting matrix \( Q \) other than positive semidefiniteness, so the de-
tectability assumption may not be satisfied. This includes the case \( Q = 0 \), when \( J^*(x) \equiv 0 \). In this case an optimal policy is \( \mu^*(x) \equiv 0 \), which may not be stable, yet the problem of finding a stable policy that minimizes the “control energy” (a cost that is quadratic on the control with no penalty on the state) among all stable policies is meaningful.

We consider the forcing function

\[
p(x) = \|x\|^2,
\]

so the \( p-\delta \)-perturbed problem satisfies the detectability condition and from classical results, \( J_{p,\delta} \) is a positive definite quadratic function \( x'P_\delta x \), where \( P_\delta \) is the unique solution of the \( \delta \)-perturbed Riccati equation

\[
P_\delta = A'(P_\delta - P_\delta B(B'P_\delta B + R)^{-1}B'P_\delta)A + Q + \delta I,
\]

within the class of positive semidefinite matrices. By Prop. 4.5.1, we have \( J_{p}(x) = x'P_\delta x \), where \( \hat{P} = \lim_{\delta \to 0} P_\delta \) is positive semidefinite, and solves the (unperturbed) Riccati equation

\[
P = A'(P - PB(B'P B + R)^{-1}B'P)A + Q.
\]

Moreover, by Prop. 4.5.4(a), \( \hat{P} \) is the largest solution among positive semidefinite matrices, since all positive semidefinite quadratic functions belong to the set \( S_p \) of Eq. (4.64). By Prop. 4.5.4(c), any stable stationary policy \( \hat{\mu} \) that is optimal among the set of stable policies must satisfy the optimality condition

\[
\hat{\mu}(x) \in \arg \min_{u \in \mathbb{R}^m} \left\{ u'Ru + (Ax + Bu)'\hat{P}(Ax + Bu) \right\}, \quad \forall x \in \mathbb{R}^n,
\]

[cf. Eq. (4.71)], or equivalently, by setting the gradient of the minimized expression to 0,

\[
(R + B'\hat{P} B)\hat{\mu}(x) = -B'\hat{P} Ax.
\]

We may solve Eq. (4.78), and check if any of its solutions \( \hat{\mu} \) is \( p \)-stable; if this is so, \( \hat{\mu} \) is optimal within the class of \( p \)-stable policies. Note, however, that in the absence of additional conditions, it is possible that some policies \( \hat{\mu} \) that solve Eq. (4.78) are \( p \)-unstable.

In the case where the pair \((A, B)\) is not stabilizable, the \( p-\delta \)-perturbed cost function \( J_{p,\delta} \) need not be real-valued, and the \( \delta \)-perturbed Riccati equation (4.77) may not have any solution (consider for example the case where \( n = 1 \), \( A = 2 \), \( B = 0 \), and \( Q = R = 1 \)). Then, Prop. 4.5.5 still applies, but the preceding analytical approach needs to be modified.

As noted earlier, the Bellman equation may have multiple solutions corresponding to different forcing functions \( p \), with each solution being unique within the corresponding set \( \mathcal{W}_p \) of Eq. (4.69), consistently with Prop. 4.5.4(a). The following is an illustrative example.
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Example 4.5.2 (An Optimal Stopping Problem)

Consider an optimal stopping problem where the state space \( X = \mathbb{R}^n \). We identify the destination with the origin of \( \mathbb{R}^n \), i.e., \( t = 0 \). At each \( x \neq 0 \), we may either stop (move to the origin) at a cost \( c > 0 \), or move to state \( \gamma x \) at cost \( \|x\| \), where \( \gamma \) is a scalar with \( 0 \leq \gamma < 1 \); see Fig. 4.5.4.† Thus the Bellman equation has the form

\[
J(x) = \begin{cases} 
\min \{ c, \|x\| + J(\gamma x) \} & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\] (4.79)

Let us consider first the forcing function

\( p(x) = \|x\| \).

Then it can be verified that all policies are \( p \)-stable. We have

\[
J^*(x) = J_p(x) = \min \left\{ c, \frac{1}{1 - \gamma} \|x\| \right\}, \quad \forall \ x \in \mathbb{R}^n,
\]

and the optimal cost function of the corresponding \( p\)-\( \delta \)-perturbed problem is

\[
\hat{J}_{p,\delta}(x) = \min \left\{ c + \delta \|x\|, \frac{1 + \delta}{1 - \gamma} \|x\| \right\}, \quad \forall \ x \in \mathbb{R}^n.
\]

† In this example, the salient feature of the policy that never stops is that it drives the system asymptotically to the destination according to an equation of the form \( x_{k+1} = f(x_k) \), where \( f \) is a contraction mapping. The example admits generalization to the broader class of optimal stopping problems where the policy that never stops has this property. For simplicity in illustrating our main point, we consider here the special case where \( f(x) = \gamma x \) with \( \gamma \in (0, 1) \).
Here the set \( S_p \) of Eq. (4.64) is given by

\[
S_p = \left\{ J \in \mathcal{J} \mid \lim_{x \to 0} J(x) = 0 \right\},
\]

and the corresponding set \( \mathcal{W}_p \) of Eq. (4.69) is given by

\[
\mathcal{W}_p = \left\{ J \in \mathcal{J} \mid J^* \leq J, \lim_{x \to 0} J(x) = 0 \right\}.
\]

Let us consider next the forcing function

\[
p^+(x) = \begin{cases} 
1 & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]

Then the \( p^+ \)-stable policies are the terminating policies. Since stopping at some time and incurring the cost \( c \) is a requirement for a \( p^+ \)-stable policy, it follows that the optimal \( p^+ \)-stable policy is to stop as soon as possible, i.e., stop at every state. The corresponding restricted optimal cost function is

\[
J^+(x) = \begin{cases} 
c & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]

The optimal cost function of the corresponding \( p^+ \)-\( \delta \)-perturbed problem is

\[
\hat{J}_{p^+, \delta}(x) = \begin{cases} 
c + \delta & \text{if } x \neq 0, \\
0 & \text{if } x = 0,
\end{cases}
\]

since in the \( p^+ \)-\( \delta \)-perturbed problem it is again optimal to stop as soon as possible, at cost \( c + \delta \). Here the set \( S_{p^+} \) is equal to \( \mathcal{J} \), and the corresponding set \( \mathcal{W}_{p^+} \) is equal to \( \left\{ J \in \mathcal{J} \mid J^+ \leq J \right\} \).

However, there are infinitely many additional solutions of Bellman’s equation between the largest and smallest solutions \( J^* \) and \( J^+ \). For example, when \( n > 1 \), functions \( J \in \mathcal{J} \) such that \( J(x) = J^*(x) \) for \( x \) in some cone and \( J(x) = J^+(x) \) for \( x \) in the complementary cone are solutions; see Fig. 4.5.4. There is also a corresponding infinite number of regions of convergence \( \mathcal{W}_p \) of VI [cf. Eq. (4.69)]. Also VI converges to \( J^* \) starting from any \( J_0 \) with \( 0 \leq J_0 \leq J^* \) [cf. Prop. 4.4.4(d)]. Figure 4.5.5 illustrates additional solutions of Bellman’s equation of a different character.

### 4.5.3 Policy Iteration Methods

Generally, the standard PI algorithm [cf. Eqs. (4.51), (4.52)] produces unclear results under our assumptions. As an illustration, in the stopping problem of Example 4.5.2, if PI is started with the policy that stops at every state, it repeats that policy, and this policy is not optimal even within the class of stable policies with respect to the forcing function \( p(x) = \|x\| \). The following example provides an instance where the PI algorithm may converge to either an optimal or a strictly suboptimal policy.
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(0) = 0

\( \hat{J}^*(x) \)

\( \hat{J}(x) \)

\( J^+(x) \)

\( c \)

\( 0 \)

\( x_0 \)

\( x \)

\( \gamma \)

\( \gamma \)

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Policy Iteration for the Case $J^* = J^+$

The PI algorithm starts with a stationary policy $\mu^0$, and generates a sequence of stationary policies $\{\mu^k\}$ via a sequence of policy evaluations to obtain $J_{\mu^k}$ from the equation

$$J_{\mu^k}(x) = g(x, \mu^k(x)) + J_{\mu^k}(f(x, \mu^k(x))), \quad x \in X,$$  \hspace{1cm} (4.80)

interleaved with policy improvements to obtain $\mu^{k+1}$ from $J_{\mu^k}$ according to

$$\mu^{k+1}(x) \in \arg\min_{u \in U(x)} \{g(x, u) + J_{\mu^k}(f(x, u))\}, \quad x \in X.$$  \hspace{1cm} (4.81)

We implicitly assume here that the minimum in Eq. (4.81) is attained for each $x \in X$, which is true under some compactness condition on either $U(x)$ or the level sets of the function $g(x, \cdot) + J_k(f(x, \cdot))$, or both.

**Proposition 4.5.6: (Convergence of PI)** Assume that $J^* = J^+$. Then the sequence $\{J_{\mu^k}\}$ generated by the PI algorithm (4.80), (4.81), satisfies $J_{\mu^k}(x) \downarrow J^*(x)$ for all $x \in X$.

**Proof:** For a stationary policy $\mu$, let $\bar{\mu}$ satisfy the policy improvement equation

$$\bar{\mu}(x) \in \arg\min_{u \in U(x)} \{g(x, u) + J_{\mu}(f(x, u))\}, \quad x \in X,$$

We have shown that

$$J_\mu(x) \geq \inf_{u \in U(x)} \{g(x, u) + J_\mu(f(x, u))\} \geq J_{\bar{\mu}}(x), \quad x \in X;$$  \hspace{1cm} (4.82)

cf. Eq. (4.30). Using $\mu^k$ and $\mu^{k+1}$ in place of $\mu$ and $\bar{\mu}$, we see that the sequence $\{J_{\mu^k}\}$ generated by PI converges monotonically to some function $J_\infty \in E^+(X)$, i.e., $J_{\mu^k} \downarrow J_\infty$. Moreover, from Eq. (4.82) we have

$$J_\infty(x) \geq \inf_{u \in U(x)} \{g(x, u) + J_\infty(f(x, u))\}, \quad x \in X,$$

as well as

$$g(x, u) + J_{\mu^k}(f(x, u)) \geq J_\infty(x), \quad x \in X, \ u \in U(x).$$

We now take the limit in the second relation as $k \to \infty$, then the infimum over $u \in U(x)$, and then combine with the first relation, to obtain

$$J_\infty(x) = \inf_{u \in U(x)} \{g(x, u) + J_\infty(f(x, u))\}, \quad x \in X.$$

Thus $J_\infty$ is a solution of Bellman’s equation, satisfying $J_\infty \geq J^*$ (since $J_{\mu^k} \geq J^*$ for all $k$) and $J_\infty \in J$ (since $J_{\mu^k} \in J$), so by Prop. 4.5.5(a), it must satisfy $J_\infty = J^*$. Q.E.D.
A Perturbed Version of Policy Iteration

We now consider PI algorithms without the condition $J^* = J^p$. We provide a version of the PI algorithm that uses the forcing function $p$ and generates a sequence $\{\mu^k\}$ of $p$-stable policies such that $J_{\mu^k} \to J_p$. Related algorithms were given in Sections 3.4 and 3.5.1. In this section, the forcing function $p$ is kept fixed, and to simplify notation, we abbreviate $J_{\mu,p,\delta}$ with $J_{\mu,\delta}$.

The following assumption requires that the algorithm generates $p$-stable policies exclusively, which can be quite restrictive. For example it is not satisfied for the problem of Example 4.5.3.

Assumption 4.5.1: For each $\delta > 0$ there exists at least one $p$-stable stationary policy $\mu$ such that $J_{\mu,\delta} \in S_p$. Moreover, given a $p$-stable stationary policy $\mu$ and a scalar $\delta > 0$, every stationary policy $\mu$ such that $\mu(x) \in \arg \min_{u \in U(x)} \{g(x, u) + J_{\mu,\delta}(f(x, u))\}$, $\forall x \in X$, (4.83) is $p$-stable, and at least one such policy exists.

The perturbed version of the PI algorithm is defined as follows. Let $\{\delta_k\}$ be a positive sequence with $\delta_k \downarrow 0$, and let $\mu^0$ be a $p$-stable policy that satisfies $J_{\mu^0,\delta_0} \in S_p$. One possibility is that $\mu^0$ is an optimal policy for the $\delta_0$-perturbed problem (cf. the discussion preceding Prop. 4.5.3). At iteration $k$, we have a $p$-stable policy $\mu^k$, and we generate a $p$-stable policy $\mu^{k+1}$ according to

$$\mu^{k+1}(x) \in \arg \min_{u \in U(x)} \{g(x, u) + J_{\mu^k,\delta_k}(f(x, u))\}, \quad x \in X, \quad (4.84)$$

Note that by Assumption 4.5.1 the algorithm is well-defined, and is guaranteed to generate a sequence of $p$-stable stationary policies.

We will use for all policies $\mu$ and scalars $\delta > 0$ the mappings $T_\mu : \mathcal{E}^+(X) \mapsto \mathcal{E}^+(X)$ and $T_{\mu,\delta} : \mathcal{E}^+(X) \mapsto \mathcal{E}^+(X)$ by

$$(T_\mu J)(x) = g(x, \mu(x)) + J(f(x, \mu(x))), \quad x \in X,$$

$$(T_{\mu,\delta} J)(x) = g(x, \mu(x)) + \delta p(x) + J(f(x, \mu(x))), \quad x \in X,$$

and the mapping $T : \mathcal{E}^+(X) \mapsto \mathcal{E}^+(X)$ given by

$$(T J)(x) = \inf_{u \in U(x)} \{g(x, u) + J(f(x, u))\}, \quad x \in X.$$

We have the following proposition.
Proposition 4.5.7: Let Assumption 4.5.1 hold. Then for a sequence of $p$-stable policies $\{\mu^k\}$ generated by the perturbed PI algorithm (4.84), we have $J_{\mu^k, \delta_k} \downarrow \hat{J}$ and $J_{\mu^k} \to \hat{J}$.

Proof: The algorithm definition (4.84) implies that for all integer $m \geq 1$ we have for all $x_0 \in X$,

$$J_{\mu^k, \delta_k}(x_0) \geq (T J_{\mu^k, \delta_k})(x_0) + \delta_k p(x_0)$$

$$= (T_{\mu^k+1, \delta_k} J_{\mu^k, \delta_k})(x_0)$$

$$\geq (T_{\mu^k+1, \delta_k}^m J_{\mu^k, \delta_k})(x_0)$$

$$\geq (T_{\mu^k+1, \delta_k}^m \bar{J})(x_0),$$

where $\bar{J}$ is the identically zero function $[\bar{J}(x) \equiv 0]$. From this relation we obtain

$$J_{\mu^k, \delta_k}(x_0) \geq \lim_{m \to \infty} (T_{\mu^k+1, \delta_k}^m \bar{J})(x_0)$$

$$= \lim_{m \to \infty} \left\{ \sum_{t=0}^{m-1} (g(x_t, \mu_t^{k+1}) + \delta_k p(x_t)) \right\}$$

$$\geq J_{\mu^{k+1}, \delta_{k+1}}(x_0),$$

as well as

$$J_{\mu^k, \delta_k}(x_0) \geq (T J_{\mu^k, \delta_k})(x_0) + \delta_k p(x_0) \geq J_{\mu^{k+1}, \delta_{k+1}}(x_0).$$

It follows that $\{J_{\mu^k, \delta_k}\}$ is monotonically nonincreasing, so that $J_{\mu^k, \delta_k} \downarrow J_\infty$ for some $J_\infty$, and

$$\lim_{k \to \infty} T J_{\mu^k, \delta_k} = J_\infty. \quad (4.85)$$

We also have, using the fact $J_\infty \leq J_{\mu^k, \delta_k}$,

$$\inf_{u \in U(x)} \{ g(x, u) + J_\infty(f(x, u)) \} \leq \lim_{k \to \infty} \inf_{u \in U(x)} \{ g(x, u) + J_{\mu^k, \delta_k}(f(x, u)) \}$$

$$\leq \inf_{u \in U(x)} \lim_{k \to \infty} \{ g(x, u) + J_{\mu^k, \delta_k}(f(x, u)) \}$$

$$= \inf_{u \in U(x)} \left\{ g(x, u) + \lim_{k \to \infty} J_{\mu^k, \delta_k}(f(x, u)) \right\}$$

$$= \inf_{u \in U(x)} \left\{ g(x, u) + J_\infty(f(x, u)) \right\}. $$

Thus equality holds throughout above, so that

$$\lim_{k \to \infty} T J_{\mu^k, \delta_k} = TJ_\infty.$$
Combining this with Eq. (4.85), we obtain $J_\infty = T J_\infty$, i.e., $J_\infty$ solves Bellman’s equation. We also note that $J_\infty \leq J_{\mu^0, \delta_0}$ and that $J_{\mu^0, \delta_0} \in S_p$ by assumption, so that $J_\infty \in S_p$. By Prop. 4.5.4(a), it follows that $J_\infty = \hat{J}_p$. Q.E.D.

Note that despite the fact $J_{\mu^k} \to \hat{J}_p$, the generated sequence $\{\mu^k\}$ may exhibit some serious pathologies in the limit. In particular, if $U$ is a metric space and $\{\mu^k\}_K$ is a subsequence of policies that converges to some $\mu$, in the sense that

$$\lim_{k \to \infty, k \in K} \mu^k(x) = \mu(x), \quad \forall x \in X,$$

it does not follow that $\mu$ is $p$-stable. In fact it is possible to construct examples where the generated sequence of $p$-stable policies $\{\mu^k\}$ satisfies $\lim_{k \to \infty} J_{\mu^k} = \hat{J}_p = J^*$, yet $\{\mu^k\}$ may converge to a $p$-unstable policy whose cost function is strictly larger than $\hat{J}_p$. Example 2.1 of the paper [BeY16] provides an instance of a stochastic shortest path problem with two states, in addition to the termination state, where this occurs.

**An Optimistic Policy Iteration Method**

Let us consider an optimistic variant of PI, where policies are evaluated inexactly, with a finite number of VIs. We use a fixed forcing function $p$. We will show that the algorithm can be used to compute $\hat{J}_p$, the restricted optimal cost function over the $p$-stable policies. The algorithm generates a sequence $\{J_k, \mu^k\}$ according to

$$T_{\mu^k} J_k = T J_k, \quad J_{k+1} = T_{\mu^k} J_k, \quad k = 0, 1, \ldots, \quad (4.86)$$

where $m_k$ is a positive integer for each $k$. We assume that a policy $\mu^k$ satisfying $T_{\mu^k} J_k = T J_k$ can be found for all $k$, but it need not be $p$-stable. However, the algorithm requires that

$$J_0 \in J, \quad J_0 \geq T J_0, \quad J_0 \in W_p. \quad (4.87)$$

This may be a restrictive assumption. We have the following proposition.

**Proposition 4.5.8: (Convergence of Optimistic PI)** Assume that there exists at least one $p$-stable policy $\pi \in \Pi_p$, and that $J_0$ satisfies Eq. (4.87). Then a sequence $\{J_k\}$ generated by the optimistic PI algorithm (4.86) belongs to $W_p$ and satisfies $J_k \downarrow \hat{J}_p$.

**Proof:** Since $J_0 \geq \hat{J}_p$ and $\hat{J}_p = T \hat{J}_p$ [cf. Prop. 4.5.5(a)], all operations on any of the functions $J_k$ with $T_{\mu^k}$ or $T$ maintain the inequality $J_k \geq \hat{J}_p$.
for all \( k \), so that \( J_k \in \mathcal{W}_p \) for all \( k \). Also the conditions \( J_0 \geq T J_0 \) and 
\[
T_{\mu^k}J_k = TJ_k
\]
imply that 
\[
J_0 = J_1 \geq T^m_0 J_0 = T_0 J_1 \geq TJ_1 = T_1 J_1 \geq \cdots \geq J_2,
\]
and continuing similarly,
\[
J_k \geq TJ_k \geq J_{k+1}, \quad k = 0, 1, \ldots
\]
Thus \( J_k \downarrow J_\infty \) for some \( J_\infty \), which must satisfy \( J_\infty \geq \hat{J}_p \), and hence belong to \( \mathcal{W}_p \). By taking limit as \( k \to \infty \) in Eq. (4.89) and using an argument similar to the one in the proof of Prop. 4.5.7, it follows that \( J_\infty = TJ_\infty \).

By Prop. 4.5.5(a), this implies that \( J_\infty \leq \hat{J}_p \). Together with the inequality \( J_\infty \geq \hat{J}_p \) shown earlier, this proves that \( J_\infty = \hat{J}_p \). Q.E.D.

As an example, for the shortest path problem of Example 4.5.3, the reader may verify that for the case where \( p(x) = 1 \), for \( x = 1 \), the optimistic PI algorithm converges in a single iteration to
\[
\hat{J}_p(x) = \begin{cases} 
1 & \text{if } x = 1, \\
0 & \text{if } x = 0,
\end{cases}
\]
provided that \( J_0 \in \mathcal{W}_p = \{ J \mid J(1) \geq 1, \ J(0) = 0 \} \). For other starting functions \( J_0 \), the algorithm converges in a single iteration to the function
\[
J_\infty(1) = \min \{ 1, J_0(1) \}, \quad J_\infty(0) = 0.
\]
All functions \( J_\infty \) of the form above are solutions of Bellman’s equation, but only \( \hat{J}_p \) is restricted optimal.

### 4.6 INFINITE SPACES STOCHASTIC SHORTEST PATH PROBLEMS

In this section we consider a stochastic discrete-time infinite horizon optimal control problem involving the system
\[
x_{k+1} = f(x_k, u_k, w_k), \quad k = 0, 1, \ldots
\]
where \( x_k \) and \( u_k \) are the state and control at stage \( k \), which belong to sets \( X \) and \( U \), \( w_k \) is a random disturbance that takes values in a countable set \( W \) with given probability distribution \( P(w_k \mid x_k, u_k) \), and \( f : X \times U \times W \to X \) is a given function. The state and control spaces \( X \) and \( U \) are arbitrary, but we assume that \( W \) is countable to bypass complex measurability issues in the choice of control (see [BeS78]). The control \( u_k \) must be chosen from a constraint set \( U(x_k) \subset U \) that may depend on the current state.
The cost for the \( k \)th stage, \( g(x_k, u_k, w_k) \), is assumed real-valued and nonnegative:

\[
0 \leq g(x_k, u_k, w_k) < \infty, \quad \forall \, x_k \in X, \, u_k \in U(x_k), \, w_k \in W, \, k = 0, 1, \ldots.
\]

We assume that \( X \) contains a special cost-free and absorbing state \( t \), referred to as the destination:

\[
f(t, u, w) = t, \quad g(t, u, w) = 0, \quad \forall \, u \in U(t), \, w \in W.
\]

Given an initial state \( x_0 \), a policy \( \pi = \{\mu_0, \mu_1, \ldots\} \) when applied to the system (4.90), generates a random sequence of state-control pairs \((x_k, \mu_k(x_k))\), \( k = 0, 1, \ldots \), with cost

\[
J_\pi(x_0) = \sum_{k=0}^{\infty} E^\pi_{x_0}\left\{g(x_k, \mu_k(x_k), w_k)\right\}, \quad x_0 \in X,
\]

where \( E^\pi_{x_0}\{\cdot\} \) denotes expectation with respect to the probability measure corresponding to initial state \( x_0 \) and policy \( \pi \). We view \( J_\pi \) as a function over \( X \), and we refer to it as the cost function of \( \pi \). For a stationary policy \( \mu \), the corresponding cost function is denoted by \( J_\mu \). The optimal cost function is defined as

\[
J^*(x) = \inf_{\pi \in \Pi} J_\pi(x), \quad x \in X,
\]

and a policy \( \pi^* \) is said to be optimal if \( J_{\pi^*}(x) = J^*(x) \) for all \( x \in X \). We are interested in the problem of finding \( J^* \) and an optimal policy. This is a special case of stochastic shortest path problem (SSP problem for short), where the cost per stage is nonnegative, but the state and control spaces are arbitrary.

We denote by \( E^+(X) \) the set of functions \( J : X \rightarrow [0, \infty] \). In our analysis, we will use the set of functions

\[
\mathcal{J} = \{ J \in E^+(X) \mid J(t) = 0 \}.
\]

Since \( t \) is cost-free and absorbing, this set contains the cost functions \( J_\pi \) of all \( \pi \in \Pi \), as well as \( J^* \).

The optimal cost function \( J^* \) satisfies the Bellman equation

\[
J(x) = \inf_{u \in U(x)} E\left\{g(x, u, w) + J(f(x, u, w))\right\}, \quad x \in X,
\]

where the expected value is with respect to the distribution \( P(w \mid x, u) \). Moreover, an optimal stationary policy (if it exists) may be obtained through the minimization in the right side of this equation (cf. Prop. 4.4.4). The VI algorithm starts from some function \( J_0 \in \mathcal{J} \), and generates a sequence \( \{J_k\} \subset \mathcal{J} \) according to

\[
J_{k+1}(x) = \inf_{u \in U(x)} E\left\{g(x, u, w) + J_k(f(x, u, w))\right\}, \quad x \in X, \, k = 0, 1, \ldots
\]
Proper Policies and the $\delta$-Perturbed Problem

We will now introduce a notion of proper policy with a definition that extends the one used for finite-state SSP in Chapter 3, and applies to nonstationary policies as well. Our analysis will focus primarily on the values of $J_\pi$ within the set

$$X_f = \{ x \in X \mid J^*(x) < \infty \},$$

since $J_\pi(x)$ is infinite for $x$ outside this set.

For a given state $x \in X$, a policy $\pi$ is said to be proper at $x$ if

$$J_\pi(x) < \infty, \quad \sum_{k=0}^{\infty} r_k(\pi, x) < \infty,$$

(4.95)

where $r_k(\pi, x)$ is the probability that $x_k \neq t$ when using $\pi$ and starting from $x_0 = x$. Note that the sum $\sum_{k=0}^{\infty} r_k(\pi, x)$ is the expected number of steps to reach the destination starting from $x$ and using $\pi$. We denote by $\hat{\Pi}_x$ the set of all policies that are proper at $x$, and we use the notation

$$\mathcal{C} = \{ (\pi, x) \mid \pi \in \hat{\Pi}_x \}. \quad (4.96)$$

We denote by $\hat{J}$ the corresponding restricted optimal cost function,

$$\hat{J}(x) = \inf_{(\pi, x) \in \mathcal{C}} J_\pi(x) = \inf_{\pi \in \hat{\Pi}_x} J_\pi(x), \quad x \in X.$$

Finally we denote by $\hat{X}$ the effective domain of $\hat{J}$, i.e.,

$$\hat{X} = \{ x \in X \mid \hat{J}(x) < \infty \}. \quad (4.97)$$

Note that $\hat{X}$ is the set of all $x$ such that $\hat{\Pi}_x$ is nonempty.

For any $\delta > 0$, let us consider the $\delta$-perturbed optimal control problem. This is the same problem as the original, except that the cost per stage is changed to

$$g(x, u, w) + \delta, \quad \forall x \neq t,$$

while $g(x, u, w)$ is left unchanged at 0 when $x = t$. Thus $t$ is still cost-free as well as absorbing in the $\delta$-perturbed problem. The $\delta$-perturbed cost function of a policy $\pi$ is denoted by $J_{\pi,\delta}$ and is given by

$$J_{\pi,\delta}(x) = J_\pi(x) + \delta \sum_{k=0}^{\infty} r_k(\pi, x). \quad (4.98)$$

We denote by $\hat{J}_\delta$ the optimal cost function of the $\delta$-perturbed problem, i.e.,

$$\hat{J}_\delta(x) = \inf_{\pi \in \Pi} J_{\pi,\delta}(x).$$

The following proposition relates the $\delta$-perturbed problem with proper policies.
Proposition 4.6.1:
(a) A policy π is proper at a state x ∈ X if and only if J_{π,δ}(x) < ∞ for all δ > 0.
(b) We have J_δ(x) < ∞ for all δ > 0 if and only if x ∈ ˆX.
(c) For every ǫ > 0, a policy π_ǫ that is ǫ-optimal for the δ-perturbed problem is proper at all x ∈ ˆX.

Proof: (a) Follows from Eq. (4.98) and the definition (4.95) of a proper policy.
(b) If x ∈ ˆX there exists a policy π that is proper at x, and by part (a), J_δ(x) ≤ J_{π,δ}(x) < ∞ for all δ > 0. Conversely, if J_δ(x) < ∞, there exists π such that J_{π,δ}(x) < ∞, implying [by part (a)] that π ∈ ˆΠ_x, so that x ∈ ˆX.
(c) We have J_{π_ǫ,δ}(x) ≤ J_δ(x) + ǫ for all x ∈ X. Hence J_{π_ǫ,δ}(x) < ∞ for all x ∈ ˆX, implying by part (a) that π_ǫ is proper at all x ∈ ˆX. Q.E.D.

The next proposition shows that the cost function J_δ of the δ-perturbed problem can be used to approximate J.

Proposition 4.6.2: We have lim_{δ↓0} J_δ(x) = J(x) for all x ∈ X. Moreover, for any ǫ > 0, a policy π_ǫ that is ǫ-optimal for the δ-perturbed problem is ǫ-optimal within the class of proper policies, i.e.
J_{π_ǫ}(x) ≤ J(x) + ǫ, ∀ x ∈ X.

Proof: Let π_ǫ be a policy that is ǫ-optimal for the δ-perturbed problem, and is also proper at all x ∈ ˆX [cf. Prop. 4.6.1(c)]. By using Eq. (4.98), we have for all δ > 0, ǫ > 0, x ∈ ˆX, and π ∈ ˆΠ_x,
J(x) − ǫ ≤ J_{π_ǫ}(x) − ǫ ≤ J_{π_ǫ,δ}(x) − ǫ ≤ J_δ(x) ≤ J_{π,δ}(x) = J(π(x) + w_{π,δ}(x),
where
w_{π,δ}(x) = δ \sum_{k=0}^{\infty} r_k(π, x) < ∞, ∀ x ∈ ˆX.

By taking the limit as ǫ ↓ 0, we obtain for all δ > 0 and π ∈ ˆΠ_x,
J(x) ≤ J_δ(x) ≤ J(π(x) + w_{π,δ}(x), ∀ x ∈ ˆΠ_x.
We have \( \lim_{\delta \downarrow 0} w_{\pi, \delta}(x) = 0 \) for all \( x \in \hat{X} \) and \( \pi \in \hat{\Pi}_x \), so by taking the limit as \( \delta \downarrow 0 \) and then the infimum over all \( \pi \in \hat{\Pi}_x \),

\[
\hat{J}(x) \leq \lim_{\delta \downarrow 0} \hat{J}_\delta(x) \leq \inf_{\pi \in \hat{\Pi}_x} J_\pi(x) = \hat{J}(x), \quad \forall x \in \hat{X},
\]

from which \( \hat{J}(x) = \lim_{\delta \downarrow 0} \hat{J}_\delta(x) \) for all \( x \in \hat{X} \). By Prop. 4.6.1(b), we also have \( \hat{J}_\delta(x) = \hat{J}(x) = \infty \) for all \( x \notin \hat{X} \), so that \( \hat{J}(x) = \lim_{\delta \downarrow 0} \hat{J}_\delta(x) \) for all \( x \in X \). We also have

\[
J_\pi(x) \leq J_{\pi, \delta}(x) \leq \hat{J}_\delta(x) + \epsilon \leq J_\pi(x) + \delta \sum_{k=0}^{\infty} r(\pi, x) + \epsilon, \quad \forall x \in \hat{X}, \pi \in \hat{\Pi}_x.
\]

By taking the limit as \( \delta \downarrow 0 \), we obtain

\[
J_{\pi, \epsilon}(x) \leq J_\pi(x) + \epsilon, \quad \forall x \in \hat{X}, \pi \in \hat{\Pi}_x.
\]

By taking the infimum over \( \pi \in \hat{\Pi}_x \), it follows that \( J_{\pi, \epsilon}(x) \leq \hat{J}(x) + \epsilon \) for all \( x \in \hat{X} \), which combined with the fact \( J_\pi(x) = J(x) = \infty \) for all \( x \notin \hat{X} \), yields the result. Q.E.D.

**Main Results**

By Prop. 4.4.4(a), \( \hat{J}_\delta \) solves Bellman’s equation for the \( \delta \)-perturbed problem, while by Prop. 4.6.2, \( \lim_{\delta \downarrow 0} \hat{J}_\delta(x) = \hat{J}(x) \). This suggests that \( \hat{J} \) solves the unperturbed Bellman equation, which is the “limit” as \( \delta \downarrow 0 \) of the \( \delta \)-perturbed version. Indeed we will show a stronger result, namely that \( \hat{J} \) is the unique solution of Bellman’s equation within the set of functions

\[
\hat{W} = \left\{ J \in J \mid J \leq \hat{J}, \ E^x_{\pi_0} \{ J(x_k) \} \to 0, \forall (\pi, x_0) \in C \right\}, \quad (4.99)
\]

where

\[
C = \{ (\pi, x) \mid \pi \in \hat{\Pi}_x \}
\]

[cf. Eq. (4.96)], \( E^x_{\pi_0} \{ \cdot \} \) denotes expected value with respect to the probability measure corresponding to initial state \( x_0 \) under policy \( \pi \), and \( E^x_{\pi} \{ J(x_k) \} \) denotes the expected value of the function \( J \) along the sequence \( \{ x_k \} \) generated starting from \( x_0 \) and using \( \pi \). The functions in \( \hat{W} \) are the ones whose expected value is decreasing to 0 along the trajectories generated by the proper policies, so they may be interpreted as a type of Lyapunov functions.

Given a policy \( \pi = \{ \mu_0, \mu_1, \ldots \} \), we denote by \( \pi_k \) the policy

\[
\pi_k = \{ \mu_k, \mu_{k+1}, \ldots \}. \quad (4.100)
\]
We first show a preliminary result.

**Proposition 4.6.3:**  
(a) For all pairs \((\pi, x_0) \in C\) and \(k = 0, 1, \ldots\), we have
\[
0 \leq E_{\pi_0}^x \{ \hat{J}(x_k) \} \leq E_{\pi_0}^x \{ J_{\pi_k}(x_k) \},
\]
where \(\pi_k\) is the policy given by Eq. (4.100).

(b) The set \(\hat{\mathcal{W}}\) of Eq. (4.99) contains \(\hat{J}\), as well as all functions \(J\) satisfying \(0 \leq J \leq h(\hat{J})\) for some function \(h : X \mapsto X\) with \(h(J) \to 0\) as \(J \to 0\).

**Proof:** (a) For any pair \((\pi, x_0) \in C\) and \(\delta > 0\), we have
\[
J_{\pi,\delta}(x_0) = E_{\pi_0}^x \left\{ J_{\pi_k,\delta}(x_k) + k\delta + \sum_{m=0}^{k-1} g(x_m, \mu_m(x_m), w_m) \right\}.
\]
Since \(J_{\pi,\delta}(x_0) < \infty\) [cf. Prop. 4.6.1(a)], it follows that \(E_{\pi_0}^x \{ J_{\pi_k,\delta}(x_k) \} < \infty\). Hence for all \(x_k\) that can be reached with positive probability using \(\pi\) and starting from \(x_0\), we have \(J_{\pi_k,\delta}(x_k) < \infty\), implying [by Prop. 4.6.1(a)] that \((\pi_k, x_k) \in C\) and hence \(\hat{J}(x_k) \leq J_{\pi_k}(x_k)\). By applying \(E_{\pi_0}^x \{ \cdot \}\) to this last inequality, the result follows.

(b) We have for all \((\pi, x_0) \in C\),
\[
J_{\pi}(x_0) = E_{\pi_0}^x \left\{ g(x_0, \mu_0(x_0), w_0) \right\} + E_{\pi_0}^x \{ J_{\pi_1}(x_1) \},
\]
and more generally, for all \(m = 0, 1, \ldots\),
\[
E_{\pi_0}^x \{ J_{\pi_m}(x_m) \} = E_{\pi_0}^x \left\{ g(x_m, \mu_m(x_m), w_m) \right\} + E_{\pi_0}^x \{ J_{\pi_{m+1}}(x_{m+1}) \},
\]
where \(\{x_m\}\) is the sequence generated starting from \(x_0\) and using \(\pi\). Using the fact \(J_{\pi}(x_0) < \infty\), it follows that all the terms in the above relations are finite, and in particular
\[
E_{\pi_0}^x \{ J_{\pi_m}(x_m) \} < \infty, \quad \forall \ (\pi, x_0) \in C, \ m = 0, 1, \ldots.
\]
By adding Eq. (4.101) for \(m = 0, \ldots, k-1\), and canceling the finite terms \(E_{\pi_0}^x \{ J_{\pi_m}(x_m) \}\) for \(m = 1, \ldots, k-1\), we obtain for all \(k = 1, 2, \ldots\),
\[
J_{\pi}(x_0) = E_{\pi_0}^x \{ J_{\pi_k}(x_k) \} + \sum_{m=0}^{k-1} E_{\pi_0}^x \left\{ g(x_m, \mu_m(x_m), w_m) \right\}, \quad \forall \ (\pi, x_0) \in C.
\]
The rightmost term above tends to $J_\pi(x_0)$ as $k \to \infty$, so we obtain
\[
E_\pi x_0 \{ J_{x_k}(x_k) \} \to 0, \quad \forall (\pi, x_0) \in \mathcal{C}.
\]
Since by part (a), $0 \leq E_\pi x_0 \{ \hat{J}(x_k) \} \leq E_\pi x_0 \{ J_{x_k}(x_k) \}$, it follows that
\[
E_\pi x_0 \{ \hat{J}(x_k) \} \to 0, \quad \forall (\pi, x_0) \in \mathcal{C},
\]
so that $\hat{J} \in \hat{W}$. This also implies that
\[
E_\pi x_0 \{ \hat{J}(x_k) \} \to 0, \quad \forall (\pi, x_0) \in \mathcal{C},
\]
if $0 \leq J \leq h(\hat{J})$. Q.E.D.

We can now prove our main result.

**Proposition 4.6.4:**

(a) $\hat{J}$ is the unique solution of the Bellman Eq. (4.93) within the set $\hat{W}$ of Eq. (4.99).

(b) (VI Convergence) If $\{J_k\}$ is the sequence generated by the VI algorithm (4.94) starting with some $J_0 \in \hat{W}$, then $J_k \to \hat{J}$.

(c) (Optimality Condition) If $\mu$ is a stationary policy that is proper at all $x \in \hat{X}$ and
\[
\hat{\mu}(x) \in \arg \min_{u \in U(x)} E \left\{ g(x, u, w) + \hat{J} \left( f(x, u, w) \right) \right\}, \quad \forall x \in \hat{X},
\]
then $\mu$ is optimal over the set of proper policies, i.e., $J_\mu = \hat{J}$. Conversely, if $\mu$ is optimal within the set of proper policies, then it satisfies the preceding condition (4.102).

**Proof:** (a), (b) By Prop. 4.6.3(b), $\hat{J} \in \hat{W}$. We will first show that $\hat{J}$ is a solution of Bellman’s equation and then show that it is the unique solution within $\hat{W}$ by showing the convergence of VI [cf. part (b)]. Since $\hat{J}_\delta$ solves the Bellman equation for the $\delta$-perturbed problem, and $\hat{J}_\delta \geq \hat{J}$ (cf. Prop. 4.6.2), we have for all $\delta > 0$ and $x \not= t$,
\[
\hat{J}_\delta(x) = \inf_{u \in U(x)} E \left\{ g(x, u, w) + \delta + \hat{J}_\delta \left( f(x, u, w) \right) \right\}
\geq \inf_{u \in U(x)} E \left\{ g(x, u, w) + \hat{J}_\delta \left( f(x, u, w) \right) \right\}
\geq \inf_{u \in U(x)} E \left\{ g(x, u, w) + \hat{J} \left( f(x, u, w) \right) \right\}.
\]
By taking the limit as \( \delta \downarrow 0 \) and using Prop. 4.6.2, we obtain

\[
\hat{J}(x) \geq \inf_{u \in U(x)} E\{ g(x, u, w) + \hat{J}(f(x, u, w)) \}, \quad \forall x \in X. \tag{4.103}
\]

For the reverse inequality, let \( \{ \delta_m \} \) be a sequence with \( \delta_m \downarrow 0 \). We have for all \( m \), \( x \neq t \), and \( u \in U(x) \),

\[
E\{ g(x, u, w) + \delta_m + \hat{J}_{\delta_m}(f(x, u, w)) \} \geq \inf_{v \in U(x)} E\{ g(x, v, w) + \delta_m + \hat{J}_{\delta_m}(f(x, v, w)) \}
= \hat{J}_{\delta_m}(x).
\]

Taking the limit as \( m \to \infty \), and using the monotone convergence theorem (to interchange limit and expectation) and the fact \( \lim_{\delta_m \downarrow 0} \hat{J}_{\delta_m} = \hat{J} \) (cf. Prop. 4.6.2), we have

\[
E\{ g(x, u, w) + \hat{J}(f(x, u, w)) \} \geq \hat{J}(x), \quad \forall x \in X, \ u \in U(x),
\]

so that

\[
\inf_{u \in U(x)} E\{ g(x, u, w) + \hat{J}(f(x, u, w)) \} \geq \hat{J}(x), \quad \forall x \in X. \tag{4.104}
\]

By combining Eqs. (4.103) and (4.104), we see that \( \hat{J} \) is a solution of Bellman’s equation.

We will next show that \( J_k \to \hat{J} \) starting from every initial \( J_0 \in \hat{W} \) [cf. part (b)]. Indeed, for \( x_0 \in \hat{X} \) and any \( \pi = \{ \mu_0, \mu_1, \ldots \} \in \hat{\Pi}_{x_0} \), let \( \{ x_k \} \) be the generated sequence starting from \( x_0 \). Since from the definition of the VI sequence \( \{ J_k \} \) [cf. Eq. (4.94)], we have

\[
J_k(x) \leq E\{ g(x, u, w) + J_{k-1}(f(x, u, w)) \}, \quad \forall x \in X, \ u \in U(x),
\]

it follows that

\[
J_k(x_0) \leq E_{\pi_0} \{ J_0(x_k) + \sum_{m=0}^{k-1} g(x_m, \mu_m(x_m), w_m) \}.
\]

Since \( J_0 \in \hat{W} \), we have \( E_{\pi_0} \{ J_0(x_k) \} \to 0 \), so by taking the limit as \( k \to \infty \) in the preceding relation, it follows that \( \limsup_{k \to \infty} J_k(x_0) \leq J(x_0) \). By taking the infimum over all \( \pi \in \hat{\Pi}_{x_0} \), we obtain \( \limsup_{k \to \infty} J_k(x_0) \leq \hat{J}(x_0) \).

Conversely, since \( \hat{J} \leq J_0 \) and \( \hat{J} \) is a solution of Bellman’s equation (as shown earlier), it follows by induction that \( \hat{J} \leq J_k \) for all \( k \). Thus \( \hat{J}(x_0) \leq \liminf_{k \to \infty} J_k(x_0) \), implying that \( J_k(x_0) \to \hat{J}(x_0) \) for all \( x_0 \in \hat{X} \). We also
Fig. 4.6.1 Illustration of the solutions of Bellman’s equation. All solutions either lie between $J^*$ and $\hat{J}$, or they lie outside the set $\hat{W}$. The VI algorithm converges to $J^*$ starting from any $J_0 \in \hat{W}$.

have $\hat{J} \leq J_k$ for all $k$, so that $\hat{J}(x_0) = J_k(x_0) = \infty$ for all $x_0 \notin \hat{X}$. This completes the proof of part (b). Finally, since $\hat{J} \in \hat{W}$ and $\hat{J}$ is a solution of Bellman’s equation, part (b) implies the uniqueness assertion of part (a).

(c) If $\mu$ is proper at all $x \in \hat{X}$ and Eq. (4.102) holds, then

$$\hat{J}(x) = E\left\{g(x, \mu(x), w) + \hat{J}(f(x, \mu(x), w))\right\}, \quad x \in X.$$  

By Prop. 4.4.4(b), this implies that $J_\mu \leq \hat{J}$, so $\mu$ is optimal over the set of proper policies. Conversely, assume that $\mu$ is proper at all $x \in \hat{X}$ and $J_\mu = \hat{J}$. Then by Prop. 4.4.4(b), we have

$$\hat{J}(x) = E\left\{g(x, \mu(x), w) + \hat{J}(f(x, \mu(x), w))\right\}, \quad x \in X,$$

and since [by part (b)] $\hat{J}$ is a solution of Bellman’s equation,

$$\hat{J}(x) = \inf_{u \in U(x)} E\left\{g(x, u, w) + \hat{J}(f(x, u, w))\right\}, \quad x \in X.$$  

Combining the last two relations, we obtain Eq. (4.102). Q.E.D.

We illustrate Prop. 4.6.4 in Fig. 4.6.1. Suppose now that the set of proper policies is sufficient in the sense that it can achieve the same optimal cost as the set of all policies, i.e., $\hat{J} = J^*$. Then, from Prop. 4.6.4, it follows that $J^*$ is the unique solution of Bellman’s equation within $\hat{W}$, and the VI algorithm converges to $J^*$ starting from any $J_0 \in \hat{W}$. Under additional conditions, such as finiteness of $U(x)$ for all $x \in X$ [cf. Prop. 4.4.4(e)], VI converges to $J^*$ starting from any $J_0 \in J$ with

$$E_{\pi_0} \left\{ J_0(x_k) \right\} \to 0, \quad \forall (\pi, x_0) \in C.$$
The Multiplicity of Solutions of Bellman’s Equation

Let us now discuss the issue of multiplicity of solutions of Bellman’s equation within the set of functions

\[ \mathcal{J} = \{ J \in \mathcal{E}^+(X) \mid J(t) = 0 \} \].

We know from Props. 4.4.4(a) and 4.6.4(a) that \( J^* \) and \( \hat{J} \) are solutions, and that all other solutions \( J \) must satisfy either \( J^* \leq J \leq \hat{J} \) or \( J \notin \hat{W} \).

In the special case of a deterministic problem (one where the disturbance \( w_k \) takes a single value), it was shown in Section 4.5 that \( \hat{J} \) is the largest solution of Bellman’s equation within \( \mathcal{J} \), so all solutions \( J \in \mathcal{J} \) satisfy \( J^* \leq J \leq \hat{J} \). Moreover, it was shown through examples that there can be any number of solutions that lie between \( J^* \) and \( \hat{J} \): a finite number, an infinite number, or none at all.

In stochastic problems, however, the situation is strikingly different. There can be an infinite number of solutions \( J \in \mathcal{J} \) such that \( J \neq \hat{J} \) and \( J \geq \hat{J} \), as shown by the following example. Of course, by Prop. 4.6.4(a), these solutions must lie outside \( \hat{W} \).

**Example 4.6.1**

Let \( X = \mathbb{R} \), \( t = 0 \), and assume that there is only one control at each state, and hence a single policy \( \pi \). The disturbance \( w_k \) takes two values: 1 and 0 with probabilities \( \alpha \in (0, 1) \) and \( 1 - \alpha \), respectively. The system equation is

\[ x_{k+1} = \frac{w_k x_k}{\alpha} \]

and there is no cost at each state and stage:

\[ g(x, u, w) \equiv 0. \]

Thus from state \( x_k \) we move to state \( x_k/\alpha \) with probability \( \alpha \) and to the termination state \( t = 0 \) with probability \( 1 - \alpha \). Here, the only admissible policy is proper, and we have

\[ J^*(x) = \hat{J}(x) = 0, \quad \forall x \in X. \]

Bellman’s equation has the form

\[ J(x) = (1 - \alpha)J(0) + \alpha J \left( \frac{x}{\alpha} \right), \quad x \in X, \]

and has an infinite number of solutions within \( \mathcal{J} \) in addition to \( J^* \) and \( \hat{J} \): any positively homogeneous function, such as, for example, \( J(x) = \gamma |x| \), \( \gamma > 0 \), is a solution. Consistently with Prop. 4.6.4(a), none of these solutions belongs to \( \hat{W} \), since \( x_k \) is either equal to \( x_0/\alpha^k \) (with probability \( \alpha^k \)) or equal to 0 (with probability \( 1 - \alpha^k \)). For example, in the case of \( J(x) = \gamma |x| \), we have

\[ E_{x_0}^{\pi} \{ J(x_k) \} = \alpha^k \gamma \left| \frac{x_0}{\alpha^k} \right| = \gamma |x_0|, \quad \forall k \geq 0, \]

so \( J(x_k) \) does not converge to 0, unless \( x_0 = 0 \). Moreover, none of these solutions seems to be significant in some discernible way.
The Case of Bounded Cost per Stage

We will now consider the case where the cost per stage \( g \) is bounded over \( X \times U \times W \), i.e.,

\[
\sup_{(x,u,w) \in X \times U \times W} g(x, u, w) < \infty.
\]

We will show that the function \( \hat{J} \) is the maximal solution of Bellman’s equation if we restrict ourselves to functions that are bounded within the effective domain \( \hat{X} \) of \( \hat{J} \) [cf. Eq. (4.97)].

We denote by \( \mathcal{B} \) the set of functions

\[
\mathcal{B} = \left\{ J \in \mathcal{J} \mid \sup_{x \in \hat{X}} J(x) < \infty \right\}.
\]

From the definition (4.95) of properness, we have

\[
J_{\pi}(x_0) \leq \left( \sup_{(x,u,w) \in X \times U \times W} g(x, u, w) \right) \cdot \sum_{k=0}^{\infty} r_k(\pi, x_0) < \infty, \quad \forall \pi \in \hat{\Pi}_{x_0},
\]

while for all \( J \in \mathcal{B} \), we have

\[
E^{x_0} \{ J(x_k) \} \leq \left( \sup_{x \in \hat{X}} J(x) \right) \cdot r_k(\pi, x_0) \to 0, \quad \forall \pi \in \hat{\Pi}_{x_0}.
\]

It follows that the set

\[
\hat{W}_b = \{ J \in \mathcal{J} \mid \hat{J} \leq J, \ J \in \mathcal{B} \}
\]

is contained in \( \hat{W} \), while the function \( \hat{J} \) belongs to \( \hat{W}_b \).

By applying Prop. 4.6.4, we see that \( \hat{J} \) is the maximal solution of Bellman’s equation within the set \( \hat{W}_b \). It follows that \( \hat{J} \) is the maximal solution of Bellman’s equation within \( \mathcal{B} \). Moreover, if \( J^* = \hat{J} \), then \( J^* \) is the unique solution of Bellman’s equation within \( \mathcal{B} \). This result is consistent with Example 4.6.1, where \( J^* \) and \( \hat{J} \) are equal and bounded, and all the additional solutions of Bellman’s equation are unbounded.

4.7 NOTES, SOURCES, AND EXERCISES

The analysis of the monotone increasing and decreasing abstract DP models of Section 4.3 was first given in the author’s papers [Ber75], [Ber77]. This analysis and its finite horizon counterpart of Section 4.2 were also presented in the monograph by Bertsekas and Shreve [BeS78].
Important examples of noncontractive infinite horizon models are the classical positive DP problems, analyzed by Blackwell [Bla65], and by Dubins and Savage [DuS65], and the negative DP problems analyzed in Strauch [Str66] (and also in Strauch’s Ph.D. thesis, written under the supervision of Blackwell). The monograph by Bertsekas and Shreve [BeS78] provides a detailed treatment of these two models, which also resolves the associated measurability questions. The paper by Yu and Bertsekas [YuB15] provides a more recent analysis that addresses some issues regarding the VI and PI algorithms that were left unresolved in the monograph [BeS78]. A simpler textbook treatment, which bypasses the measurability questions, is given in the author’s textbook [Ber12a, Chapter 4].

The compactness condition that guarantees convergence of VI to $J^*$ under Assumption I (cf. Prop. 4.3.14) was obtained by the author in [Ber73] for reachability problems (see Exercise 4.5), and in [Ber75], [Ber77] for negative DP models; also by Schal [Sch75] and Whittle [Whi80]. A more refined analysis of the question of convergence of VI to $J^*$ is possible. This analysis provides a necessary and sufficient condition for convergence and improves over the compactness condition of Prop. 4.3.14. In particular, the following characterization is shown in [Ber77], Prop. 11 (see also [BeS78], Prop. 5.9):

For a set $C \subset X \times U \times \mathbb{R}$, let $\Pi(C)$ be the projection of $C$ onto $X \times \mathbb{R}$:

$$\Pi(C) = \{ (x, \lambda) \mid (x, u, \lambda) \in C \text{ for some } u \in U(x) \},$$

and denote also

$$\overline{\Pi(C)} = \{ (x, \lambda) \mid \lambda_m \to \lambda \text{ for some sequence } \{\lambda_m\} \text{ with } \{ (x, \lambda_m) \} \subset C \}.$$

Consider the sets $C_k \subset X \times U \times \mathbb{R}$ given by

$$C_k = \{ (x, \lambda) \mid H(x, u, T^k \bar{J}) \leq \lambda, x \in X, u \in U(x) \}, \quad k = 0, 1, \ldots.$$

Then under Assumption I we have $T^k \bar{J} \to J^*$ if and only if

$$\overline{\Pi\left( \bigcap_{k=0}^{\infty} C_k \right)} = \bigcap_{k=0}^{\infty} \Pi(C_k).$$

Moreover we have $T^k \bar{J} \to J^*$ and in addition there exists an optimal stationary policy if and only if

$$\Pi\left( \bigcap_{k=0}^{\infty} C_k \right) = \bigcap_{k=0}^{\infty} \overline{\Pi(C_k)}. \quad (4.105)$$

For a connection with Prop. 4.3.14, it can be shown that compactness of

$$U_k(x, \lambda) = \{ u \in U(x) \mid H(x, u, T^k \bar{J}) \leq \lambda \}$$

implies Eq. (4.105) (see [Ber77], Prop. 12, or [BeS78], Prop. 5.10).
Optimistic PI and $\lambda$-PI under Assumption D have not been considered prior to the first edition of this book, and the corresponding analysis of Section 4.3.3 is new. See [Bel96], [ThS10a], [ThS10b], [Ber11b], [Sch11] for analyses of $\lambda$-PI for discounted and SSP problems. The connection of $\lambda$-PI for finite-state Markovian decision problems and the proximal algorithm was established recently in the author’s paper [Ber16b]. An extension of $\lambda$-PI, called $\Lambda$-PI, uses a different parameter $\lambda_i$ for each state $i$, and is discussed in Section 5 of the paper by Yu and Bertsekas [YuB12]. Using the analysis of [Ber16b], $\Lambda$-PI may be viewed as a diagonally scaled version of the proximal algorithm, i.e., one that uses a different penalty parameter for each proximal term.

The definition of regularity of Section 4.4 was introduced in the author’s paper [Ber15]. Section 4.4.2 makes use of ideas from the paper by Yu and Bertsekas [YuB15], which considers stochastic optimal control problems in a more general setting where universally measurable policies are allowed. The analyses of discounted models of Section 4.4.3, and convergent models of Section 4.4.4. under the condition $J^*(x) \geq J(x) > -\infty$ for all $x \in X$, are new. A survey of stochastic optimal control problems under convergence conditions that are more general than the ones considered here is given by Feinberg [Fei02]. A recent analysis of convergent models for stochastic optimal control, which illustrates the broad range of pathological behaviors that can occur without the condition $J^* \geq J$, is given in the paper by Yu [Yu15].

The issue of the connection of optimality with stability (and also with controllability and observability) was raised by Kalman [Kal60] in the context of linear-quadratic problems. The set of solutions of the Riccati equation has been extensively investigated starting with the papers by Willems [Wil71] and Kucera [Kuc72], [Kuc73], which were followed up by several other works; see the book by Lancaster and Rodman [LaR95] for a comprehensive treatment. In these works, the “largest” solution of the Riccati equation is referred to as the “stabilizing” solution, and the stability of the corresponding policy is shown, although the author could not find an explicit statement regarding the optimality of this policy within the class of all linear stable policies. Also the lines of analysis of these works are tied to the structure of the linear-quadratic problem and are unrelated to the analysis of Section 4.5, which follows the author’s paper [Ber17a].

Proper policies for infinite-state SSP problems have been considered earlier, notably in the works of Pliska [Pli78], and James and Collins [JaC06], where they are called transient. There are a few differences between the frameworks of [Pli78], [JaC06] and Section 4.6, which impact on the results obtained. In particular, the paper [Pli78] uses a similar (but not identical) definition of properness to the one of Section 4.6, but assumes that all policies are proper, that $g$ is bounded, and that $J^*$ is real-valued. The paper [JaC06] uses the properness definition of [Pli78], and extends the analysis of [BeT91] from finite state space to infinite state space (ad-
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dressing also measurability issues). Moreover, [JaC06] allows the cost per stage $g$ to take both positive and negative values. However, [JaC06] uses assumptions that guarantee that improper policies cannot be optimal and that $J^* = J$, while $J^*$ is real-valued. Our analysis of Section 4.6 is due to the author’s paper [Ber17b], and is most closely related to the SSP analysis under the weak conditions of Section 3.5.1, where we assumed that the state space is finite, but allowed $g$ to take both positive and negative values. The extension of our results of Section 4.6 to SSP problems where $g$ takes both positive and negative values may be possible, but our line of analysis relies strongly on the nonnegativity of $g$ and cannot be extended without major modifications.

**EXERCISES**

4.1 (Example of Nonexistence of an Optimal Policy Under D)

This is an example of a deterministic stopping problem where Assumption D holds, and an optimal policy does not exist, even though only two controls are available at each state (stop and continue). The state space is $X = \{1, 2, \ldots \}$. Continuation from state $x$ leads to state $x + 1$ with certainty and no cost, while the stopping cost is $-1 + (1/x)$, so that there is an incentive to delay stopping at every state. Here for all $x$, $J(x) = 0$, and

$$H(x, u, J) = \begin{cases} J(x + 1) & \text{if } u = \text{continue}, \\ -1 + (1/x) & \text{if } u = \text{stop}. \end{cases}$$

Show that $J^*(x) = -1$ for all $x$, but there is no policy (stationary or not) that attains the optimal cost starting from $x$.

**Solution:** Since a cost is incurred only upon stopping, and the stopping cost is greater than -1, we have $J_\mu(x) > -1$ for all $x$ and $\mu$. On the other hand, starting from any state $x$ and stopping at $x + n$ yields a cost $-1 + \frac{1}{x + n}$, so by taking $n$ sufficiently large, we can attain a cost arbitrarily close to -1. Thus $J^*(x) = -1$ for all $x$, but no policy can attain this optimal cost.

4.2 (Counterexample for Optimality Condition Under D)

For the problem of Exercise 4.1, show that the policy $\mu$ that never stops is nonoptimal and satisfies $T_\mu J^* = TJ^*$.

**Solution:** We have $J^*(x) = -1$ and $J_\mu(x) = 0$ for all $x \in X$. Thus $\mu$ is nonoptimal, yet attains the minimum in Bellman’s equation

$$J^*(x) = \min \left\{ J^*(x + 1), -1 + \frac{1}{x} \right\}$$
for all $x$.

**4.3 (Counterexample for Optimality Condition Under I)**

Let 

$$X = \mathbb{R}, \quad U(x) \equiv (0, 1], \quad J(x) \equiv 0,$$

$$H(x, u, J) = |x| + J(u x), \quad \forall x \in X, u \in U(x).$$

Let $\mu(x) = 1$ for all $x \in X$. Then $J_{\mu}(x) = \infty$ if $x \neq 0$ and $J_{\mu}(0) = 0$. Verify that $T_{\mu}J_{\mu} = T J_{\mu}$. Verify also that $J^*_{\mu}(x) = |x|$, and hence $\mu$ is not optimal.

**Solution:** The verification of $T_{\mu}J_{\mu} = T J_{\mu}$ is straightforward. To show that $J^*_{\mu}(x) = |x|$, we first note that $|x|$ is a fixed point of $T$, so by Prop. 4.3.2, $J^*_{\mu}(x) \leq |x|$. Also $(T J)(x) = |x|$ for all $x$, while under Assumption I, we have $J^* \geq T J$, so $J^*(x) \geq |x|$. Hence $J^*(x) = |x|$.

**4.4 (Solution by Mathematical Programming)**

This exercise shows that under Assumptions I and D, it is possible to devise a computational method based on mathematical programming when $X = \{1, \ldots, n\}$.

(a) Under Assumption I, show that $J^*$ is the unique solution of the following optimization problem in $z = (z_1, \ldots, z_n)$:

$$\text{minimize } \sum_{i=1}^n z_i$$

subject to $z_i \geq \bar{J}(i), \quad z_i \geq \inf_{u \in U(i)} H(i, u, z), \quad i = 1, \ldots, n.$

(b) Under Assumption D, show that $J^*$ is the unique solution of the following optimization problem in $z = (z_1, \ldots, z_n)$:

$$\text{maximize } \sum_{i=1}^n z_i$$

subject to $z_i \leq \bar{J}(i), \quad z_i \leq H(i, u, z), \quad i = 1, \ldots, n, \quad u \in U(i).$

**Note:** Generally, these programs may not be linear or even convex.

**Solution:** (a) Any feasible solution $z$ of the given optimization problem satisfies $z \geq \bar{J}$ as well as $z_i \geq \inf_{u \in U(i)} H(i, u, z)$ for all $i = 1, \ldots, n$, so that $z \geq T z$. It follows from Prop. 4.4.9 that $z \geq J^*$, which implies that $J^*$ is an optimal solution of the given optimization problem. Also $J^*$ is the unique optimal solution since if $z$ is feasible and $z \neq J^*$, the inequality $z \geq J^*$ implies that $\sum_i z_i > \sum_i J^*(i)$, so $z$ cannot be optimal.

(b) Any feasible solution $z$ of the given optimization problem satisfies $z \leq \bar{J}$ as well as $z_i \leq H(i, u, z)$ for all $i = 1, \ldots, n$ and $u \in U(i)$, so that $z \leq T z$. It follows
from Prop. 4.3.6 that $z \leq J^*$, which implies that $J^*$ is an optimal solution of the given optimization problem. Similar to part (a), $J^*$ is the unique optimal solution.

4.5 (Infinite Time Reachability [Ber72])

This exercise provides an instance of an interesting problem where the mapping $H$ is naturally extended real-valued. Consider a dynamic system

$$x_{k+1} = f(x_k, u_k, w_k),$$

where $w_k$ is viewed as an uncertain disturbance that may be any point in a set $W(x_k, u_k)$ (this is known in the literature as an “unknown but bounded” disturbance, and is the basis for a worst case/minimax treatment of uncertainty in the control of uncertain dynamic systems). We introduce an abstract DP model where the objective is to find a policy that keeps the state $x_k$ of the system within a given set $X$ at all times, for all possible values of the sequence $\{w_k\}$. This is a common objective, which arises in a variety of control theory contexts, including model predictive control.

Let

$$\bar{J}(x) = \begin{cases} 0 & \text{if } x \in X, \\ \infty & \text{otherwise}, \end{cases}$$

and

$$H(x, u, J) = \begin{cases} 0 & \text{if } J(x) = 0, \ u \in U(x), \text{ and } J(f(x, u, w)) = 0, \ \forall \ w \in W(x, u), \\ \infty & \text{otherwise}. \end{cases}$$

(a) Show that Assumption I holds, and that the optimal cost function has the form

$$J^*(x) = \begin{cases} 0 & \text{if } x \in X^*, \\ \infty & \text{otherwise}, \end{cases}$$

where $X^*$ is some subset of $X$.

(b) Consider the sequence of sets $\{X_k\}$, where

$$X_k = \{x \in X \mid (T^k \bar{J})(x) = 0\}.$$  

Show that $X_{k+1} \subseteq X_k$ for all $k$, and that $X^* \subseteq \cap_{k=0}^\infty X_k$. Show also that convergence of VI (i.e., $T^k \bar{J} \to J^*$) is equivalent to $X^* = \cap_{k=0}^\infty X_k$.

(c) Show that $X^* = \cap_{k=0}^\infty X_k$ and there exists an optimal stationary policy if the sets

$$\hat{U}_k(x) = \{u \in U(x) \mid f(x, u, w) \in X_k, \ \forall \ w \in W(x, u)\}$$

are compact for all $k$ greater than some index $\tilde{k}$. Hint: Use Prop. 4.3.14.

**Solution:** Let $\hat{E}(X)$ be the subset of $\mathcal{E}(X)$ that consists of functions that take only the two values 0 and $\infty$, and for all $J \in \hat{E}(X)$ denote

$$D(J) = \{x \in X \mid J(x) = 0\}.$$
Note that for all \( J \in \hat{E}(X) \) we have \( T_\mu J \in \hat{E}(X) \), \( TJ \in \hat{E}(X) \), and that

\[
D(T_\mu J) = \{ x \in X \mid x \in D(J), f(x, \mu(x), w) \in D(J), \forall w \in W(x, \mu(x)) \},
\]

\[
D(TJ) = \cup_{\mu \in \mathcal{M}} D(T_\mu J).
\]

(a) For all \( J \in \hat{E}(X) \), we have \( D(T_\mu J) \subseteq D(J) \) and \( T_\mu J \geq J \), so condition (1) of Assumption I holds, and it is easily verified that the remaining two conditions of Assumption I also hold. We have \( J \in \hat{E}(X) \), so for any policy \( \pi = \{ \mu_0, \mu_1, \ldots \} \), we have \( T_{\mu_0} \cdots T_{\mu_k} J \in \hat{E}(X) \). It follows that \( J_\pi \), given by

\[
J_\pi = \lim_{k \to \infty} T_{\mu_0} \cdots T_{\mu_k} J,
\]

also belongs to \( \hat{E}(X) \), and the same is true for \( J^* = \inf_{x \in \Pi} J_\pi \). Thus \( J^* \) has the given form with \( D(J^*) = X^* \).

(b) Since \( \{ T^k J \} \) is monotonically nondecreasing we have \( D(T^{k+1} J) \subseteq D(T^k J) \), or equivalently \( X_{k+1} \subseteq X_k \) for all \( k \). Generally for a sequence \( \{ J_k \} \subseteq \hat{E}(X) \), if \( J_k \uparrow J \), we have \( J \in \hat{E}(X) \) and \( D(J) = \cap_{k=0}^\infty D(J_k) \). Thus convergence of VI (i.e., \( T^k J \uparrow J^* \)) is equivalent to \( D(J^*) = \cap_{k=0}^\infty D(J_k) \) or \( X^* = \cap_{k=0}^\infty X_k \).

(c) The compactness condition of Prop. 4.3.14 guarantees that \( T^k J \uparrow J^* \), or equivalently by part (b), \( X^* = \cap_{k=0}^\infty X_k \). This condition requires that the sets

\[
U_k(x, \lambda) = \{ u \in U(x) \mid H(x, u, T^k J) \leq \lambda \}
\]

are compact for every \( x \in X \), \( \lambda \in \mathbb{R} \), and for all \( k \) greater than some integer \( T \). It can be seen that \( U_k(x, \lambda) \) is equal to the set

\[
\hat{U}_k(x) = \{ u \in U(x) \mid f(x, u, w) \in X_k, \forall w \in W(x, u) \}
\]

given in the statement of the exercise.

4.6 (Exceptional Linear-Quadratic Problems)

Consider the deterministic linear-quadratic problem of Section 3.5.4 and Example 4.5.1. Assume that there is a single control variable \( u_k \), and two state variables, \( x^1_k \) and \( x^2_k \), which evolve according to

\[
x^{1}_{k+1} = \gamma x^{1}_k + bu_k, \quad x^{2}_{k+1} = x^{1}_k + x^{2}_k + u_k,
\]

where \( \gamma > 1 \). The cost of stage \( k \) is quadratic of the form

\[
g((x^{1}_k)^2 + (x^{2}_k)^2 + (u_k)^2).
\]

Consider the four cases of pairs of values \((b, q)\) where \( b \in \{0, 1\} \) and \( q \in \{0, 1\} \). For each case, use the theory of Section 4.5 to find the optimal cost function \( J^* \) and the optimal cost function over stable policies \( J^*_+ \), and to describe the convergence behavior of VI.
Solution: When \( b = 1 \) and \( q = 1 \), the classical controllability and observability conditions are satisfied, and we have \( J^* = J^+ \), while there exists an optimal policy that is linear and stable (so \( J^* \) and \( J^+ \) are real-valued and positive definite quadratic). Moreover, the VI algorithm converges to \( J^* \) starting from any \( J_0 \geq 0 \) [even extended real-valued \( J_0 \) with \( J_0(0) = 0 \)].

When \( b = 0 \) and \( q = 0 \), we clearly have \( J^*(x) \equiv 0 \), while \( J^+(x^1, x^2) = \infty \) for \( x^1 \neq 0 \), while \( J^+(0, x^2) \) is finite for all \( x^2 \), but positive for \( x^2 \neq 0 \) (since for \( x^1 = 0 \), the problem becomes essentially one-dimensional, and similar to the one of Section 3.5.4). The VI algorithm converges to \( J^+ \) starting from any positive semidefinite quadratic initial condition \( J_0 \) with \( J_0(0, x^2) = 0 \) and \( J_0 \neq J^* \).

When \( b = 0 \) and \( q = 1 \), we have \( J^* = J^+ \), but \( J^* \) and \( J^+ \) are not real-valued. In particular, since \( x^1_k \) stays constant under all policies when \( b = 0 \), we have \( J^*(x^1, x^2) = J^+(x^1, x^2) = \infty \) for \( x^1 \neq 0 \). Moreover, for an initial state with \( x^1_0 = 0 \), the problem becomes essentially a one-dimensional problem that satisfies the classical controllability and observability conditions, and we have \( J^*(0, x^2) = J^+(0, x^2) \) for all \( x^2 \). The VI algorithm takes the form

\[
J_{k+1}(0, x^2) = \min_u \left\{ (x^2)^2 + (u)^2 + J_k(0, x^2 + u) \right\},
\]

\[
J_{k+1}(x^1, x^2) = \min_u \left\{ (x^1)^2 + (x^2)^2 + (u)^2 + J_k(\gamma x^1, x^1 + x^2 + u) \right\}, \quad \text{if } x^1 \neq 0.
\]

It can be seen that the VI iterates \( J_k(0, x^2) \) evolve as in the case of a single state variable problem, where \( x^1 \) is fixed at 0. For \( x^1 \neq 0 \), the VI iterates \( J_k(x^1, x^2) \) diverge to \( \infty \).

When \( b = 1 \) and \( q = 0 \), we have \( J^*(x) \equiv 0 \), while \( 0 < J^+(x) < \infty \) for all \( x \neq 0 \). Similar to Example 4.5.1, the VI algorithm converges to \( J^+ \) starting from any initial condition \( J_0 \geq J^+ \). The functions \( J^* \) and \( J^+ \) are real-valued and satisfy Bellman’s equation, which has the form

\[
J(x^1, x^2) = \min_u \left\{ (u)^2 + J(\gamma x^1 + u, x^1 + x^2 + u) \right\}.
\]

However, Bellman’s equation has additional solutions, other than \( J^* \) and \( J^+ \). One of these is

\[
\hat{J}(x^1, x^2) = P(x^1)^2,
\]

where \( P = \gamma^2 - 1 \) (cf. the example of Section 3.5.4).