Incremental Gradient, Subgradient, and Proximal Methods for Convex Optimization

Dimitri P. Bertsekas

Laboratory for Information and Decision Systems
Massachusetts Institute of Technology

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Recall the Classical Subgradient and Proximal Algorithms

Convex Optimization Problem

$$\text{minimize} \quad f(x) \qquad \text{subject to} \qquad x \in X,$$

where $f : \Re^n \mapsto \Re$ is convex, and X is closed and convex.

Classical subgradient projection algorithm: Typical iteration

$$x_{k+1} = P_X\big(x_k - \alpha_k \tilde{\nabla} f(x_k)\big)$$

where α_k is a positive stepsize and $\tilde{\nabla}$ denotes (any) subgradient.

Classical proximal algorithm: Typical iteration

$$x_{k+1} = \arg\min_{x \in X} \left\{ f(x) + \frac{1}{2\alpha_k} \|x - x_k\|^2 \right\}$$

where α_k is a positive parameter.

- Proximal has more solid convergence properties, but requires more overhead.
- Proximal algorithm $\overset{\text{duality}}{\Longleftrightarrow}$ augmented Lagrangian method.

Problems with Many Additive Cost Components

$$\text{minimize} \quad \sum_{i=1}^{m} f_i(x) \qquad \text{subject to} \qquad x \in X,$$

where $f_i : \Re^n \mapsto \Re$ are convex, and X is closed and convex.

Incremental algorithms (long history, early 90s-present): Typical iteration

- Choose an index $i_k \subset \{1, \ldots, m\}$.
- Perform a subgradient iteration or a proximal iteration:

$$x_{k+1} = P_X\left(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_k)\right)$$

or

$$x_{k+1} = \arg\min_{x \in X} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k}\|x - x_k\|^2 \right\}$$

Motivation is to avoid processing all the cost components at each iteration

Separable Convex Optimization Problem

$$\text{minimize} \quad \sum_{i=1}^{m} f_i(x^i) \qquad \text{subject to} \qquad x^i \in X_i,\ i = 1, \ldots, m, \qquad \sum_{i=1}^{m} h_i(x^i) = 0$$

where $f_i : \Re_i^n \mapsto \Re$ are convex, $h_i : \Re_i^n \mapsto \Re^r$ are linear, $X_i \subset \Re_i^n$ are closed and convex.

Dual problem decomposes

$$\text{maximize} \quad \sum_{i=1}^{m} q_i(\lambda) \qquad \text{subject to} \qquad \lambda \in \Re^r$$

where q_i is a “component" dual function:

$$q_i(\lambda) = \inf_{x^i \in X_i} \left\{ f_i(x^i) + \lambda' h_i(x^i) \right\}$$

- The subgradient method exploits the separable structure (Lagrangian relaxation)
- The proximal algorithm yields the augmented Lagrangian method but destroys the separable structure
- Incremental versions of the proximal algorithm yield incremental augmented Lagrangian methods that exploit the separable structure

References for this Overview Talk

- Joint and individual works with A. Nedic and M. Wang.
- Focus on convergence, rate of convergence, component formation, and component selection.

- Work on incremental gradient methods and extended Kalman filter for least squares, 1994-1997 (DPB).
- Work on incremental subgradient methods with A. Nedic, 2000-2010.
- Work on incremental proximal methods, 2010-2012 (DPB).
- Work on incremental constraint projection methods with M. Wang, 2012-2014 (following work by A. Nedic in 2011).
- Work on incremental augmented Lagrangian methods 2015 (DPB).

General references:

- Convex Optimization Algorithms book 2015 (DPB).
- Nonlinear Programming: 3rd edition 2016 (DPB).

Outline

Incremental Subgradient Methods

- Problem: $\min_{x \in X} \sum_{i=1}^{m} f_i(x)$, where f_i and X are convex
- Long history: LMS (Widrow-Hoff, 1960, for linear least squares w/out projection), former Soviet Union literature 1960s, stochastic approximation literature 1960s, neural network literature 1970s

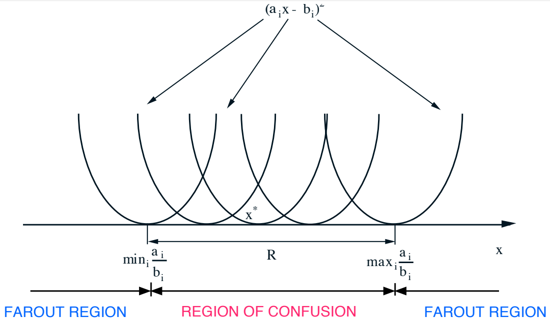
Basic incremental subgradient method

$$x_{k+1} = P_X\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_k)\big)$$

- Stepsize selection possibilities:
  - $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$
  - α_k: Constant
  - Dynamically chosen (based on estimate of optimal cost)
- Index i_k selection possibilities:
  - Cyclically
  - Fully randomized/equal probability $1/m$
  - Reshuffling/randomization within a cycle (frequent practical choice)

Convergence Mechanism

Quadratic One-Dimensional Example: $\min_{x \in \Re} \sum_{i=1}^{m} (c_i x - b_i)^2$



- Conceptually, the idea generalizes to higher dimensions, but is hard to treat/quantify analytically
- Adapting the stepsize α_k to the farout and confusion regions is an important issue
- Shaping the confusion region is an important issue

Incremental Proximal Method

Select index i_k and set

$$x_{k+1} = \arg\min_{x \in X} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k}\|x - x_k\|^2 \right\}$$

Many similarities with incremental subgradient

- Similar stepsize choices
- Similar index selection schemes
- Can be written as

$$x_{k+1} = P_X\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_{k+1})\big)$$

where $\tilde{\nabla} f_{i_k}(x_{k+1})$ is a special subgradient at x_{k+1} (index advanced by 1)

Compared to incremental subgradient

- Likely more stable
- May be harder to implement

Incremental Subgradient-Proximal Methods

Typical iteration

Choose $i_k \in \{1, \dots, m\}$ and do a subgradient or a proximal iteration

$$x_{k+1} = P_X\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_k)\big) \quad \text{or} \quad x_{k+1} = \arg\min_{x \in X} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k} \|x - x_k\|^2 \right\}$$

where α_k is a positive stepsize and $\tilde{\nabla}$ denotes (any) subgradient.

- Idea: Use proximal when easy to implement; use subgradient otherwise
- A very flexible implementation
- The proximal iterations still require diminishing α_k for convergence

Convergence Analysis

Under Lipschitz continuity-type assumptions (Nedic and Bertsekas, 2000):

- Convergence to the optimum for diminishing stepsize.
- Convergence to a neighborhood of the optimum for constant stepsize.
- Faster convergence for randomized index selection (relative to a worst-case cyclic choice).

Incremental Aggregated Gradient Method

$$x_{k+1} = P_X\left(x_k - \alpha_k \sum_{i=1}^{m} \tilde{\nabla} f_i(x_{\ell_i})\right)$$

where $\tilde{\nabla} f_i(x_{\ell_i})$ is a “delayed" subgradient of f_i at some earlier iterate x_{ℓ_i} with

$$k - b \le \ell_i \le k, \qquad \forall\, i, k.$$

- Key idea: Replace current subgradient components with earlier computed versions
- Only one component subgradient may be computed per iteration
- Proposed for nondifferentiable f_i and diminishing stepsize by Bertsekas, Nedic, and Borkar (2001)
- Key Work (Blatt, Hero, and Gauchman, 2008): Differentiable strongly convex f_i, no constraints, constant stepsize, and linear convergence.
- This is a gradient method with error proportional to the stepsize.
- A fundamentally different convergence mechanism (relies on differentiability and aims at cost function descent (no region of confusion).
- Intense recent activity by many researchers (Gurbuzbalaban, Ozdaglar, Parrilo, 2015).

Incremental Aggregated Proximal Method

Select index i_k and set

$$x_{k+1} \in \arg\min_{x \in X} \left\{ f_{i_k}(x) + \sum_{i \neq i_k} \tilde{\nabla} f_i(x_{\ell_i})'(x - x_k) + \frac{1}{2\alpha_k}\|x - x_k\|^2 \right\}$$

and $\tilde{\nabla} f_i(x_{\ell_i})$ is a “delayed" subgradient of f_i at some earlier iterate x_{ℓ_i} with

$$k - b \leq \ell_i \leq k, \qquad \forall\, i, k.$$

Equivalently,

$$x_{k+1} \in \arg\min_{x \in X} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k}\|x - z_k\|^2 \right\},$$

where

$$z_k = x_k - \alpha_k \sum_{i \neq i_k} \tilde{\nabla} f_i(x_{\ell_i}).$$

If f is differentiable and strongly convex, linear convergence can be shown with constant but sufficiently small α_k (DPB 2015).

Separable Convex Optimization: A Summary

$$\text{minimize} \quad \sum_{i=1}^{m} f_i(x^i) \qquad \text{subject to} \qquad x^i \in X_i,\ i = 1, \ldots, m, \qquad \sum_{i=1}^{m} h_i(x^i) = 0$$

where $f_i : \Re_i^n \mapsto \Re$ are convex, $h_i : \Re_i^n \mapsto \Re^r$ are linear, $X_i \subset \Re_i^n$ are closed and convex.

Dual problem decomposes

$$\text{maximize} \quad \sum_{i=1}^{m} q_i(\lambda) \qquad \text{subject to} \qquad \lambda \in \Re^r$$

where q_i is a “component" dual function:

$$q_i(\lambda) = \inf_{x^i \in X_i} \left\{ f_i(x^i) + \lambda' h_i(x^i) \right\}$$

- The subgradient method exploits the separable structure (Lagrangian relaxation)
- The proximal algorithm yields the augmented Lagrangian method but destroys the separable structure
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Proximal - Augmented Lagrangian Relation

Proximal Algorithm for the Dual Problem

$$\lambda_{k+1} \in \arg\max_{\lambda \in \Re^r} \left\{ \sum_{i=1}^{m} q_i(\lambda) - \frac{1}{2\alpha_k} \|\lambda - \lambda_k\|^2 \right\}$$

Dualization using Fenchel duality –> augmented Lagrangian method

Introduce the augmented Lagrangian function

$$L_\alpha(x, \lambda) = \sum_{i=1}^{m} f_i(x^i) + \lambda' \sum_{i=1}^{m} h_i(x^i) + \frac{\alpha}{2} \left\| \sum_{i=1}^{m} h_i(x^i) \right\|^2$$

where $\alpha > 0$ is a parameter. For a sequence $\{\alpha_k\}$ and a starting λ_0, set

$$x_{k+1} \in \arg \min_{x^i \in X_i,\, i=1,\ldots,m} L_{\alpha_k}(x, \lambda_k)$$

Update λ according to

$$\lambda_{k+1} = \lambda_k + \alpha_k \sum_{i=1}^{m} h_i(x_{k+1}^i)$$

A major flaw: min of $L_{\alpha_k}(x, l_k)$ is not separable.

Incremental Augmented Lagrangian Method

Incremental Proximal Algorithm for the Dual Problem

At iteration k, pick index i_k, and set

$$\lambda_{k+1} \in \arg\max_{\lambda \in \Re^r} \left\{ q_{i_k}(\lambda) - \frac{1}{2\alpha_k} \|\lambda - \lambda_k\|^2 \right\}$$

Dualization using Fenchel duality –> Incremental augmented Lagrangian method

Pick index i_k, and update the single component x^{i_k} according to

$$x_{k+1}^{i_k} \in \arg\min_{x^{i_k} \in X_{i_k}} \left\{ f_{i_k}(x^{i_k}) + \lambda_k' h_{i_k}(x^{i_k}) + \frac{\alpha_k}{2} \left\| h_{i_k}(x^{i_k}) \right\|^2 \right\},$$

while keeping the others unchanged, $x_{k+1}^i = x_k^i$ for all $i \neq i_k$. Update λ according to

$$\lambda_{k+1} = \lambda_k + \alpha_k h_{i_k}(x_{k+1}^{i_k})$$

Incremental Aggregated Proximal Algorithm for the Dual Problem

At iteration k, pick index i_k, and set

$$\lambda_{k+1} \in \arg\max_{\lambda \in \Re^r} \left\{ q_{i_k}(\lambda) - \frac{1}{2\alpha_k} \|\lambda - z_k\|^2 \right\},$$

where

$$z_k = \lambda_k + \alpha_k \sum_{i \neq i_k} \tilde{\nabla} q_i(\lambda_{\ell_i})$$

Dualization using Fenchel duality –> Incremental aggregated augmented Lagrangian method

- Pick index i_k, and update the single component x^{i_k} according to

$$x_{k+1}^{i_k} \in \arg \min_{x^{i_k} \in X_{i_k}} \left\{ f_{i_k}(x^{i_k}) + \lambda_k' h_{i_k}(x^{i_k}) + \frac{\alpha_k}{2} \left\| h_{i_k}(x^{i_k}) + \sum_{i \neq i_k} h_i(x_{\ell_i}^i) \right\|^2 \right\}$$

  while keeping the others unchanged, $x_{k+1}^i = x_k^i$ for all $i \neq i_k$.
- Update λ according to

$$\lambda_{k+1} = \lambda_k + \alpha_k \left(h_{i_k}(x_{k+1}^{i_k}) + \sum_{i \neq i_k} h_i(x_{\ell_i}^i) \right)$$

  Here $h_i(x_{\ell_i}^i)$, $i \neq i_k$, come from earlier iterations.

Comparison with Alternating Direction Methods of Multipliers (ADMM)

ADMM Iteration for Separable Problems (DPB 1989)

Perform a separate augmented Lagrangian minimization over x^i, for each $i = 1, \ldots, m$,

$$x_{k+1}^i \in \arg\min_{x^i \in X_i} \left\{ f_i(x^i) + \lambda_k' h_i(x^i) + \frac{\alpha}{2} \left\| h_i(x^i) - h_i(x_k^i) + \frac{1}{m} \sum_{j=1}^m h_j(x_k^j) \right\|^2 \right\},$$

and then update λ_k according to

$$\lambda_{k+1} = \lambda_k + \frac{\alpha}{m} \sum_{i=1}^m h_i(x_{k+1}^i)$$

Comparison with Incremental Aggregated Augmented Lagrangian

- The two methods involve fairly similar operations
- ADMM has guaranteed convergence for any constant α, and under weaker conditions (dual differentiability and strong convexity are not required)
- IAAL has stepsize restrictions
- At each iteration, all components x^i are updated in ADMM, but a single component x^i is updated in IAAL (m times greater overhead per iteration)

Incremental Methods with Constraint Projection

$$\text{minimize} \quad \sum_{i=1}^{m} f_i(x) \qquad \text{subject to} \qquad x \in \cap_{\ell=1}^{q} X_\ell,$$

where $f_i : \Re^n \mapsto \Re$ are convex, and the sets X_ℓ are closed and convex.

Incremental constraint projection algorithm

- Choose indexes $i_k \in \{1, \ldots, m\}$ and $\ell_k \in \{1, \ldots, q\}$.
- Perform a subgradient iteration or a proximal iteration

$$x_{k+1} = P_{X_{\ell_k}}\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_k)\big) \quad \text{or} \quad x_{k+1} = \arg\min_{x \in X_{\ell_k}} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k} \|x - x_k\|^2 \right\}$$

where α_k is a positive stepsize and $\tilde{\nabla}$ denotes (any) subgradient.

Connection to feasibility/alternating projection methods.

Incremental Random Projection Method

Problem

$$\text{minimize} \quad \sum_{i=1}^{m} f_i(x) \qquad \text{subject to} \qquad x \in X = \cap_{\ell=1}^{q} X_\ell,$$

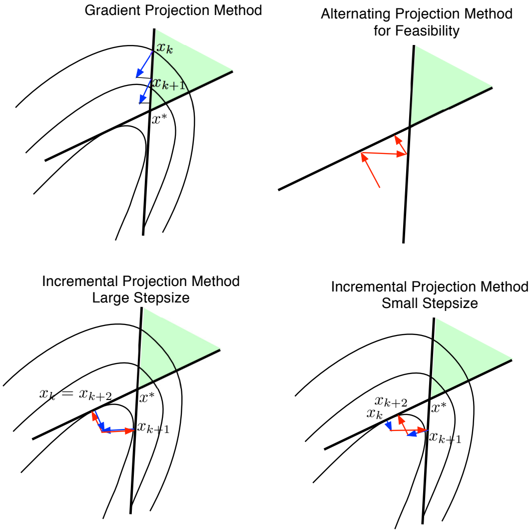
Typical iteration

- Choose indexes $i_k \in \{1, \ldots, m\}$ and $\ell_k \in \{1, \ldots, q\}$.
- Set
$$x_{k+1} = P_{X_{\ell_k}}\big(x_k - \alpha_k \tilde{\nabla} f_{i_k}(\bar{x}_k)\big)$$
- $\bar{x}_k = x_k$ (subgradient iteration) or $\bar{x} = x_{k+1}$ (proximal iteration).
- $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$ (diminishing stepsize is essential).

Two-way progress

- Progress to feasibility: The projection $P_{X_{\ell_k}}(\cdot)$.
- Progress to optimality: The “subgradient/proximal" iteration $x_k - \alpha_k \tilde{\nabla} f_{i_k}(\bar{x}_k)$.

Visualization of Convergence



Progress to feasibility should be faster than progress to optimality. Gradient stepsizes α_k should be $<<$ than the feasibility stepsize of 1.

Sampling Schemes for Constraint Index ℓ_k

Nearly independent sampling

$$\inf_{k\geq 0} \text{Prob}(\ell_k = X_\ell \mid \mathcal{F}_k) > 0, \qquad \ell = 1, \ldots, q,$$

where $\mathcal{F}_k$ is the history of the algorithm up to time k.

Cyclic sampling

Deterministic or random reshuffling every q iterations.

Most distant constraint sampling

$$\ell_k = \arg \max_{\ell=1,\ldots,q} \big\| x_k - P_{X_\ell}(x_k) \big\|$$

Markov sampling

Generate ℓ_k as the state of an ergodic Markov chain with states $1, \ldots, q$.

Sampling Schemes for Cost Component Index i_k

Random independent uniform sampling

Each index $i \in \{1, \ldots, m\}$ is chosen with equal probability $1/m$, independently of earlier choices.

Cyclic sampling

Deterministic or random reshuffling every m iterations.

Markov sampling

Generate i_k as the state of a Markov chain with states $1, \ldots, m$, and steady state distribution $\{1/m, \ldots, 1/m\}$.

Convergence Theorem

Assuming Lipschitz continuity of the cost, linear regularity of the constraint, and nonemptiness of the optimal solution set, $\{x_k\}$ converges to some optimal solution x^* w.p. 1, under any combination of the preceding sampling schemes.

Idea of the convergence proof

There are two convergence processes taking place:

- Progress towards feasibility, which is fast (geometric thanks to the linear regularity assumption).
- Progress towards optimality, which is slower (because of the diminishing stepsize α_k).
- This two-time scale convergence analysis idea is encoded in a coupled supermartingale convergence theorem, which governs the evolution of two measures of progress

$$\mathbf{E}[\text{dist}^2(x_k, X)] : \text{ Distance to the constraint set, which is fast}$$

$$\mathbf{E}[\text{dist}^2(x_k, X^*)] : \text{ Distance to the optimal solution set, which is slow}$$

Concluding Remarks

- Incremental methods exhibit interesting convergence behavior, and can lead to great efficiencies for large-sum cost functions
- Incremental proximal methods enhance reliability and can be combined seamlessly with incremental gradient/subgradient methods
- Incremental proximal methods when dualized yield incremental augmented Lagrangian methods that can take advantage of constrained problem separability
- Constraint projection variants provide flexibility and enlarge the range of potential applications
- Incremental methods are amenable to distributed asynchronous implementation

Thank you!