Cover photography by Dimitri Bertsekas.
Stars over the Stata Center at MIT (built on the location of the old Building 20 where Claude Shannon had his first office as a professor in 1956).

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Professor Bertsekas' teaching and research have spanned several fields, including deterministic optimization, dynamic programming and stochastic control, large-scale and distributed computation, artificial intelligence, and data communication networks. He has authored or coauthored numerous research papers and eighteen books, several of which are currently used as textbooks in MIT classes, including “Dynamic Programming and Optimal Control,” “Data Networks,” “Introduction to Probability,” and “Nonlinear Programming.”

Professor Bertsekas was awarded the INFORMS 1997 Prize for Research Excellence in the Interface Between Operations Research and Computer Science for his book “Neuro-Dynamic Programming” (co-authored with John Tsitsiklis), the 2001 AACC John R. Ragazzini Education Award, the 2009 INFORMS Expository Writing Award, the 2014 AACC Richard Bellman Heritage Award, the 2014 INFORMS Khachiyan Prize for Life-Time Accomplishments in Optimization, and the 2015 MOS/SIAM George B. Dantzig Prize. In 2018 he shared with his coauthor, John Tsitsiklis, the 2018 INFORMS John von Neumann Theory Prize for the contributions of the research monographs “Parallel and Distributed Computation” and “Neuro-Dynamic Programming.” Professor Bertsekas was elected in 2001 to the United States National Academy of Engineering for “pioneering contributions to fundamental research, practice and education of optimization/control theory, and especially its application to data communication networks.”
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Contents

1. Dynamic Programming Principles
   1.1. Deterministic Dynamic Programming .................. p. 2
      1.1.1. Basic Finite Horizon Problem Formulation ........ p. 2
      1.1.2. The Dynamic Programming Algorithm ............... p. 5
      1.1.3. Approximation in Value Space .................... p. 7
   1.2. Stochastic Dynamic Programming ....................... p. 10
      1.2.1. Finite Horizon Problems ......................... p. 10
      1.2.2. Infinite Horizon Problems - An Overview .......... p. 14
   1.3. Examples, Variations, and Simplifications ............. p. 20
      1.3.1. Discrete Deterministic Optimization ............... p. 21
      1.3.2. Problems with a Termination State ............... p. 25
      1.3.3. State Augmentation, Time Delays, and Forecasts .. p. 29
      1.3.4. Partial State Information and Belief States ..... p. 32
   1.4. Reinforcement Learning and Optimal Control - Some .... p. 35
      Terminology ........................................ p. 35
   1.5. Notes and Sources ................................ p. 37

2. Rollout and Policy Improvement
   2.1. Approximation in Value and Policy Space ............ p. 43
      2.1.1. Approximation in Value Space - One-Step and .... p. 43
      Multistep Lookahead ................................ p. 43
      2.1.2. Approximation in Policy Space ................... p. 47
      2.1.3. Combined Approximation in Value and Policy Space .. p. 49
   2.2. General Issues of Approximation in Value Space ...... p. 53
      2.2.1. Model-Based and Model-Free Implementations ...... p. 53
      2.2.2. Off-Line and On-Line Implementations ............ p. 54
      2.2.3. Methods for Cost-to-Go Approximation .......... p. 56
      2.2.4. Methods for Simplification of the Lookahead ..... p. 58
      Minimization ........................................ p. 58
      2.2.5. Simplification of the Lookahead Minimization ... by Q-Factor Approximation ............... p. 59
## 2.3. Rollout and the Policy Improvement Principle
- **2.3.1. On-Line Rollout for Deterministic Discrete Optimization**... p. 64
- **2.3.2. Using Multiple Base Heuristics - Parallel Rollout**... p. 72
- **2.3.3. The Fortified Rollout Algorithm**... p. 73
- **2.3.4. Truncated Rollout with Multistep Lookahead and Terminal Cost Approximation**... p. 76
- **2.3.5. Rollout with Small Stage Costs and Long Horizon** - Continuous-Time Rollout... p. 78
- **2.3.6. Rollout with an Expert**... p. 88

## 2.4. Stochastic Rollout and Monte Carlo Tree Search
- **2.4.1. Simulation-Based Implementation of the Rollout Algorithm**... p. 94
- **2.4.2. Rollout and Monte Carlo Tree Search**... p. 98
- **2.4.3. Randomized Policy Improvement by Monte Carlo Tree Search**... p. 102
- **2.4.4. Rollout Parallelization**... p. 103
- **2.4.5. The Effect of Errors in Rollout - Variance Reduction**... p. 104

## 2.5. Rollout for Infinite-Spaces Problems - Optimization Heuristics
- **2.5.1. Rollout for Infinite-Spaces Deterministic Problems**... p. 107
- **2.5.2. Rollout Based on Stochastic Programming**... p. 111

## 2.6. Notes and Sources
- p. 114

## 3. Specialized Rollout Algorithms

### 3.1. Model Predictive Control
- **3.1.1. Target Tubes and Constrained Controllability**... p. 127
- **3.1.2. Model Predictive Control with Terminal Cost**... p. 131
- **3.1.3. Variants of Model Predictive Control**... p. 132

### 3.2. Multiagent Rollout
- **3.2.1. Multiagent Coupling Through Constraints**... p. 145
- **3.2.2. Multiagent Rollout for Separable and Multiarmed Bandit Problems**... p. 147
- **3.2.3. Multiagent Model Predictive Control**... p. 150
- **3.2.4. Asynchronous Distributed Multiagent Rollout**... p. 151

### 3.3. Constrained Rollout for Deterministic Optimization
- **3.3.1. State-Constrained Rollout and Target Tubes**... p. 156
- **3.3.2. Rollout with Trajectory Constraints**... p. 160
- **3.3.3. Constrained Multiagent Rollout**... p. 169

### 3.4. Constrained Rollout - Combinatorial and Discrete Optimization
- **3.4.1. A General Discrete Optimization Problem**... p. 172
3.4.2. Multidimensional Assignment . . . . . . . . . p. 179
3.5. Surrogate Dynamic Programming and Rollout . . . p. 187
3.5.1. Rollout for Bayesian Optimization . . . . . . . p. 189
3.6. Rollout for Minimax Control . . . . . . . . . . . . p. 193
3.7. Notes and Sources . . . . . . . . . . . . . . . . . p. 196

4. Learning Values and Policies
4.1. Approximation Architectures . . . . . . . . . . . . . . p. 204
4.1.1. Feature-Based Architectures . . . . . . . . . . . . . p. 205
4.1.2. Training of Linear and Nonlinear Architectures . . p. 215
4.2. Neural Networks . . . . . . . . . . . . . . . . . . . . p. 219
4.2.1. Training of Neural Networks . . . . . . . . . . . . p. 223
4.2.2. Multilayer and Deep Neural Networks . . . . . . . p. 224
4.3. Training of Cost Functions in Approximate DP . . . . p. 226
4.3.1. Fitted Value Iteration . . . . . . . . . . . . . . . . p. 226
4.3.2. Q-Factor Parametric Approximation . . . . . . . . p. 228
4.3.3. Advantage Updating - Approximating Q-Factor . . . .
        Differences . . . . . . . . . . . . . . . . . . . . . . p. 230
4.3.4. Differential Training of Cost Differences for Rollout p. 233
4.4. Training of Policies in Approximate DP . . . . . . . . p. 235
4.4.1. Perpetual Rollout with Value and Policy Networks - .
        Multiprocessor Parallelization . . . . . . . . . . . . p. 239
4.5. Notes and Sources . . . . . . . . . . . . . . . . . . . p. 240

5. Infinite Horizon: Distributed and Multiagent Algorithms
5.1. Stochastic Shortest Path and Discounted Problems . . p. 248
5.2. Exact and Approximate Policy Iteration . . . . . . . . . p. 260
5.2.1. Policy Iteration and Rollout . . . . . . . . . . . . . p. 261
5.2.2. Optimistic and Multistep Policy Iteration - .
        Truncated Rollout . . . . . . . . . . . . . . . . . . . p. 265
5.2.3. Policy Iteration for Q-Factors . . . . . . . . . . . . p. 268
5.2.4. Multiagent Rollout and Policy Iteration . . . . . . . p. 270
5.2.5. Approximation in Value Space . . . . . . . . . . . . p. 277
5.2.6. Performance Bounds for Truncated Rollout and .
        Approximate Policy Iteration . . . . . . . . . . . . . p. 279
5.3. Abstract View of Infinite Horizon Problems . . . . . . p. 290
5.4. Multiagent Value and Policy Iteration . . . . . . . . . . p. 301
5.4.1. Convergence to an Agent-by-Agent Optimal Policy . p. 305
5.4.2. Optimistic Multiagent PI Algorithms . . . . . . . . . p. 310
5.5. Asynchronous Distributed Value Iteration . . . . . . . . p. 313
5.5.1. State Space Partitioning . . . . . . . . . . . . . . . p. 314
5.5.2. Asynchronous Convergence Theorem . . . . . . . . . p. 315
5.6. Asynchronous Distributed Policy Iteration . . . . . . . . p. 318
5.6.1. Randomized Asynchronous Optimistic Policy . . . .
Contents

Iteration . . . . . . . . . . . . . . . . . . . p. 320

5.6.2. Asynchronous Optimistic Policy Iteration with a . . . . . . .
Uniform Fixed Point . . . . . . . . . . . . . p. 323

5.6.3. Approximate Policy Iteration - Asynchronous . . . . . . . .
Multiprocessor Parallelization . . . . . . . . p. 330

5.7. Notes and Sources . . . . . . . . . . . . . . p. 332

References . . . . . . . . . . . . . . . . . . . p. 337

Index . . . . . . . . . . . . . . . . . . . . . . p. 357
Preface

We know the past but cannot control it. We control the future but cannot know it.

Claude Shannon

In this research monograph we discuss the solution of large and challenging multistage decision problems using methods of reinforcement learning (RL for short), also referred to by other names such as approximate dynamic programming and neuro-dynamic programming. We will focus on a subset of methods which are based on the idea of policy iteration, i.e., starting from some policy and generating one or more improved policies.

If just one improved policy is generated, this is called rollout, which, based on broad and consistent computational experience, appears to be one of the simplest and most reliable of all RL methods. Rollout is also well-suited for on-line model-free implementation and on-line replanning. Approximate policy iteration can be viewed as repeated application of rollout. This is one of the most prominent types of RL methods. It can be implemented using data generated by the system itself, a process known as self-learning, and value and policy approximation architectures, including neural networks.

Approximate policy iteration is more ambitious than rollout, but it is a strictly off-line method, and it is generally far more computationally intensive (of course rollout may also require a lot of on-line computation). This motivates the use of parallel and distributed computation. One of the purposes of the monograph is to discuss distributed (possibly asynchronous) methods that relate to rollout and policy iteration, both in the context of an exact and an approximate implementation involving neural networks or other approximation architectures.

One of the contributions of the monograph is to develop variants of rollout and policy iteration for problems with a multiagent structure, where the control consists of multiple components, each associated with a separate agent. In particular, we introduce a new approach to lookahead
simplification through the use of multiagent rollout, which allows the 
dramatic reduction of the computational requirements for one-step lookahead 
when the control consists of multiple components, and connects with the 
theme of distributed asynchronous implementation.

Multiagent rollout also has a strong connection with a well-developed 
body of research with a long history: the theory of teams and the notion of 
person-by-person optimality. In particular, we develop an infinite horizon 
dynamic programming methodology, which includes value and policy iter-
ation methods that converge to a person-by-person optimal policy. While 
our multiagent schemes are based on fully shared agent information, they 
are also well suited as a starting point for approximations, in the context of 
on-line autonomous decision making by multiple agents each coordinating 
in varying degrees with the other agents. In this context, agent information 
that is not shared by other agents, is appropriately estimated, with the 
estimates being treated as if they were exact.

Several of the ideas that we develop in some depth in this mono-
graph have been central in the implementation of recent high profile suc-
cesses, such as the AlphaZero program for playing chess, Go, and other 
games. In addition to the fundamental process of successive policy itera-
tion/improvement, this program includes the use of deep neural networks 
for representation of both value functions and policies, the extensive use 
of large scale parallelization, and the simplification of lookahead minimiza-
tion, through methods involving Monte Carlo tree search and pruning of 
the lookahead tree. In this monograph, we also focus on policy iteration, 
value and policy neural network representations, parallel and distributed 
computation, and lookahead simplification. Thus while there are significant 
differences, the principal design ideas that form the core of this monograph 
are shared by the AlphaZero architecture, except that we develop these 
ideas in a broader and less application-specific framework.

Another subject that we deal with in some detail is model predic-
tive control (MPC for short), one of the most prominent control system 
design methods at present. One of the reasons is that classical forms of 
MPC are closely related to (and indeed can be viewed as) rollout algo-
rithms, thereby providing a connection with reinforcement learning, which 
is beneficial in two ways. On one hand the MPC context provides rich 
crossfertilization opportunities with the analytical and algorithmic ideas 
of rollout and RL; for example the notion of sequential improvement in 
rollout is intimately connected to Lyapunov stability analysis in MPC, and 
the target tube ideas that are central in MPC may prove useful in the 
context of constrained rollout and policy iteration. On the other hand the 
dynamic programming and RL methodologies point the way to extensions 
of MPC based on self-learning, approximate policy iteration, simulation, 
the treatment of stochastic and set membership uncertainty, and the use 
of distributed computation.

In our development of several of the topics of this book we rely on
Preface

methodology that is covered in greater depth in the 1996 neuro-dynamic programming book [BeT96] (jointly written with J. Tsitsiklis) as well as the author’s recent RL book [Ber19a]. However, we aim to develop rollout and approximate policy iteration beyond the books [BeT96] and [Ber19a]. In particular, we present new research, relating to distributed asynchronous computation, partitioned architectures, and multiagent systems. We also indicate how our methods are well-suited for several types of challenging large scale optimization problems, such as combinatorial/discrete optimization, as well as partially observed Markov decision problems (POMDP).

This monograph took shape in the fall of 2019 and was based on two separate but related lines of research for distributed large-scale computation:

(a) My work on rollout, policy iteration, and value iteration for multi-agent DP problems, including those arising in discrete deterministic optimization settings [Ber19c], [Ber19d], [Ber20].

(b) My work on distributed policy iteration algorithms with state space-partitioned architectures. These ideas were extended, implemented, and applied to some large-scale POMDP problems in collaboration with my Arizona State University (ASU) colleagues Sushmita Bhattacharya, Sahil Badyal, Thomas Wheeler, and Stephanie Gil [BBW20]. This work is also connected with my joint research on asynchronous distributed state space-partitioned policy iteration with Huizhen Yu [BeY10], [BeY12], [YuB13], which is presented in Section 5.6 of this monograph.

Most of the book was written while teaching a research-oriented course at ASU starting in January 2020. The hospitable and stimulating environment at ASU contributed much to my productivity during this period, and for this I am very thankful to several colleagues and students, including Stephanie Gil, Giulia Pedrielli, and Petr Sulc, and my teaching assistant, Sushmita Bhattacharya. I have also appreciated fruitful interactions with colleagues and students outside ASU, particularly Yuchao Li, who also provided valuable proofreading support.

Finally, I would like to dedicate this monograph to the creative genius of Claude Shannon, the father of information theory, but also the father of computer chess. His approximate dynamic programming ideas, which predated the work of Bellman, live on inside the AlphaZero program, the most impressive success story of reinforcement learning up to now.

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July 2020