**Recall Approximation in Value Space**

**Approximate Min Discretization**

\[
\min_{u_k} \ E \left\{ g_k(x_k, u_k, w_k) + \tilde{J}_{k+1}(x_{k+1}) \right\}
\]

**First Step**

**“Future”**

**Approximate Cost-to-Go \( \tilde{J}_{k+1} \)**

- Problem approximation
- Rollout, Model Predictive Control
- Parametric approximation
- Neural nets
- Aggregation

**ONE-STEP LOOKAHEAD**

At State \( x_k \)

DP minimization

\[
\min_{u_k, \mu_{k+1}, \ldots, \mu_{k+\ell-1}} E \left\{ g_k(x_k, u_k, w_k) + \sum_{m=k+1}^{k+\ell-1} g_k(x_m, \mu_m(x_m), w_m) + \tilde{J}_{k+\ell}(x_{k+\ell}) \right\}
\]

**Lookahead Minimization**

**MULTISTEP LOOKAHEAD**

Cost-to-go Approximation
The Pure Form of Rollout

At State $x_k$
DP minimization

\[
\min_{u_k, \mu_{k+1}, \ldots, \mu_{k+\ell-1}} E \left\{ g_k(x_k, u_k, w_k) + \sum_{m=k+1}^{k+\ell-1} g_k(x_m, \mu_m(x_m), w_m) + \tilde{J}_{k+\ell}(x_{k+\ell}) \right\}
\]

First $\ell$ Steps
Lookahead Minimization

“Future”
Heuristic Cost
Run the Base Policy

Use a suboptimal/heuristic policy at the end of limited lookahead

- The heuristic is called base policy (or default policy).
- The lookahead policy is called rollout policy.
- The aim of rollout is policy improvement (i.e., rollout policy performs better than the base policy); true under some assumptions. In practice: good performance, very reliable, very simple to implement.
- Rollout in its "standard" forms involves simulation and on-line implementation.
- The simulation can be prohibitively expensive (so further approximations may be needed); particularly for stochastic problems and multistep lookahead.
Connection with problem approximation

- Suppose the base heuristic is an optimal policy for the approximating problem.
- Then rollout is lookahead with problem approximation: the optimal cost of the approximating problem is used as lookahead function.
- True for both one-step and multistep lookahead.

Connection with policy iteration/self learning - Infinite horizon problems

- Rollout can be viewed as one-step policy iteration (more on this later).
- Cost improvement property of rollout is based on the fundamental cost improvement property of policy iteration (more on this later).
- Policy iteration can be viewed as "perpetual" rollout, i.e., every so often replace the base policy with the current rollout policy (or an approximation thereof).
At state $x_k$, for every pair $(x_k, u_k)$, $u_k \in U_k(x_k)$, we generate a Q-factor

$$\tilde{Q}_k(x_k, u_k) = g_k(x_k, u_k) + H_{k+1}(f_k(x_k, u_k))$$

using the base heuristic $[H_{k+1}(x_{k+1})]$ is the heuristic cost starting from $x_{k+1}$.

- We select the control $u_k$ with minimal Q-factor.
- We move to next state $x_{k+1}$, and continue.
- Multistep lookahead versions (length of lookahead limited by the branching factor of the lookahead tree).
Traveling Salesman Example of Rollout with a Greedy Heuristic

- $N$ cities $c = 0, \ldots, N - 1$; each pair of distinct cities $c$, $c'$, has traversal cost $g(c, c')$.
- Find a minimum cost tour that visits each city once and returns to the initial city.
- Recall that it can be viewed as a shortest path/deterministic DP problem. States are the partial tours, i.e., the sequences of ordered collections of distinct cities exponentially growing size of state space.
- Nearest neighbor heuristic; chooses the best one-hop extension of a partial tour.
- Rollout algorithm: Start at some city; given a partial tour \{\(c_0, \ldots, c_k\)\} of distinct cities, select as next city \(c_{k+1}\) the one that yielded the minimum cost tour under the nearest neighbor heuristic.
Special conditions must hold to guarantee that the rollout policy has no worse performance than the base heuristic.

Two such conditions are **sequential consistency** and **sequential improvement**.

A sequentially improving heuristic is also sequentially consistent.

Any heuristic can be modified to become sequentially improving.

The base heuristic is sequentially consistent if it “stays the course”

- If the heuristic generates the sequence
  \[ \{x_k, x_{k+1}, \ldots, x_N\} \]
  starting from state \(x_k\), it also generates the sequence
  \[ \{x_{k+1}, \ldots, x_N\} \]
  starting from state \(x_{k+1}\).

- The base heuristic is sequentially consistent if and only if it can be implemented with a legitimate DP policy \(\{\mu_0, \ldots, \mu_{N-1}\}\).

- Greedy heuristics are sequentially consistent.
Sequential improvement holds if for all $x_k$ (Best heuristic Q-factor $\leq$ Heuristic cost):

$$
\min_{u_k \in U_k(x_k)} \left[ g_k(x_k, u_k) + H_{k+1}(f_k(x_k, u_k)) \right] \leq H_k(x_k),
$$

where $H_k(x_k)$ is the cost of the trajectory generated by the heuristic starting from $x_k$. True for a sequentially consistent heuristic [$H_k(x_k)$ is the Q-factor of the heuristic at $x_k$].

Cost improvement property for a sequentially improving heuristic

Let the rollout policy be $\tilde{\pi} = \{\tilde{\mu}_0, \ldots, \tilde{\mu}_{N-1}\}$, and let $J_{k, \tilde{\pi}}(x_k)$ denote its cost starting from $x_k$. Then for all $x_k$ and $k$, $J_{k, \tilde{\pi}}(x_k) \leq H_k(x_k)$.

Proof by induction: It holds for $k = N$, since $J_{N, \tilde{\pi}} = H_N = g_N$. Assume that it holds for index $k + 1$.

$$
J_{k, \tilde{\pi}}(x_k) = g_k(x_k, \tilde{\mu}_k(x_k)) + J_{k+1, \tilde{\pi}}(f_k(x_k, \tilde{\mu}_k(x_k))) \\
\leq g_k(x_k, \tilde{\mu}_k(x_k)) + H_{k+1}(f_k(x_k, \tilde{\mu}_k(x_k))) \\
= \min_{u_k \in U_k(x_k)} \left[ g_k(x_k, u_k) + H_{k+1}(f_k(x_k, u_k)) \right] \\
\leq H_k(x_k)
$$
Walk on a line of length $2N$ starting at position 0. At each of $N$ steps, move one unit to the left or one unit to the right.

Objective is to land at a position $i$ of small cost $g(i)$ after $N$ steps.

Question: Consider a base heuristic that takes steps to the right only. How will the rollout perform compared to the base heuristic?

Compare with a superheuristic/combination of two heuristics: 1) Move only to the right, and 2) Move only to the left. Base heuristic chooses the path of best cost.
Fortified Rollout: Restores Cost Improvement for Base Heuristics that are not Sequentially Consistent

- Upon reaching state $x_k$ it stores the permanent trajectory
  \[ \overline{P}_k = \{ x_0, u_0, \ldots, u_{k-1}, x_k \} \]
  that has been constructed up to stage $k$, called, and it also stores a tentative trajectory
  \[ \overline{T}_k = \{ x_k, \overline{u}_k, \overline{x}_{k+1}, \overline{u}_{k+1}, \ldots, \overline{u}_{N-1}, \overline{x}_N \} \]
- The tentative trajectory is such that $\overline{P}_k \cup \overline{T}_k$ is the best end-to-end trajectory computed up to stage $k$ of the algorithm.
- At each step follow the best trajectory.
Multistep Rollout with Terminal Cost Approximation

- Saves computation but the cost improvement property is lost.
- We can prove cost improvement, assuming sequential consistency and a special property of the terminal cost function approximation that resembles sequential improvement (more on this when we discuss infinite horizon rollout).
- It is not necessarily true that longer lookahead leads to improved performance; but usually true (similar counterexamples as in the last lecture).
- It is not necessarily true that increasing the length of the rollout leads to improved performance (some examples indicate this). Moreover, long rollout is costly.
- Experimentation with length of rollout and terminal cost function approximation are recommended.
At State $x_k$

DP minimization

\[ \min_{u_k, \mu_k+1, \ldots, \mu_k+\ell-1} \mathbb{E} \left\{ g_k(x_k, u_k, w_k) + \sum_{m=k+1}^{k+\ell-1} g_k(x_m, \mu_m(x_m), w_m) + \tilde{J}_{k+\ell}(x_{k+\ell}) \right\} \]

First $\ell$ Steps

Lookahead Minimization

"Future"

Heuristic Cost

Run the Base Policy

Consider the pure case (no truncation, no terminal cost approximation)

- Assume that the base heuristic is a legitimate policy $\pi = \{\mu_0, \ldots, \mu_{N-1}\}$ (i.e., is sequentially consistent, in the context of deterministic problems).

- Let $\tilde{\pi} = \{\mu_0, \ldots, \mu_{N-1}\}$ be the rollout policy. Then cost improvement is obtained

\[ J_{k, \tilde{\pi}}(x_k) \leq J_{k, \pi}(x_k), \quad \text{for all } x_k \text{ and } k. \]

- Essentially identical induction proof as for the sequentially improving case (see the text).
Announced by Tesauro in 1996.

Truncated rollout with cost function approximation provided by TD-Gammon (earlier program involving a neural network trained by a form of policy iteration).

Plays better than TD-Gammon, and better than any human.

Too slow for real-time play (without parallel hardware), due to excessive simulation time.
We assumed equal effort for evaluation of Q-factors of all controls at a state $x_k$.

**Drawbacks:**
- The trajectories may be too long because the horizon length $N$ is large (or infinite, in an infinite horizon context).
- Some of the controls $u_k$ may be clearly inferior to others, and may not be worth as much sampling effort.
- Some of the controls $u_k$ that appear to be promising, may be worth exploring better through multistep lookahead.

Monte Carlo tree search (MCTS) is a “randomized" form of lookahead

- MCTS aims to trade off computational economy with a hopefully small risk of degradation in performance.
- It involves **adaptive simulation** (simulation effort adapted to the perceived quality of different controls).
- Aims to balance **exploitation** (extra simulation effort on controls that look promising) and **exploration** (adequate exploration of the potential of all controls).
Monte Carlo Tree Search - Adaptive Simulation

Find a control \( \tilde{u}_k \) that minimizes the approximate Q-factor

\[
\tilde{Q}_k(x_k, u_k) = E \left\{ g_k(x_k, u, w_k) + \tilde{J}_{k+1} (f_k(x_k, u, w_k)) \right\}
\]

over \( u_k \in U_k(x_k) \), by averaging samples of \( \tilde{Q}_k(x_k, u_k) \).

Assume that \( U_k(x_k) \) contains \( m \) elements, denoted \( 1, \ldots, m \)

- After the \( n \)th sampling period we have \( Q_{i,n} \), the empirical mean of the Q-factor of control \( i \) (total sample value divided by total number of samples).
- How do we use the estimates \( Q_{i,n} \) to select the control to sample next?
A good sampling policy balances exploitation (sample controls that seem most promising, i.e., a small $Q_{i,n}$) and exploration (sample controls with small sample count).

- A popular strategy: Sample next the control $i$ that minimizes the sum $Q_{i,n} + R_{i,n}$ where $R_{i,n}$ is an exploration index.
- $R_{i,n}$ is based on a confidence interval formula and depends on the sample count $s_i$ of control $i$ (which comes from analysis of multiarmed bandit problems).
- The UCB rule (upper confidence bound) sets $R_{i,n} = -c\sqrt{\log n/s_i}$, where $c$ is a positive constant, selected empirically (values $c \approx \sqrt{2}$ are suggested, assuming that $Q_{i,n}$ is normalized to take values in the range $[-1, 0]$).
- MCTS with UCB rule has been extended to multistep lookahead.
About the Next Lecture

We will cover:

- Model predictive control
- Approximation architectures
- Training approximation architectures

PLEASE READ AS MUCH OF SECTIONS 2.5, 3.1 AS YOU CAN

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