3D Integral Viscous Method for General Configurations

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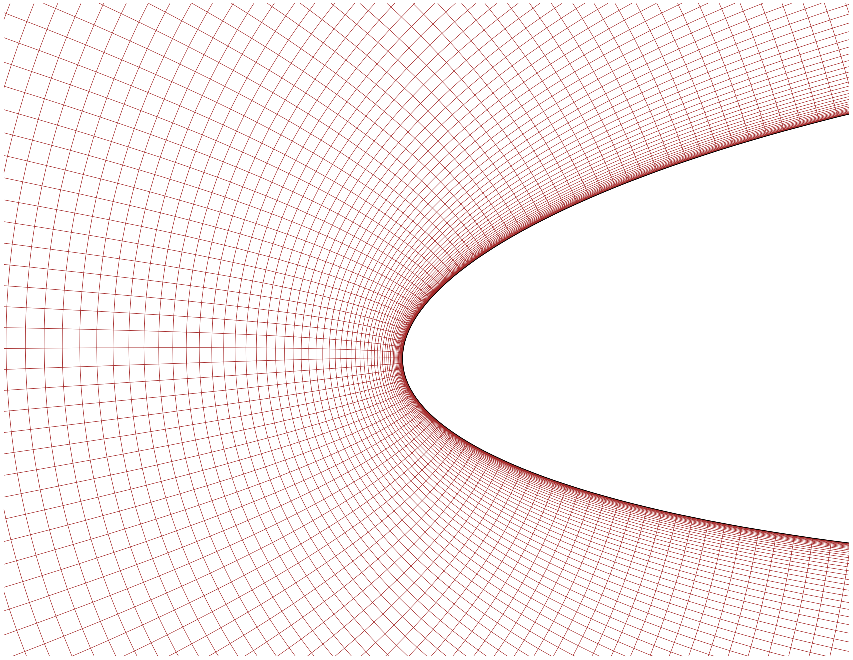
Historical Motivation

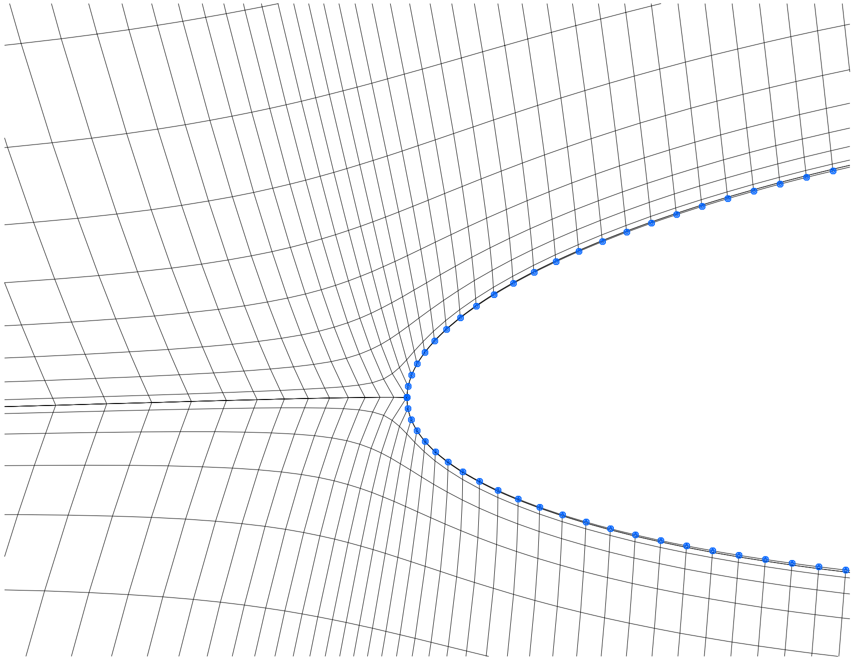
2D Inviscid+Integral-BL (IBL) methods have proven very effective

- Enormously faster than alternative Navier-Stokes for similar accuracy
- Can exploit any inviscid flow solver
- Compatible with inverse design methods
- Compatible with virtual displacements for linearized aeroelasticity



<u>Navier Stokes</u>

$\sim$ 500000 variables
$\sim$ 1 hr. runtime

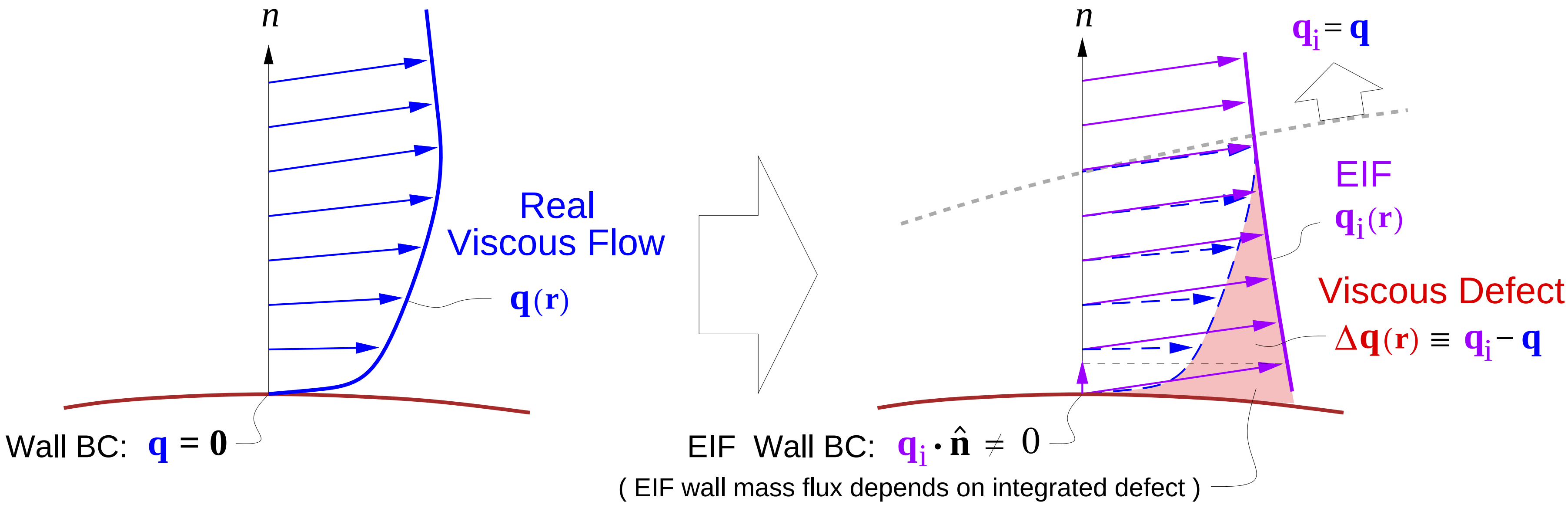


<u>Potential+BL</u> (MSIS, TRANAIR-2D)

$\sim$ 5000 variables
$\sim$ 1 sec. runtime

EIF/Defect Formulation – I

- Real flow decomposed into irrotational Equivalent Inviscid Flow $\mathbf{q}_i$ and rotational Viscous Defect $\Delta\mathbf{q}$
- By definition, $\mathbf{q}_i = \mathbf{q}$ outside the rotational viscous layers



- Originally employed by LeBalleur with boundary layer methods
- But does <u>not</u> depend on thin boundary layer approximations

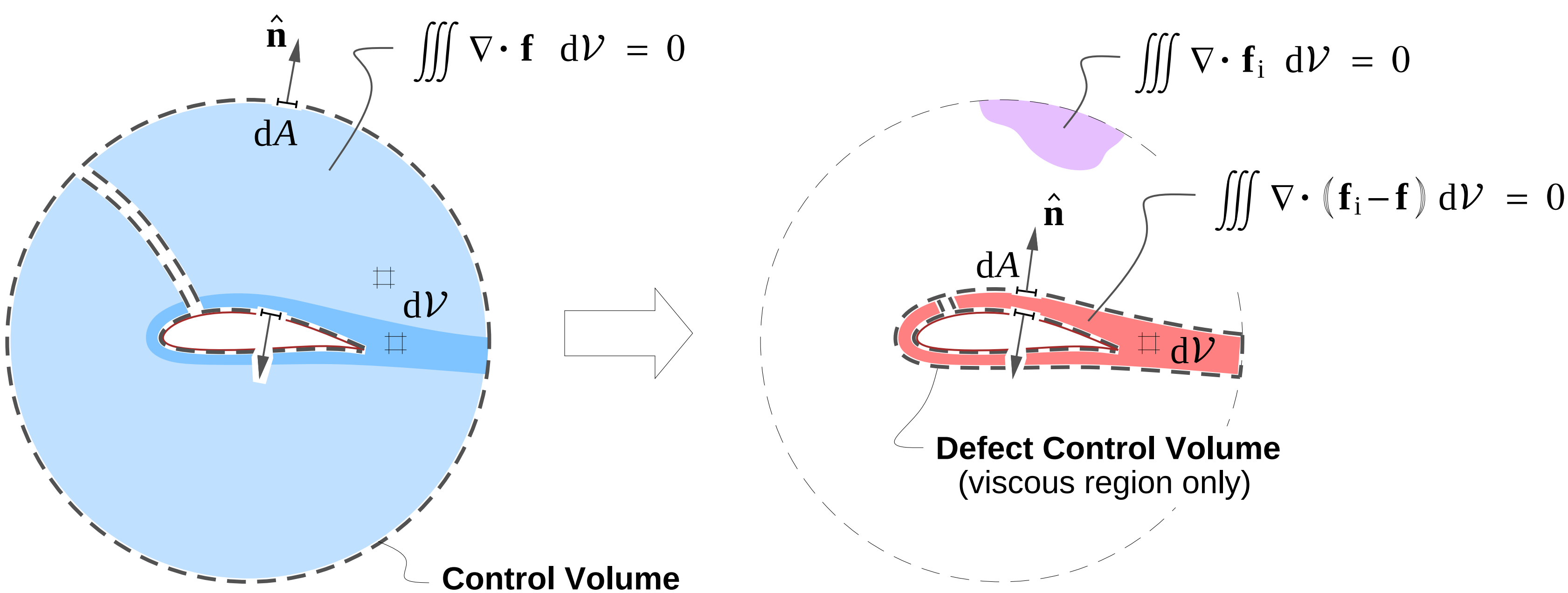
EIF/Defect Formulation – II

- Governing conservation laws for fluxes $\mathbf{f}$ split into EIF, Defect parts

$$\mathbf{f} \;=\; \mathbf{f}_\mathrm{i} \;-\; (\mathbf{f}_\mathrm{i} - \mathbf{f})$$

$$\boxed{\nabla \cdot \mathbf{f} \;=\; 0} \quad \Rightarrow \quad \boxed{\nabla \cdot \mathbf{f}_\mathrm{i} \;=\; 0} \;,\; \boxed{\nabla \cdot (\mathbf{f}_\mathrm{i} - \mathbf{f}) \;=\; 0}$$

- Inviscid solutions are economical, well automated
- Defect solutions needed only on compact viscous-region domain



EIF/Defect Formulation – III

Defect conservation law on differential volume spanning viscous layer:

$$\iiint \nabla\cdot(\mathbf{f}_{\rm i}-\mathbf{f})\,{\rm d}\mathcal{V} \;=\; \iint\left[\int_{y_{\rm w}}^{y_e}\nabla\cdot(\mathbf{f}_{\rm i}-\mathbf{f})\,{\rm d}y\right]{\rm d}x\,{\rm d}z$$

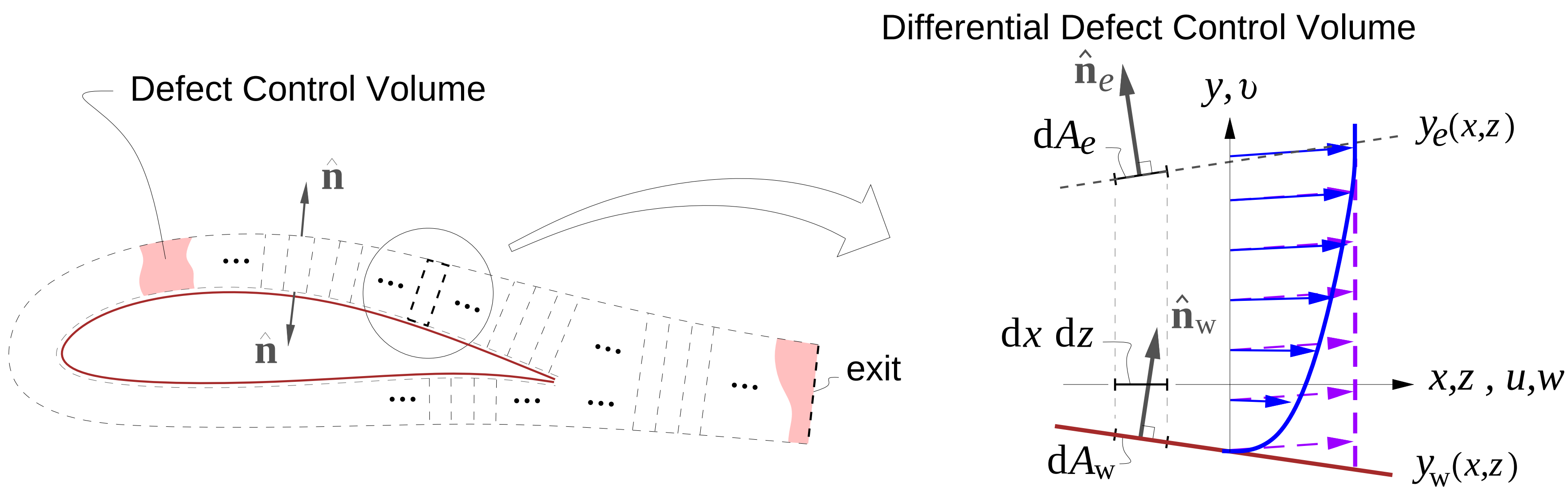
$$= \iint \widetilde{\nabla}\cdot\left[\int_{y_{\rm w}}^{y_e}\left(\tilde{\mathbf{f}}_{\rm i}-\tilde{\mathbf{f}}\right){\rm d}y\right]{\rm d}x\,{\rm d}z$$

$$+\; \cancel{\iint(\mathbf{f}_{\rm i}-\mathbf{f})_e\cdot\hat{\mathbf{n}}_e\,{\rm d}A_e}$$

$$-\; \iint(\mathbf{f}_{\rm i}-\mathbf{f})_{\rm w}\cdot\hat{\mathbf{n}}_{\rm w}\,{\rm d}A_{\rm w} \;=\; 0$$

$$\tilde{\mathbf{f}} \;=\; {\rm f}_x\,\hat{\mathbf{x}} \;+\; {\rm f}_z\,\hat{\mathbf{z}} \qquad \text{(in-plane flux)}$$

$$\widetilde{\nabla}() \;=\; \frac{\partial()}{\partial x}\,\hat{\mathbf{x}} \;+\; \frac{\partial()}{\partial z}\,\hat{\mathbf{z}} \qquad \text{(in-plane gradient)}$$



Defect Formulation

Defect equations on differential volumes spanning viscous layer:

$$\int(\mathsf{mass}_\mathrm{i} - \mathsf{mass})\,\mathrm{d}y \quad\rightarrow\quad \widetilde{\nabla}\cdot\mathbf{M} - \rho_{\mathrm{i_w}}\mathbf{q}_{\mathrm{i_w}}\cdot\hat{\mathbf{n}}_\mathrm{w} \;=\; 0$$

$$\int(\boldsymbol{\mathsf{mom}}_\mathrm{i} - \boldsymbol{\mathsf{mom}})\,\mathrm{d}y \quad\rightarrow\quad \widetilde{\nabla}\cdot\bar{\bar{\mathbf{J}}} - \rho_{\mathrm{i_w}}\mathbf{q}_{\mathrm{i_w}}\cdot\hat{\mathbf{n}}_\mathrm{w}\,\mathbf{q}_{\mathrm{i_w}} - \boldsymbol{\tau}_\mathrm{w} + (p_{\mathrm{i_w}} - p_\mathrm{w})\hat{\mathbf{n}}_\mathrm{w} \;=\; \mathbf{0}$$

$$\int(\mathbf{q}_\mathrm{i}\cdot\boldsymbol{\mathsf{mom}}_\mathrm{i} - \mathbf{q}\cdot\boldsymbol{\mathsf{mom}})\,\mathrm{d}y \quad\rightarrow\quad \widetilde{\nabla}\cdot\mathbf{E} - \rho_{\mathrm{i_w}}\mathbf{q}_{\mathrm{i_w}}\cdot\hat{\mathbf{n}}_\mathrm{w}\,q^2_{\mathrm{i_w}} - \rho_\mathrm{i}\mathbf{Q}\cdot\widetilde{\nabla}q^2_\mathrm{i} - 2\mathcal{D} \;=\; 0$$

$$\int(\mathbf{q}_\mathrm{i}\times\boldsymbol{\mathsf{mom}}_\mathrm{i} - \mathbf{q}\times\boldsymbol{\mathsf{mom}})\cdot\hat{\mathbf{y}}\,\mathrm{d}y \quad\rightarrow\quad \widetilde{\nabla}\cdot\mathbf{E}^\circ - \rho_{\mathrm{i_w}}\mathbf{q}_{\mathrm{i_w}}\cdot\hat{\mathbf{n}}_\mathrm{w}\,q^2_{\mathrm{i_w}}\psi_{\mathrm{i_w}} - \rho_\mathrm{i}\mathbf{Q}^\circ\cdot\widetilde{\nabla}q^2_\mathrm{i}\ldots - 2\mathcal{D}^\circ \;=\; 0$$

In-plane flow angle: $\psi \;\equiv\; \arctan(w/u)$

Integral defects:

$$\mathbf{M} \;\equiv\; \int(\rho_\mathrm{i}\mathbf{q}_\mathrm{i} - \rho\mathbf{q})\,\mathrm{d}y$$

$$\bar{\bar{\mathbf{J}}} \;\equiv\; \int(\rho_\mathrm{i}\mathbf{q}_\mathrm{i}\,\mathbf{q}_\mathrm{i}^T - \rho\mathbf{q}\,\mathbf{q}^T)\,\mathrm{d}y$$

$$\mathbf{E} \;\equiv\; \int(\rho_\mathrm{i}\mathbf{q}_\mathrm{i}\,q_\mathrm{i}^2 - \rho\mathbf{q}\,q^2)\,\mathrm{d}y$$

$$\mathbf{E}^\circ \;\equiv\; \int(\rho_\mathrm{i}\mathbf{q}_\mathrm{i}\,q_\mathrm{i}^2\psi_\mathrm{i} - \rho\mathbf{q}\,q^2\psi)\,\mathrm{d}y$$

$$\mathbf{Q} \;\equiv\; \int(\mathbf{q}_\mathrm{i} - \mathbf{q})\,\mathrm{d}y$$

$$\mathbf{Q}^\circ \;\equiv\; \int(\mathbf{q}_\mathrm{i}\psi_\mathrm{i} - \mathbf{q}\psi)\,\mathrm{d}y$$

Dissipation integrals:

$$\mathcal{D} \;\equiv\; \int(\bar{\bar{\boldsymbol{\tau}}}\cdot\nabla)\,\mathbf{q}\,\mathrm{d}y$$

$$\mathcal{D}^\circ \;\equiv\; \int(\bar{\bar{\boldsymbol{\tau}}}\cdot\nabla)\,[\mathbf{q}(\psi_\mathrm{i} - \psi)]\,\mathrm{d}y$$

Defect Formulation

Integral mass equation gives EIF wall mass flux for inviscid solver:

$$\rho_{\mathrm{i_w}} \mathbf{q}_{\mathrm{i_w}} \cdot \hat{\mathbf{n}}_{\mathrm{w}} \;=\; \widetilde{\nabla} \cdot \mathbf{M}$$

Also used to eliminate wall flux terms from remaining equations:

$$\widetilde{\nabla} \cdot \bar{\bar{\mathbf{J}}} - \mathbf{q}_{\mathrm{i_w}} \widetilde{\nabla} \cdot \mathbf{M} \;-\; \boldsymbol{\tau}_{\mathrm{w}} + (p_{\mathrm{i_w}} - p_{\mathrm{w}}) \hat{\mathbf{n}}_{\mathrm{w}} \;=\; \mathbf{0}$$

$$\widetilde{\nabla} \cdot \mathbf{E} \;-\; q_{\mathrm{i_w}}^2 \widetilde{\nabla} \cdot \mathbf{M} - \rho_{\mathrm{i}} \mathbf{Q} \cdot \widetilde{\nabla} q_{\mathrm{i}}^2 \;-\; 2\mathcal{D} \;=\; 0$$

$$\widetilde{\nabla} \cdot \mathbf{E}^{\circ} - \psi_{\mathrm{i_w}} \big(\widetilde{\nabla} \cdot \mathbf{E} - \rho_{\mathrm{i}} \mathbf{Q} \cdot \widetilde{\nabla} q_{\mathrm{i}}^2 \big) - \rho_{\mathrm{i}} \mathbf{Q}^{\circ} \cdot \widetilde{\nabla} q_{\mathrm{i}}^2 + \tfrac{1}{2} \rho_{\mathrm{i}} \mathbf{Q} \times \widetilde{\nabla} q_{\mathrm{i}}^2 \;-\; 2\mathcal{D}^{\circ} \;=\; 0$$

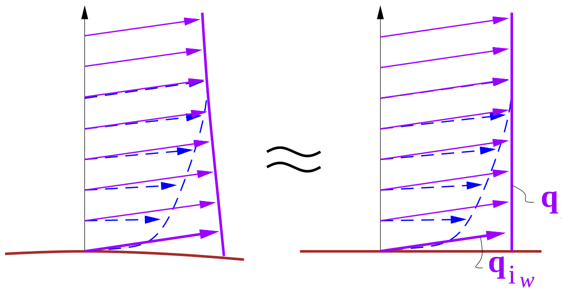
Consistency with Classical Theory

Present method's x-momentum equation:

$$\widetilde{\nabla} \cdot \mathbf{J}_x - u_{\mathrm{i_w}} \widetilde{\nabla} \cdot \mathbf{M} - \tau_{x_\mathrm{w}} + (p_{\mathrm{i_w}} - p_\mathrm{w})\, n_{x_\mathrm{w}} = 0 \qquad (*)$$

With the small-curvature BL approximations,

$$\mathbf{q}_{\mathrm{i_w}} = \mathbf{q}_\mathrm{i} = \text{constant in } y$$
$$p_{\mathrm{i_w}} = p_\mathrm{i} = \text{constant in } y$$



equation $(*)$ can be restated as

$$\widetilde{\nabla} \cdot (\mathbf{J}_x - u_\mathrm{i}\, \mathbf{M}) + \mathbf{M} \cdot \widetilde{\nabla} u_\mathrm{i} - \tau_{x_\mathrm{w}} = 0$$

In the 2D case this becomes

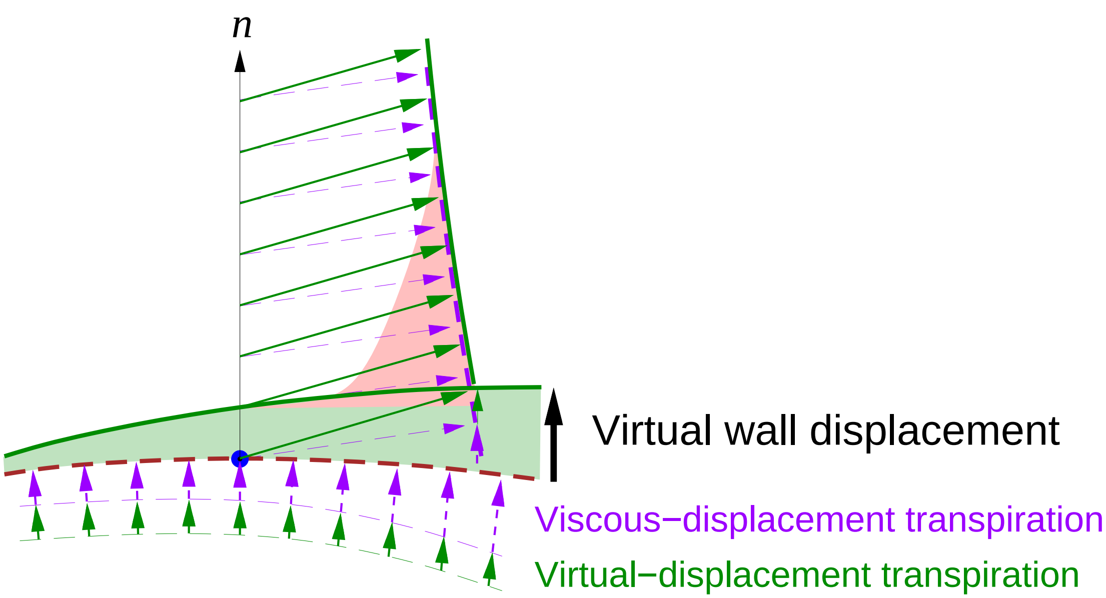
$$\frac{\mathrm{d}(\mathrm{J} - u_\mathrm{i}\, \mathrm{M})}{\mathrm{d}x} + \mathrm{M} \frac{\mathrm{d}u_\mathrm{i}}{\mathrm{d}x} - \tau_\mathrm{w} = 0$$

which is the same as the von Karman integral momentum equation:

$$\frac{\mathrm{d}(\rho_\mathrm{i}\, u_\mathrm{i}^2\, \theta)}{\mathrm{d}x} + \rho_\mathrm{i}\, u_\mathrm{i}\, \delta^* \frac{\mathrm{d}u_\mathrm{i}}{\mathrm{d}x} - \tau_\mathrm{w} = 0$$

Attractive Features of Defect Formulation

- Fully exploits any inviscid solver
  - Panel (volume grid not required at all)
  - Full Potential
  - Euler
- Can represent small geometry changes, steady or unsteady
  - Inverse or optimization design change without regridding
  - Linearized aeroelasticity — static, forced dynamic, flutter



Central Ideas of Present Work

- Assume EIF is represented by any fast inviscid formulation ...
  - Vortex Lattice (AVL)
  - Panel (PANAIR, PMARC, QUADPAN, etc.)
  - Full Potential (TRANAIR)
  - Euler (CART-3D, etc.)
- Develop complementary "surface-only" method (IBL3) to represent Viscous Defect for any EIF formulation
- Use modern sparse-matrix methods (ILU, GMRES ...) to solve overall EIF+Defect problem

IBL3 Formulation

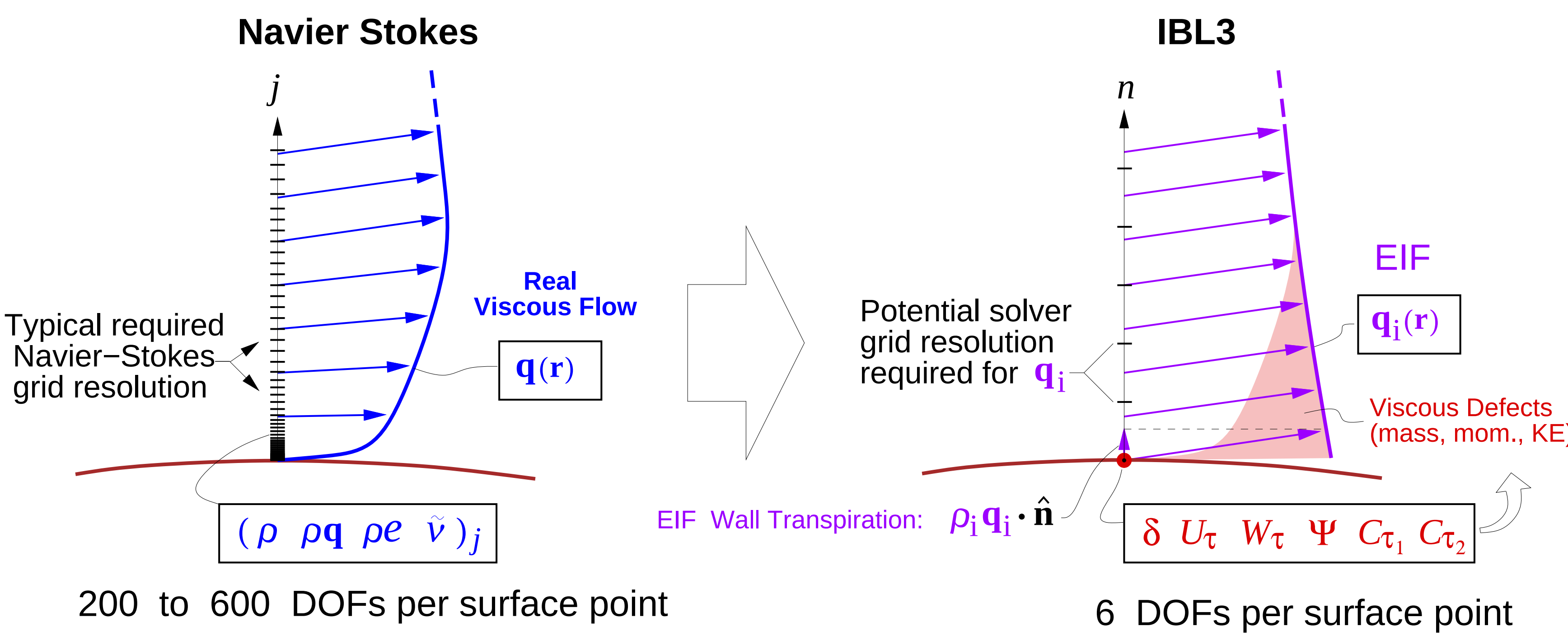
- Viscous defects represented by assumed profiles, parameterized by

$$\delta \quad U_\tau \quad W_\tau \quad \Psi \quad , \qquad \text{roughly equivalent to} \qquad \delta_1^* \quad \theta_{11} \quad \delta_2^* \quad \theta_{12}$$

- TS, CF-wave amplitudes and Reynolds stresses parameterized by

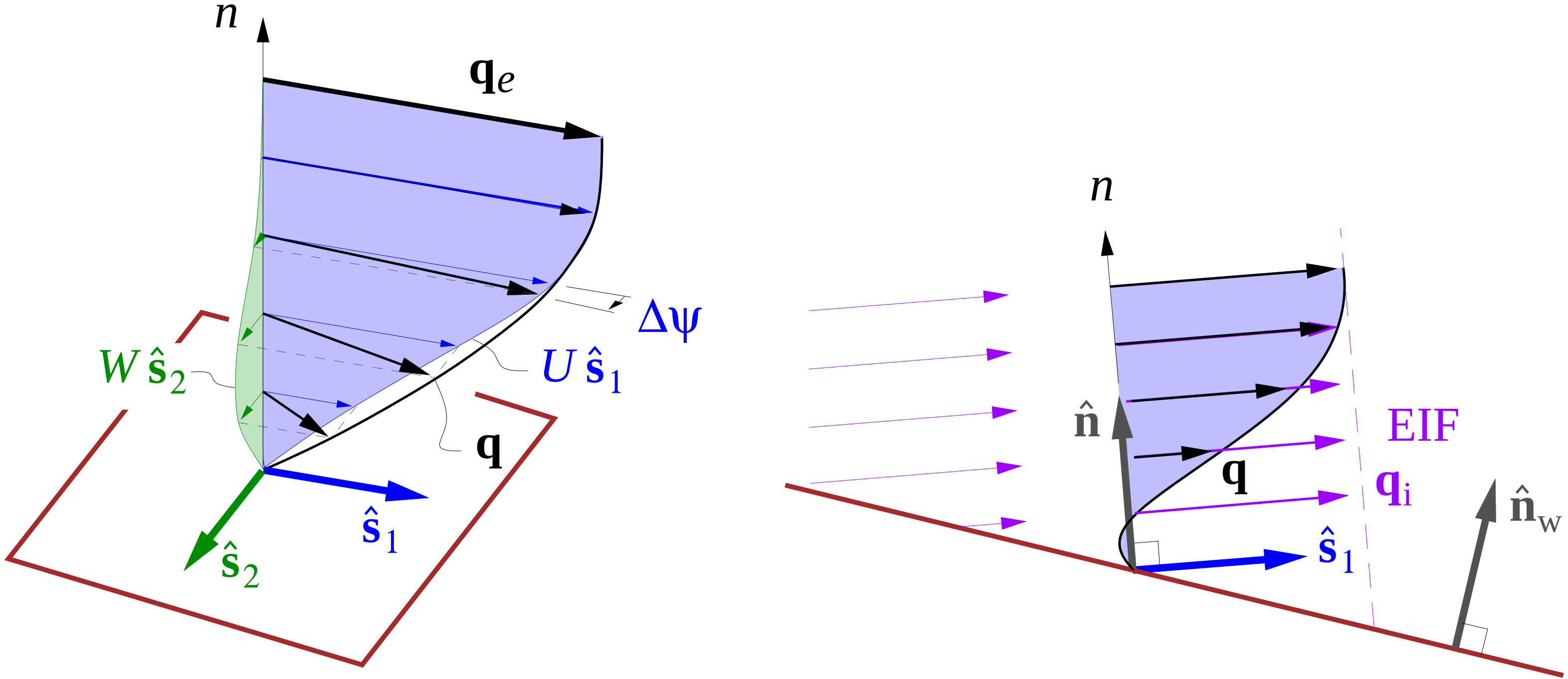
$$C_{\tau_1} \quad C_{\tau_2} \quad , \qquad \text{equivalent to} \qquad \|u_1'\| \quad \|u_2'\|$$

- Enormous reduction in number of unknowns from RANS



IBL3 Profile Construction

Velocity profile $\mathbf{q}(n)$ formed from streamwise and crossflow profiles U, W along $\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2$ basis vectors



$$\hat{\mathbf{s}}_1 = \frac{\mathbf{q}_i}{|\mathbf{q}_i|} \quad , \quad \hat{\mathbf{s}}_2 = \frac{\mathbf{q}_i \times \hat{\mathbf{n}}_w}{|\mathbf{q}_i \times \hat{\mathbf{n}}_w|} \quad , \quad q_i = |\mathbf{q}_i|$$

$$\mathbf{q}(n) = q_i \, (U\,\hat{\mathbf{s}}_1 + W\,\hat{\mathbf{s}}_2)$$

Assumed Laminar Profiles

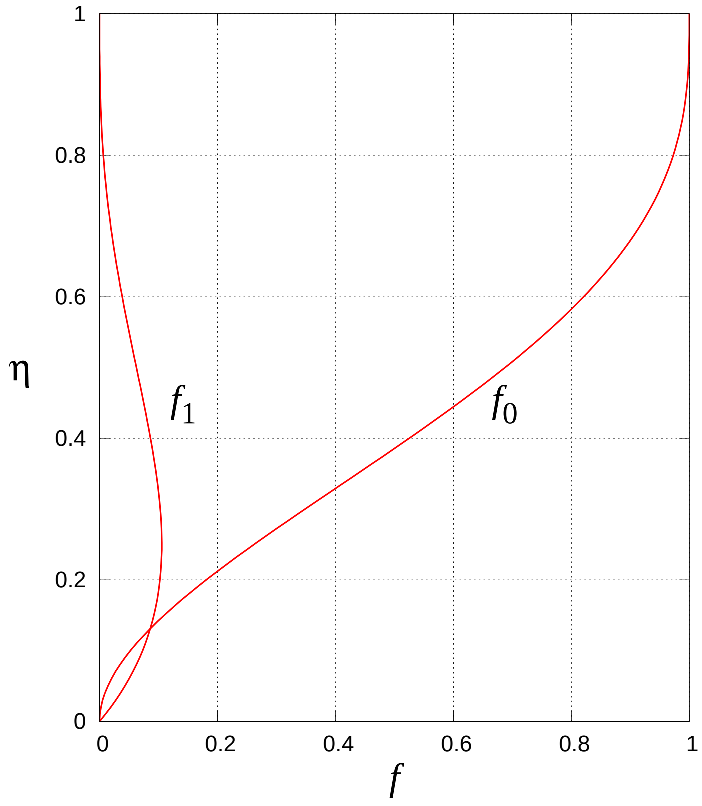
$$U\,(y\,;\,\delta\,,U_\tau\,,W_\tau) \;=\; A\,f_1(\eta) \;+\; f_0(\eta)$$

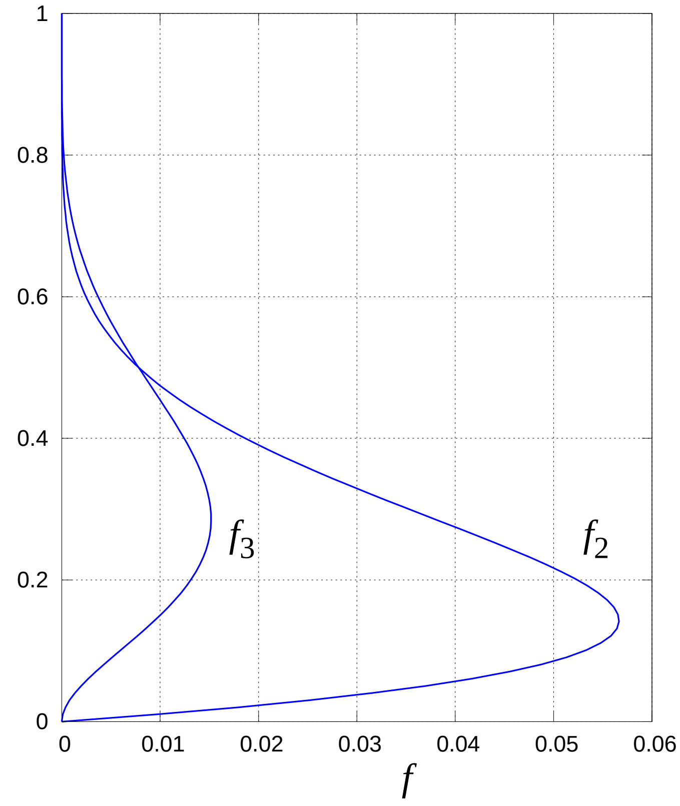
$$W\,(y\,;\,\delta\,,U_\tau\,,W_\tau\,,\Psi) \;=\; B\,f_2(\eta) \;+\; \Psi\,f_3(\eta)$$

$$\eta \;=\; y/\delta$$

$$A \;=\; U_\tau\,Re_\delta\,(U_\tau^2+W_\tau^2)^{1/2}$$

$$B \;=\; W_\tau\,Re_\delta\,(U_\tau^2+W_\tau^2)^{1/2}$$





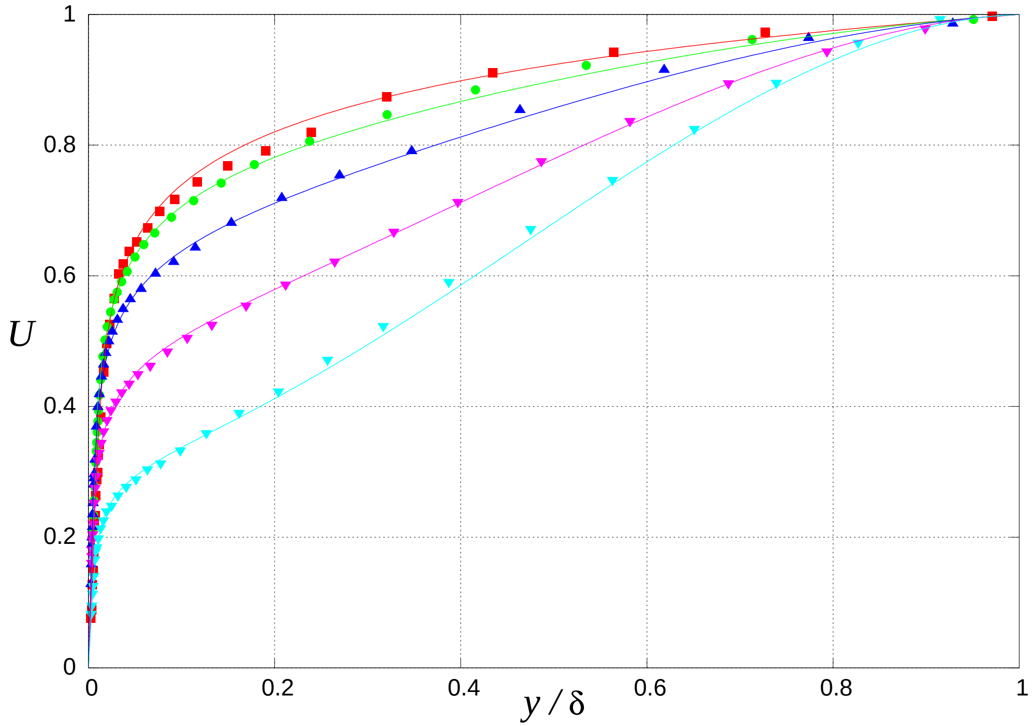
Assumed Turbulent Profiles

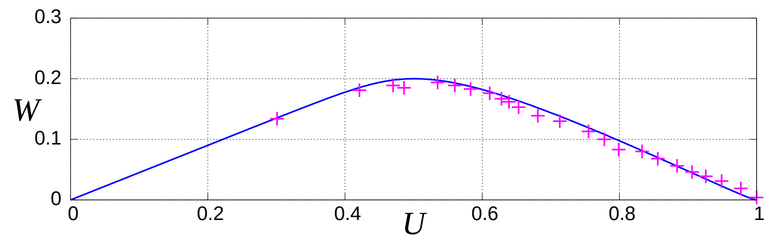
$$U(y\,;\delta\,,U_\tau\,,W_\tau\,,\Psi) \;=\; U_\tau\; u_S^+(y^+) \;+\; K\cos(\alpha-\Psi(1-\eta)^2)\; g_o(\eta)$$

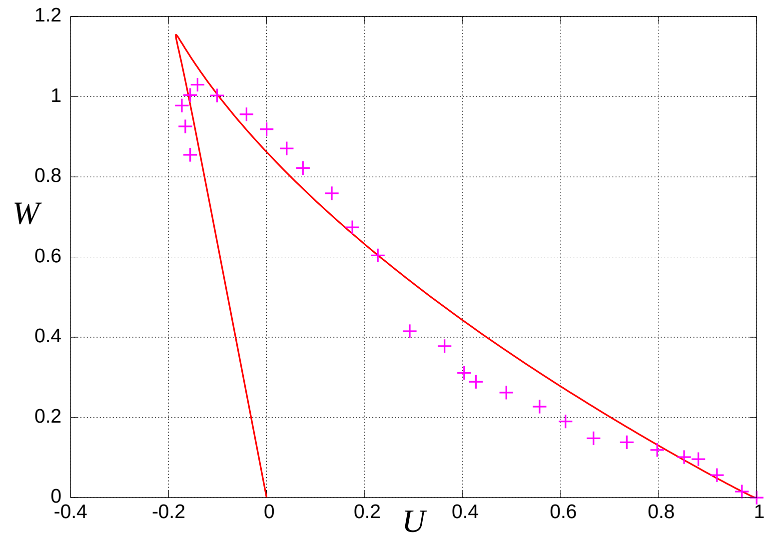
$$W(y\,;\delta\,,U_\tau\,,W_\tau\,,\Psi) \;=\; W_\tau\; u_S^+(y^+) \;-\; K\sin(\alpha-\Psi(1-\eta)^2)\; g_o(\eta)$$

$$K(\delta\,,U_\tau\,,W_\tau) \;=\; \left[\,(W_\tau\, u_S^+(\delta^+))^2 + (1-U_\tau\, u_S^+(\delta^+))^2\,\right]^{1/2}$$

$$\alpha(\delta\,,U_\tau\,,W_\tau) \;=\; \arctan\left[\,(W_\tau\, u_S^+(\delta^+))/(1-U_\tau\, u_S^+)\,\right]$$







Density and Viscosity Profiles

Density profile R given via Crocco-Busemann enthalpy profile:

$$\frac{h}{h_\mathrm{i}} \;=\; \frac{\rho_\mathrm{i}}{\rho} \;\equiv\; \frac{1}{R} \;=\; 1 \;+\; \Delta\mathcal{H}_\mathrm{w}\left(1-U\right) \;+\; \mathcal{E}'\left(1-U^2-W^2\right)$$

Eckert number: $$\mathcal{E}' \;\equiv\; r\,\frac{\gamma-1}{2}M_\mathrm{i}^2$$

Wall overheat ratio: $$\Delta\mathcal{H}_\mathrm{w} \;\equiv\; \frac{h_\mathrm{w}-h_\mathrm{aw}}{h_\mathrm{i}} \;=\; \frac{h_\mathrm{w}}{h_\mathrm{i}} \;-\; (1+\mathcal{E}')$$

Viscosity profile $\mathcal{M}$ given by Sutherland's formula:

$$\frac{\mu}{\mu_\mathrm{i}} \;\equiv\; \mathcal{M} = \left(\frac{h}{h_\mathrm{i}}\right)^{3/2}\frac{h_\mathrm{i}+h_S}{h+h_S}$$

Integral Thicknesses

Integral thicknesses in terms of assumed $U, W, R, \mathcal{M}$ profiles:

$$
\begin{aligned}
\delta_1^* \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(1 - RU\big) \; \mathrm{d}\eta \\
\delta_2^* \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(0 - RW\big) \; \mathrm{d}\eta \\
\phi_{11} \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(1 - RU^2\big) \; \mathrm{d}\eta \\
\phi_{12} \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(1 - RUW\big) \; \mathrm{d}\eta \\
\phi_{21} \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(0 - RUW\big) \; \mathrm{d}\eta \\
\phi_{22} \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(0 - RW^2\big) \; \mathrm{d}\eta \\
\phi_1^* \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(1 - RU(U^2 + W^2)\big) \; \mathrm{d}\eta \\
\phi_2^* \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(0 - RW(U^2 + W^2)\big) \; \mathrm{d}\eta \\
\phi_1^\circ \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(0 - RU(U^2 + W^2)\Delta\psi\big) \; \mathrm{d}\eta \\
\phi_2^\circ \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(0 - RW(U^2 + W^2)\Delta\psi\big) \; \mathrm{d}\eta \\
\delta_1' \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(1 - U\big) \; \mathrm{d}\eta \\
\delta_2' \, (\delta\,, U_\tau\,, W_\tau\,, \Psi\,, q_\mathrm{i}) &= \delta \int \big(0 - W\big) \; \mathrm{d}\eta
\end{aligned}
$$

Thicknesses and their derivatives evaluated on-the-fly using Gaussian quadrature

Skin Friction and Dissipation

$$\tfrac{1}{2}C_{f_1}\,(U_\tau\,,W_\tau\,,q_{\rm i}) = \mathcal{M}_{\rm w}\,U_\tau\,(U_\tau^2+W_\tau^2)^{1/2}$$

$$\tfrac{1}{2}C_{f_2}\,(U_\tau\,,W_\tau\,,q_{\rm i}) = \mathcal{M}_{\rm w}\,W_\tau\,(U_\tau^2+W_\tau^2)^{1/2}$$

laminar: $$C_{\mathcal{D}}\,(U_\tau\,,W_\tau\,,\Psi\,,q_{\rm i}) = \int \mathcal{M}\left[\left(\frac{{\rm d}U}{{\rm d}\eta}\right)^2 + \left(\frac{{\rm d}W}{{\rm d}\eta}\right)^2\right]{\rm d}\eta$$

turbulent: $$C_{\mathcal{D}}\,(U_\tau\,,W_\tau\,,\Psi\,,C_{\tau_1}\,,C_{\tau_2}\,,q_{\rm i}) = \tfrac{1}{2}C_{f_1}\,U_s \;+\; \tfrac{1}{2}C_{f_2}\,W_s \\ +\,C_{\tau_1}(1-U_s) \;+\; C_{\tau_2}W_s$$

Integral Defects (Fluxes) and Residuals

$$\mathbf{q}_1 = q_\mathrm{i}\,\hat{\mathbf{s}}_1 \quad , \qquad \mathbf{q}_2 = q_\mathrm{i}\,\hat{\mathbf{s}}_2$$

$$\mathbf{M}\,(\delta\,,U_\tau\,,W_\tau\,,\Psi\,,\mathbf{q}_\mathrm{i}) = \rho_\mathrm{i}\left(\delta_1^*\,\mathbf{q}_1 + \delta_2^*\,\mathbf{q}_2\right)$$

$$\bar{\bar{\mathbf{J}}}\,(\delta\,,U_\tau\,,W_\tau\,,\Psi\,,\mathbf{q}_\mathrm{i}) = \rho_\mathrm{i}\left(\phi_{11}\,\mathbf{q}_1\,\mathbf{q}_1^T + \phi_{12}\,\mathbf{q}_1\,\mathbf{q}_2^T + \phi_{21}\,\mathbf{q}_2\,\mathbf{q}_1^T + \phi_{22}\,\mathbf{q}_2\,\mathbf{q}_2^T\right)$$

$$\mathbf{E}\,(\delta\,,U_\tau\,,W_\tau\,,\Psi\,,\mathbf{q}_\mathrm{i}) = \rho_\mathrm{i}\,q_\mathrm{i}^2\left(\phi_1^*\,\mathbf{q}_1 + \phi_2^*\,\mathbf{q}_2\right)$$

$$\mathbf{E}^\circ(\delta\,,U_\tau\,,W_\tau\,,\Psi\,,\mathbf{q}_\mathrm{i}) = \rho_\mathrm{i}\,q_\mathrm{i}^2\left(\phi_1^\circ\,\mathbf{q}_1 + \phi_2^\circ\,\mathbf{q}_2\right)$$

$$\mathbf{Q}\,(\delta\,,U_\tau\,,W_\tau\,,\Psi\,,\mathbf{q}_\mathrm{i}) = \left(\delta_1'\,\mathbf{q}_1 + \delta_2'\,\mathbf{q}_2\right)$$

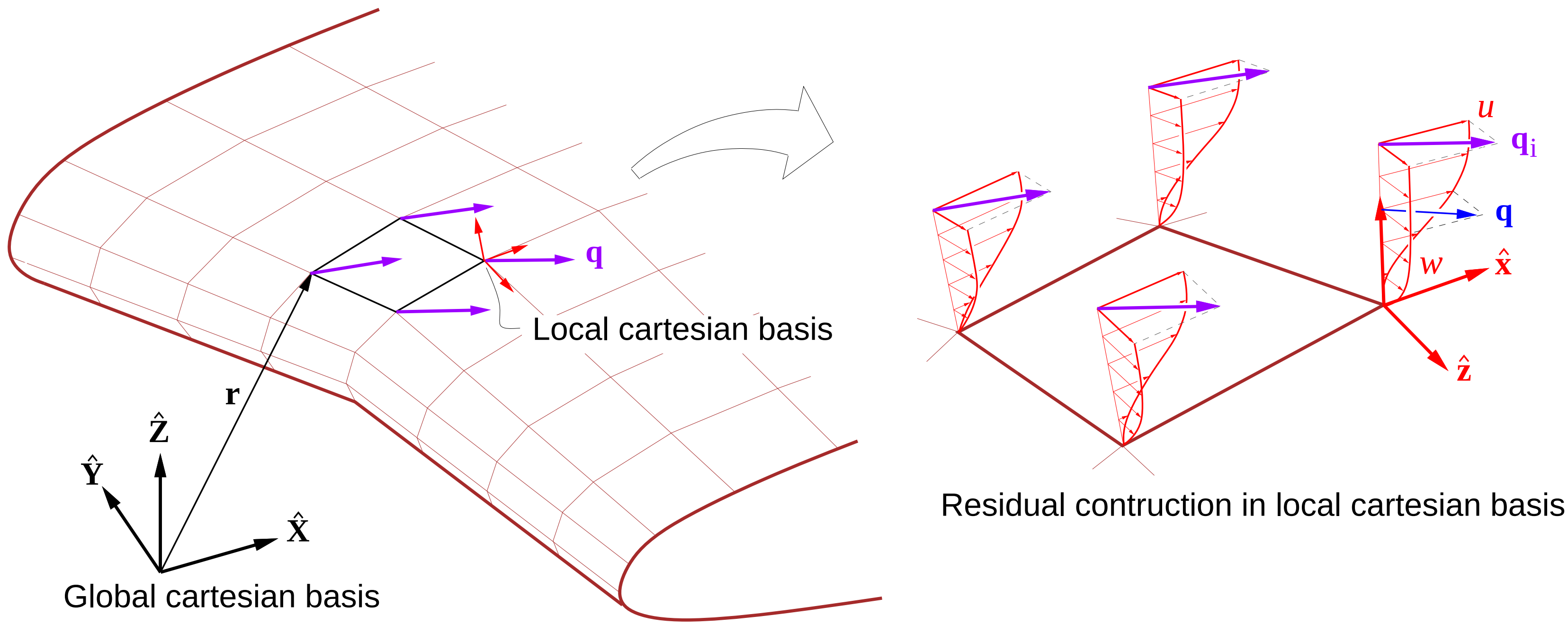
$$\boldsymbol{\tau}\,(\delta\,,U_\tau\,,W_\tau\,,\Psi\,,\mathbf{q}_\mathrm{i}) = \rho_\mathrm{i}\,q_\mathrm{i}\left(C_{f_1}\,\mathbf{q}_1 + C_{f_2}\,\mathbf{q}_2\right)$$

$$\mathcal{D}\,(\delta\,,U_\tau\,,W_\tau\,,\Psi\,,\mathbf{q}_\mathrm{i}\,,C_{\tau_1}\,,C_{\tau_2}) = \rho_\mathrm{i}\,q_\mathrm{i}^3\,C_\mathcal{D}$$

IBL3 Equation Discretization – I

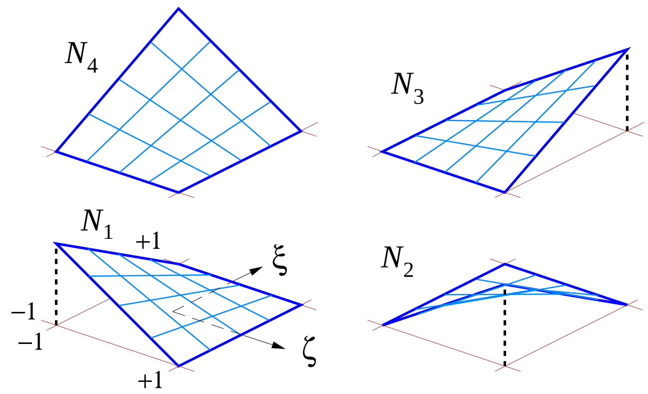
- Surface finite element discretization allows arbitrary geometry
- Geometry $\mathbf{r}$, velocities $\mathbf{q}$, defects $\mathbf{M}, \bar{\bar{\mathbf{J}}} \ldots$ computed in global basis XYZ
- Fluxes put into local cartesian basis xyz for residual construction

$$
\begin{aligned}
u &= \mathbf{q} \cdot \hat{\mathbf{x}} \quad , \quad w = \mathbf{q} \cdot \hat{\mathbf{z}} \\
\mathrm{M}_x &= \mathbf{M} \cdot \hat{\mathbf{x}} \quad , \quad \mathrm{M}_z = \mathbf{M} \cdot \hat{\mathbf{z}} \\
\mathrm{J}_{xx} &= \hat{\mathbf{x}} \cdot \bar{\bar{\mathbf{J}}} \cdot \hat{\mathbf{x}} \quad , \quad \mathrm{J}_{xz} = \hat{\mathbf{x}} \cdot \bar{\bar{\mathbf{J}}} \cdot \hat{\mathbf{z}} \\
\mathrm{J}_{zx} &= \hat{\mathbf{z}} \cdot \bar{\bar{\mathbf{J}}} \cdot \hat{\mathbf{x}} \quad , \quad \mathrm{J}_{zz} = \hat{\mathbf{z}} \cdot \bar{\bar{\mathbf{J}}} \cdot \hat{\mathbf{z}}
\end{aligned}
$$



IBL3 Equation Discretization – II

Finite-element discretization in local xyz basis



Variable interpolation:

$$u_{(\xi,\zeta)} = \sum_i u_i \, N_{i(\xi,\zeta)}$$

$$\mathrm{M}_{x(\xi,\zeta)} = \sum_i (\mathrm{M}_x)_i \, N_{i(\xi,\zeta)} \qquad \text{etc.}$$

Weighted residuals:

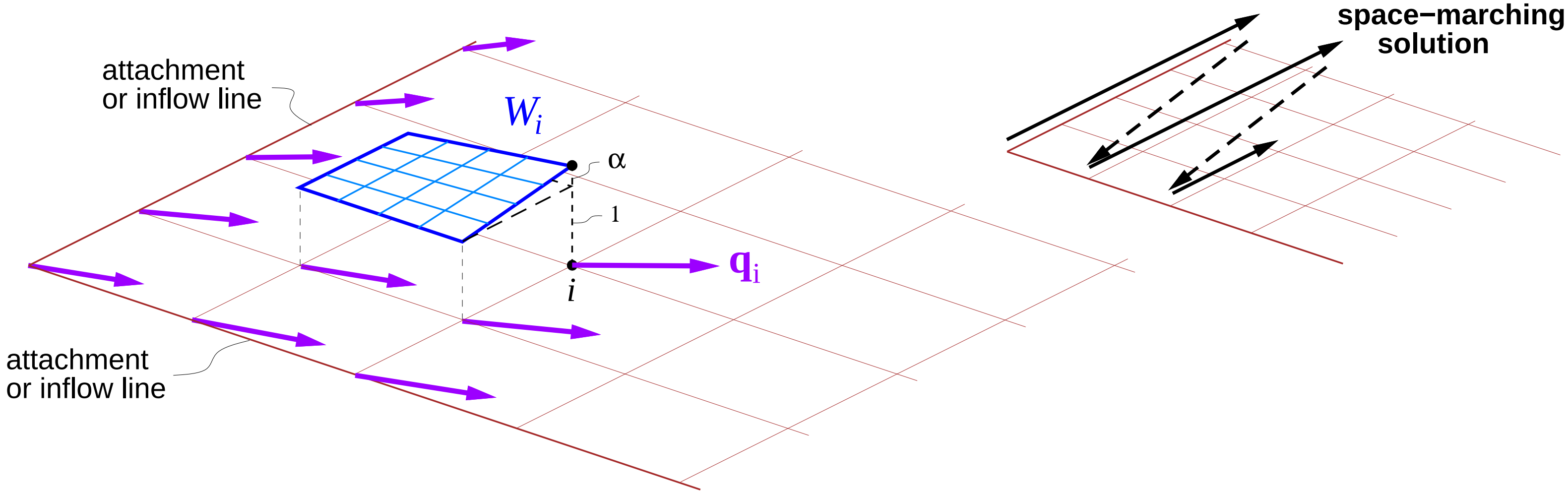
$$\mathcal{R}_i^x \, (\delta, U_\tau, W_\tau, \Psi, C_{\tau_1}, C_{\tau_2}, \mathbf{q}_\mathrm{i})_j \equiv \iint \left\{ \widetilde{\nabla} \cdot \mathbf{J}_x - u_{\mathrm{i_w}} \widetilde{\nabla} \cdot \mathbf{M} - \tau_{x_\mathrm{w}} \right\} W_i \, J \, \mathrm{d}\xi \, \mathrm{d}\zeta$$

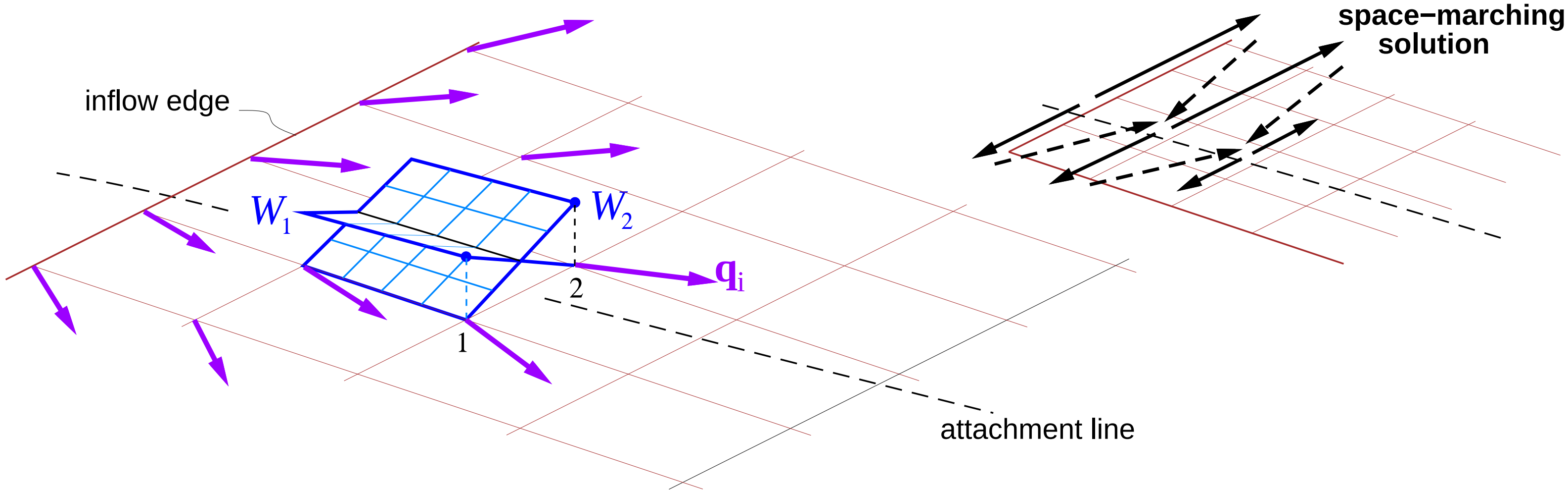
$$\mathcal{R}_i^z \, (\delta, U_\tau, W_\tau, \Psi, C_{\tau_1}, C_{\tau_2}, \mathbf{q}_\mathrm{i})_j \equiv \iint \left\{ \widetilde{\nabla} \cdot \mathbf{J}_z - w_{\mathrm{i_w}} \widetilde{\nabla} \cdot \mathbf{M} - \tau_{z_\mathrm{w}} \right\} W_i \, J \, \mathrm{d}\xi \, \mathrm{d}\zeta \qquad \text{etc.}$$

Weight function $W_{i(\xi,\zeta)}$ depends on the chosen solution method . . .

IBL3 Equation Discretization – III

Weight functions for space-marching solution:

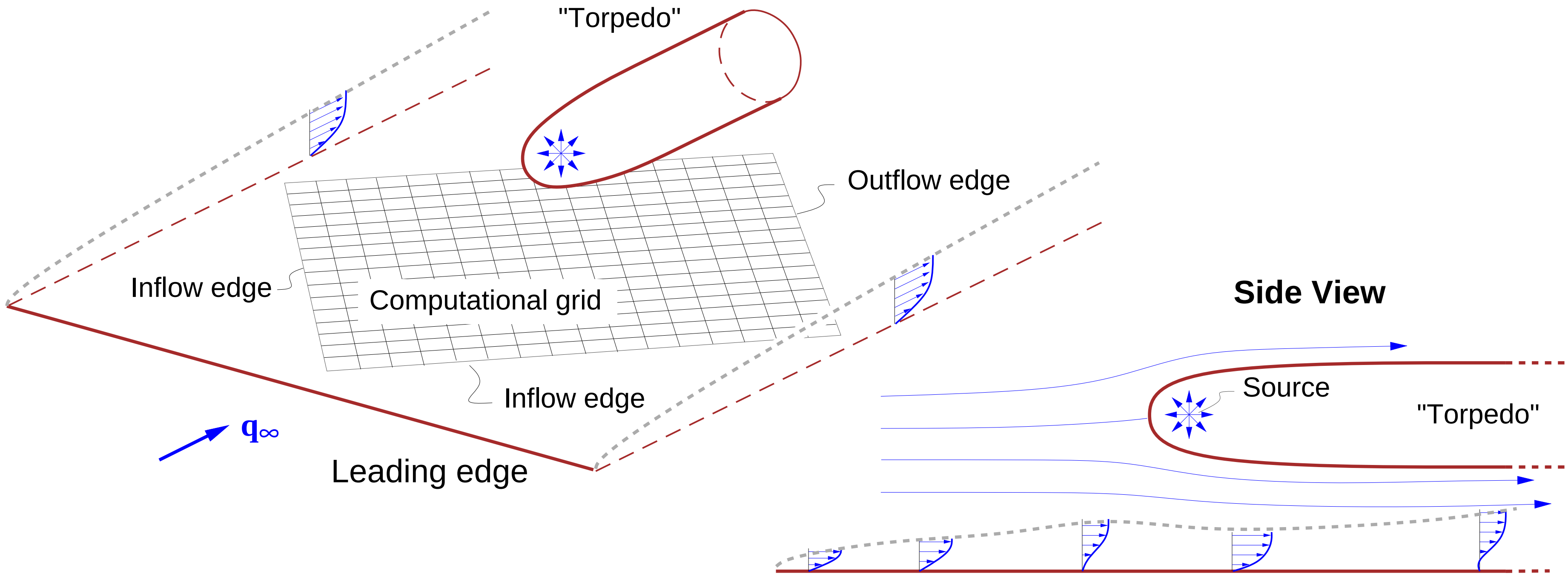




- Requires obeying characteristic directions (difficult)
- Impossible to use for separated flow

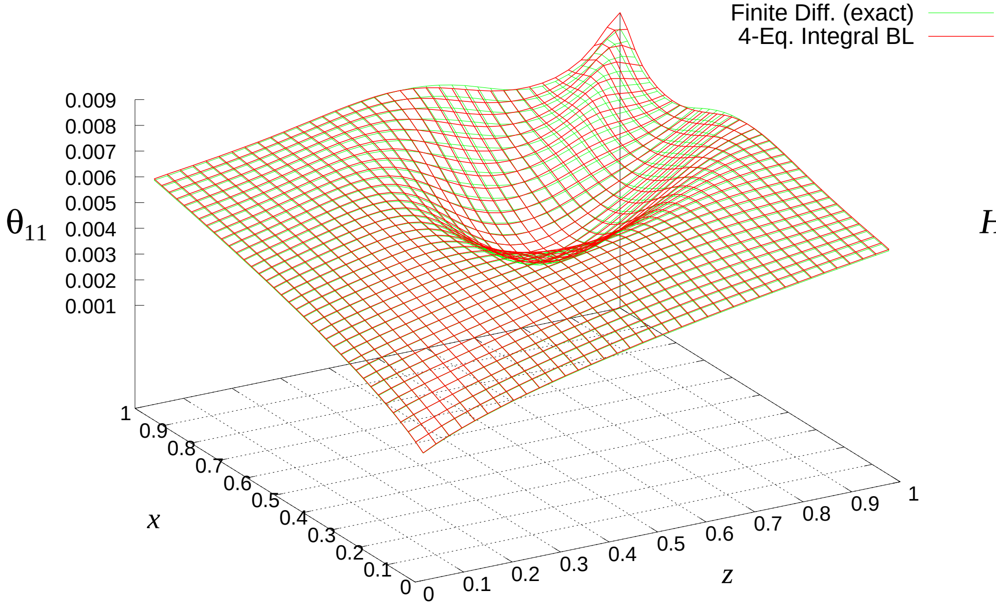
Torpedo Test Case

“Torpedo” over a wall boundary layer

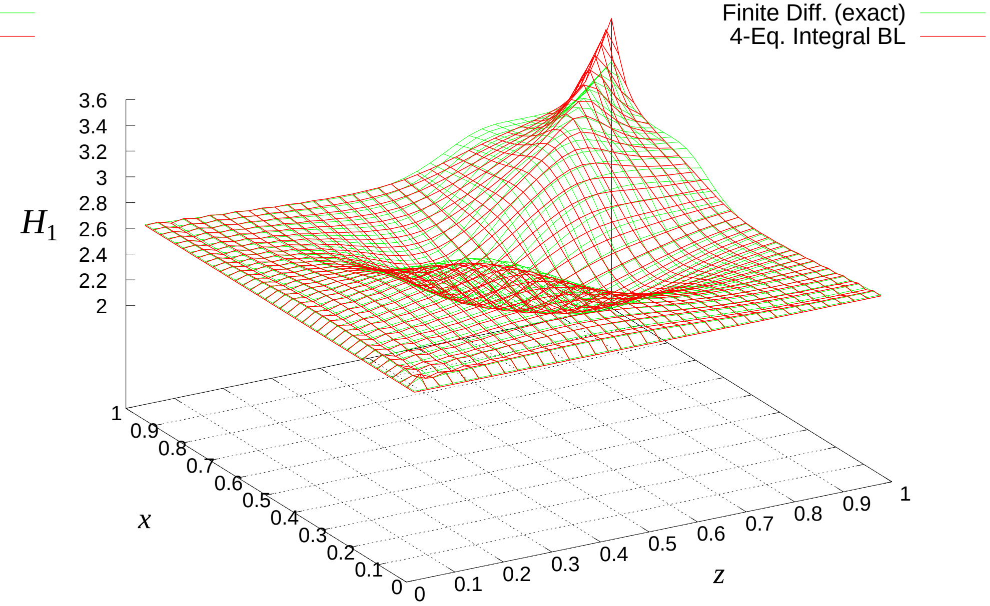


Torpedo Test Case

Streamwise momentum thickness

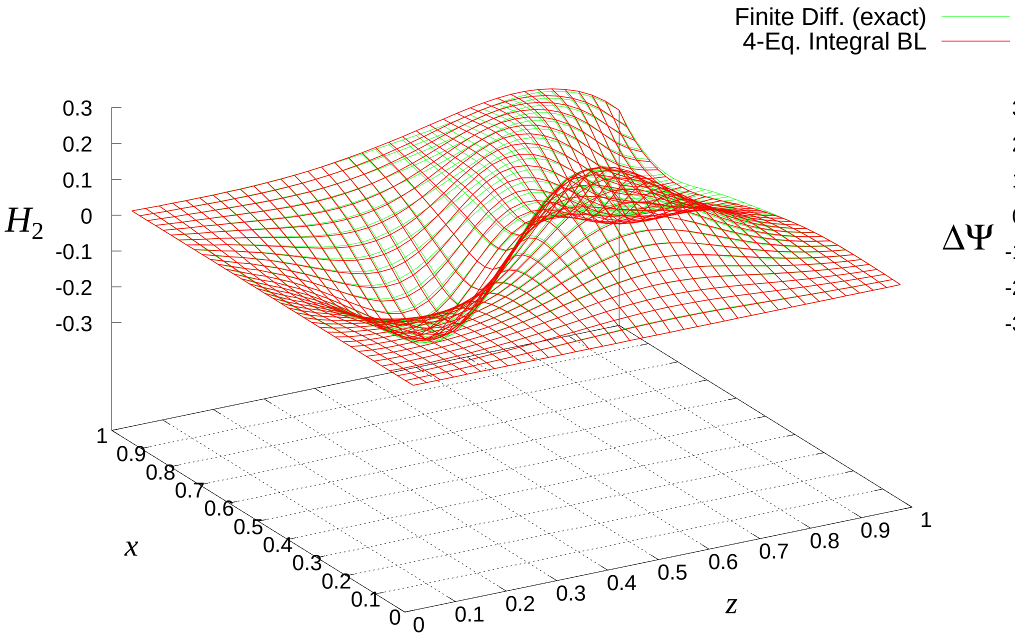


Streamwise shape parameter

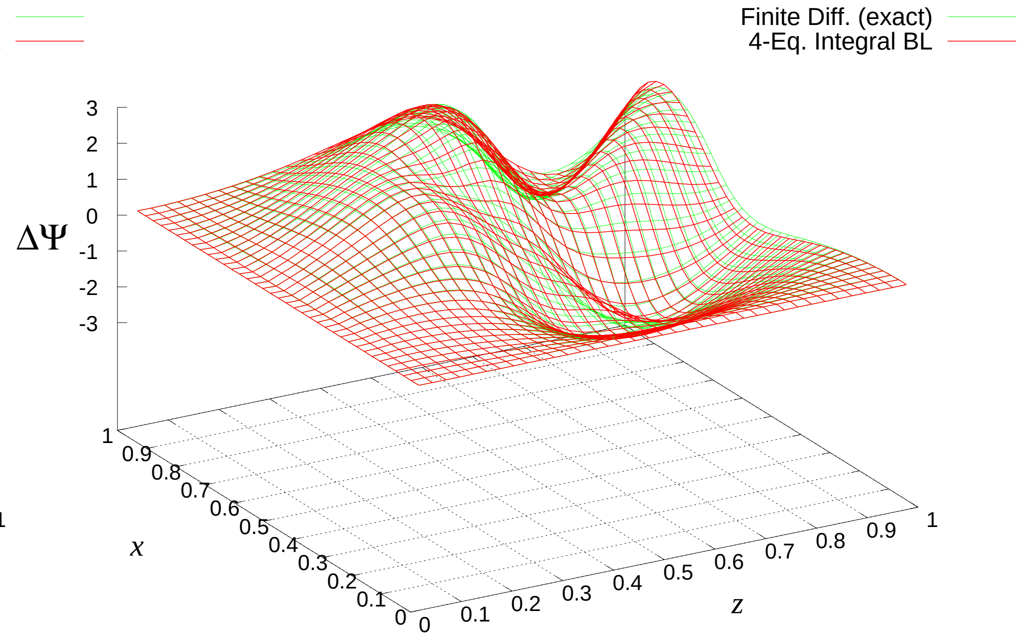


Torpedo Test Case

Crossflow shape parameter

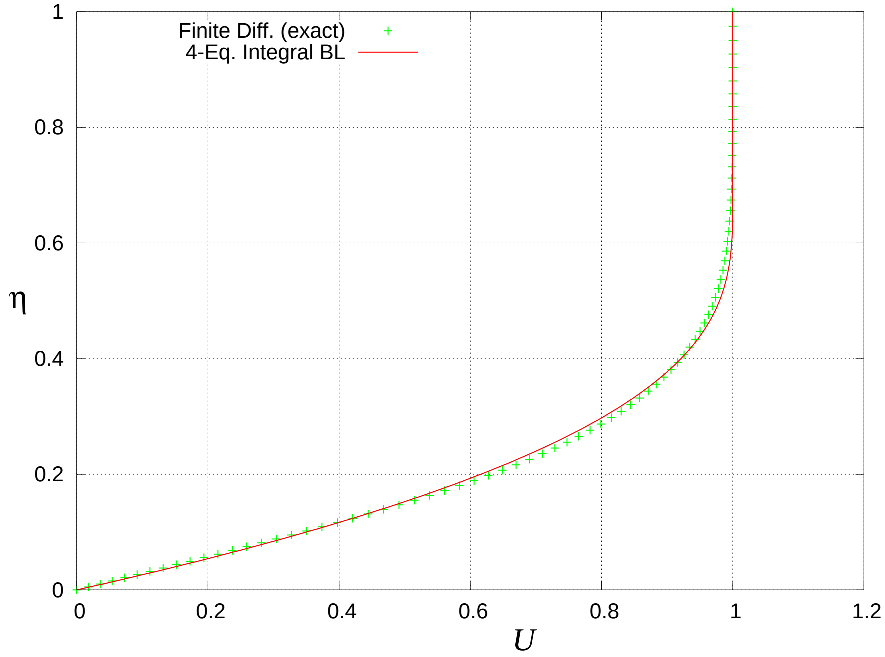


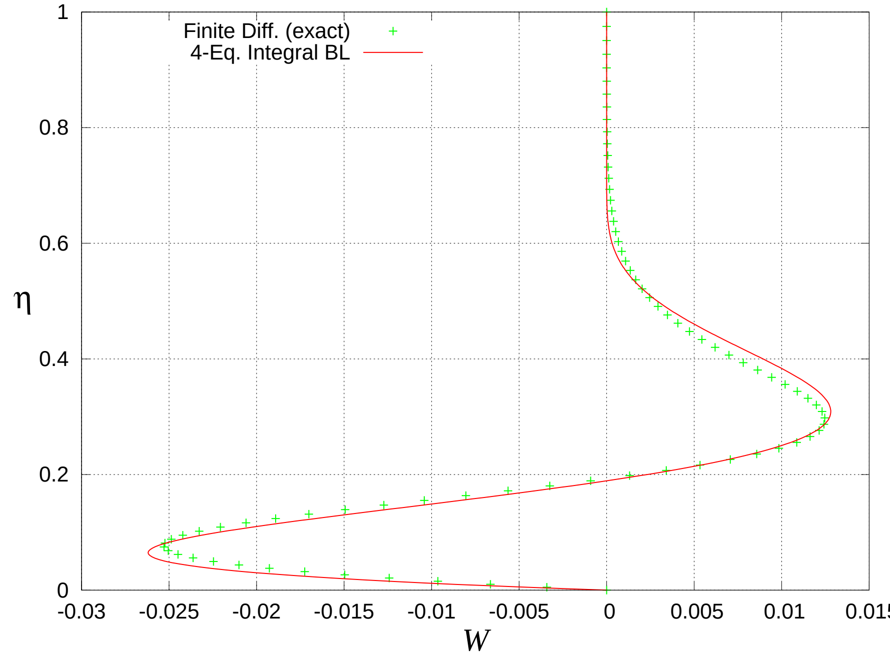
Profile crossover parameter



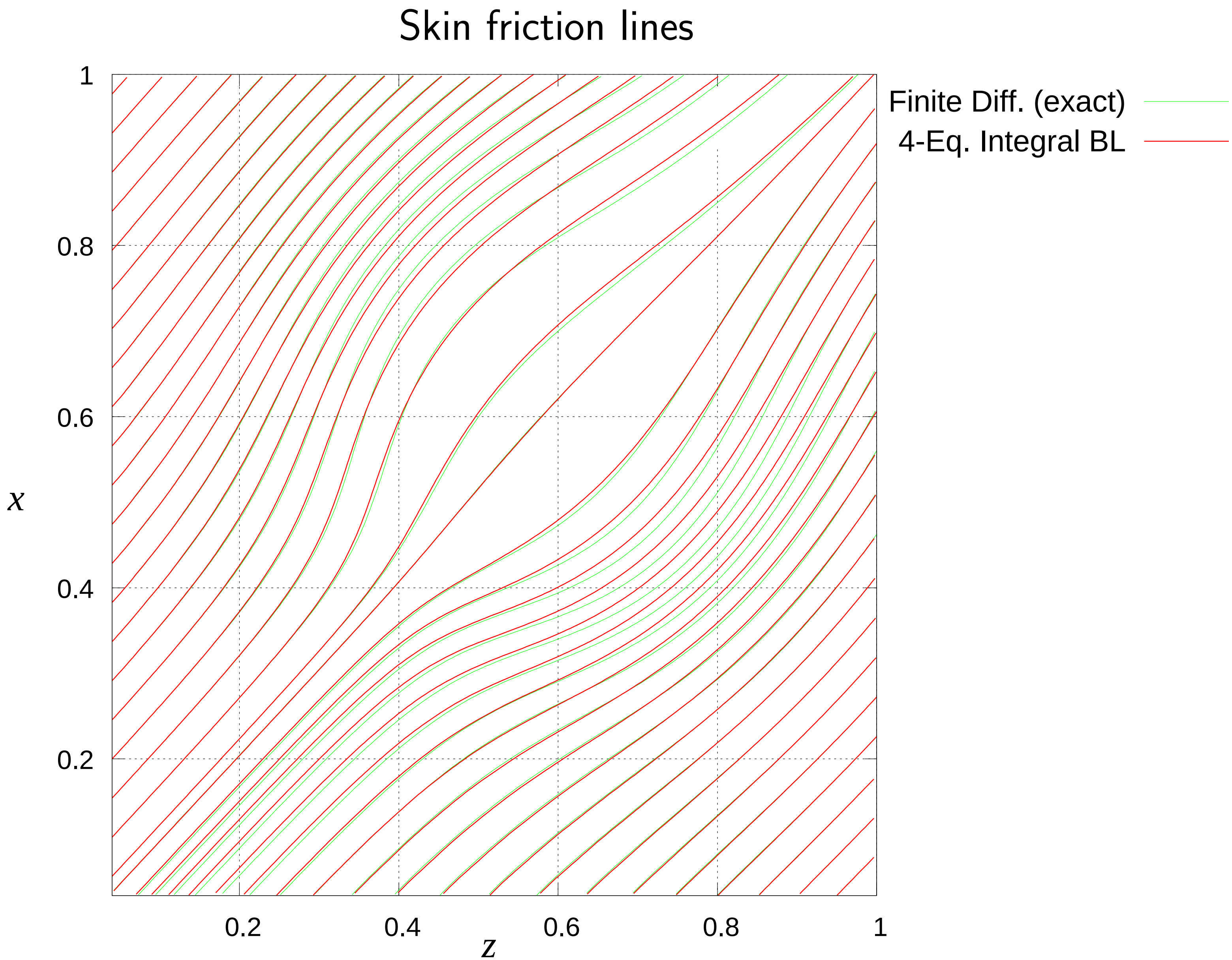
Torpedo Test Case

Streamwise and crossflow profiles





Torpedo Test Case



3D BL Numerical Problem Comparison

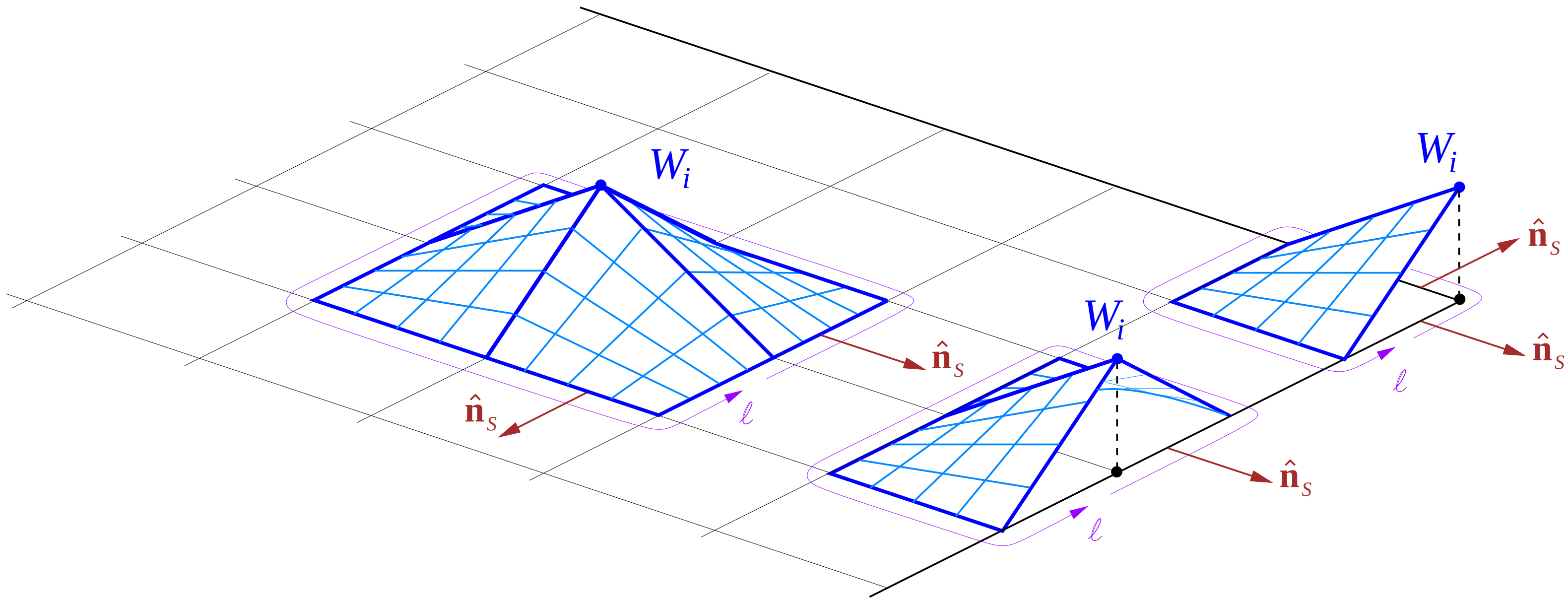
Space-marching solution with prescribed $\mathbf{q}_\mathrm{i}$

45×45 surface grid

	Finite Difference	IBL3
DOFs	740 000	8 100
Runtime	45 s	0.5 s

IBL3 Equation Discretization – IV

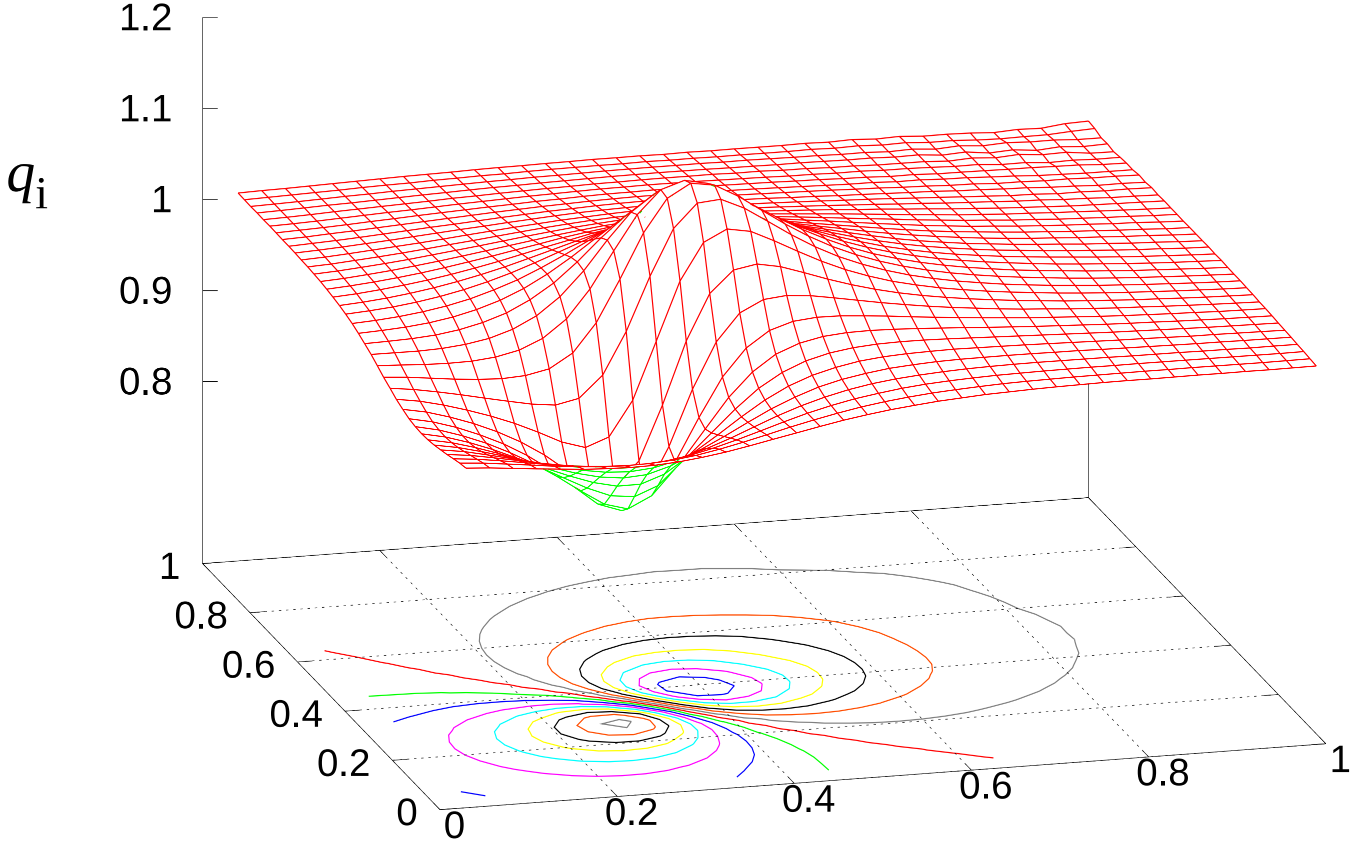
Weight functions for arbitrary characteristic directions:



- Requires addition of numerical dissipation
- Equations solved or time-marched "all at once" (not space-marched)

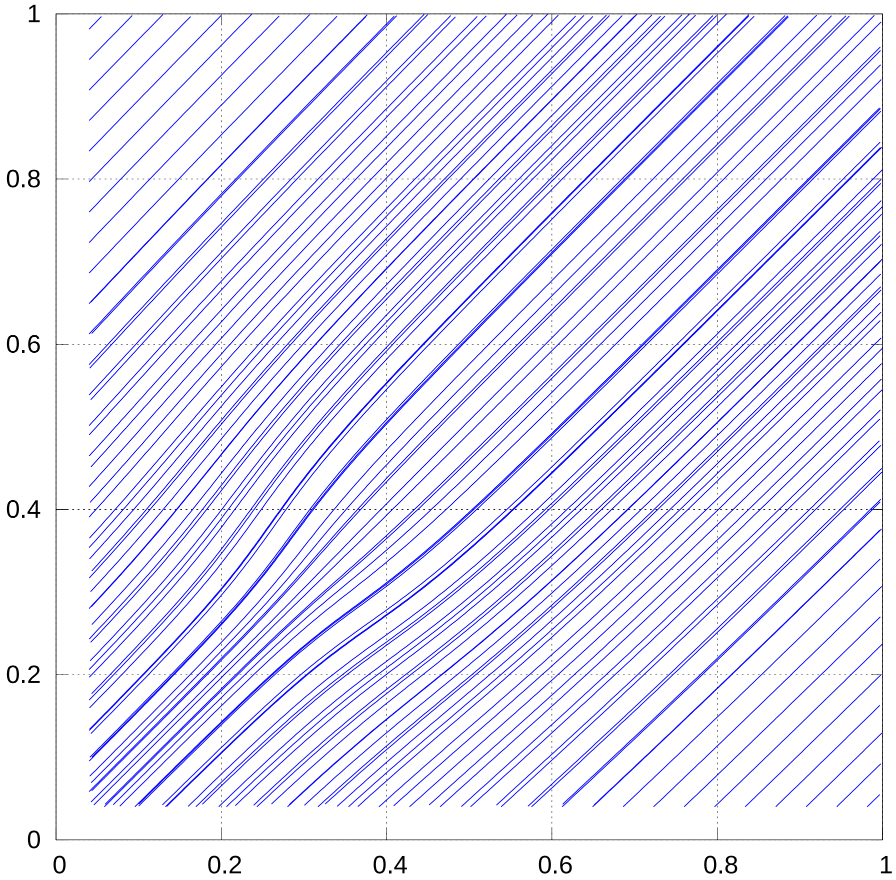
Torpedo Test Case – Separated Flow

- Laminar low speed flow, $Re = 10^5$
- Torpedo sufficiently large to produce separation on wall
- IBL3 strongly coupled with potential flow via wall-source model
- 6400 viscous DOFs + 1600 inviscid DOFs
- Simultaneous Newton solution (quasi-time marching)

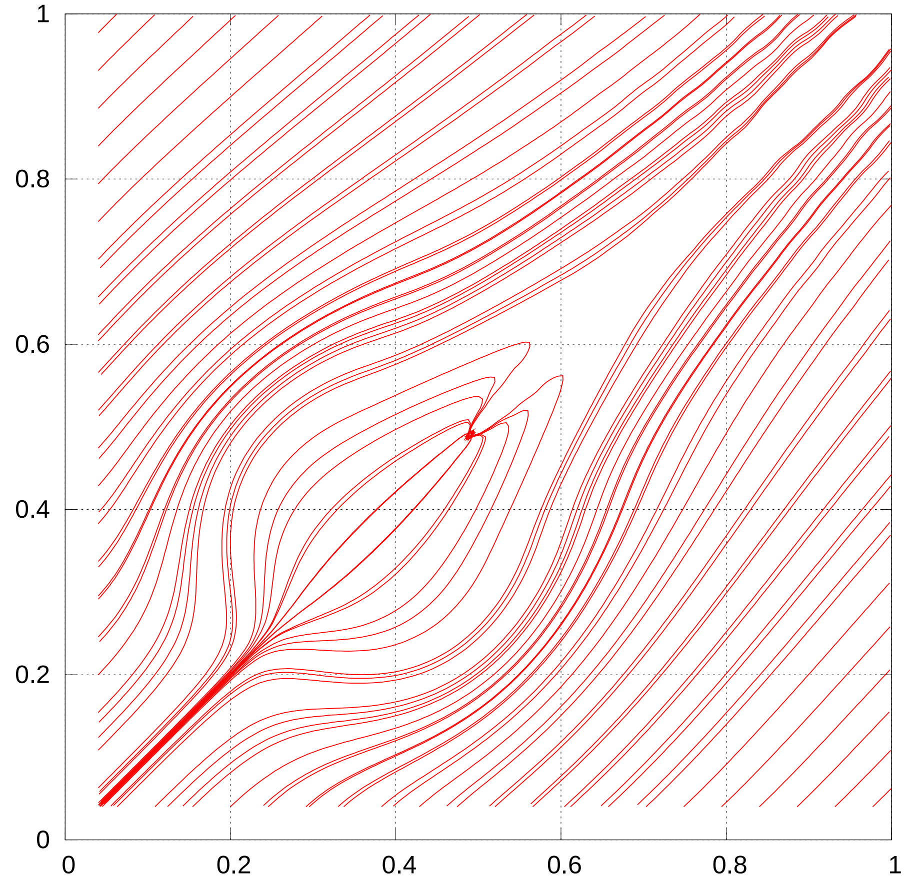


Torpedo Test Case – Separated Flow

EIF streamlines



Skin friction lines



Capturing a closed recirculation zone is unique for a 3D BL method

Unsteady Extension

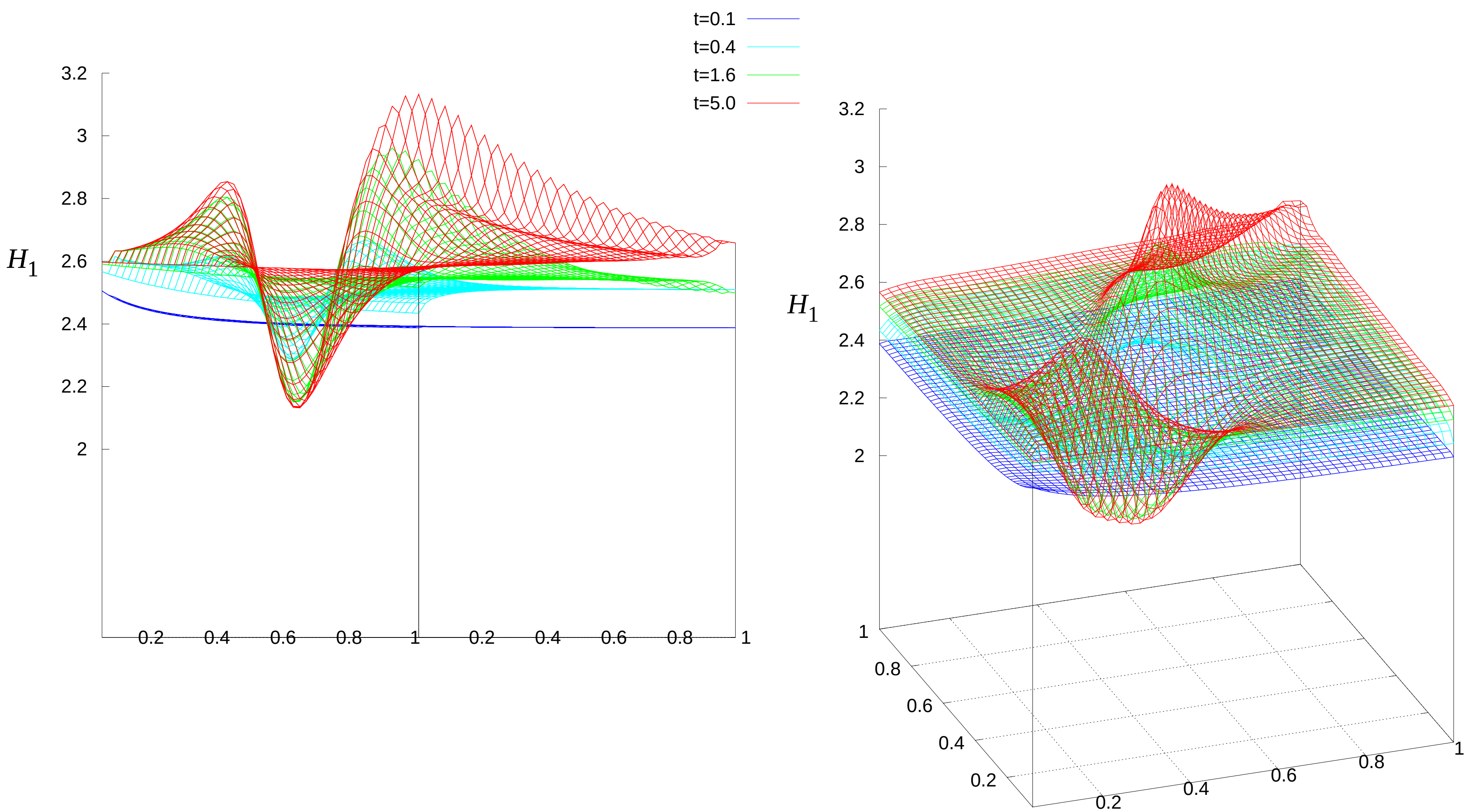
General IBL3 formulation includes . . .

- Unsteady terms, allowing . . .
  - Robust time-marching startup, $\Delta t \rightarrow \infty$ recovers steady solution
  - Time-domain unsteady (nonlinear)
  - Frequency-domain unsteady (linearized)
- Artificial dissipation
  - Necessary to stabilize FEM discretization of hyperbolic IBL equations
  - Captures converging-characteristic "shocks" (separation lines)
  - Conservative — can only <u>redistribute</u> momentum defect (drag)

$$\underline{\frac{\partial \mathbf{M}}{\partial t} - \mathbf{q}_{\mathrm{iw}} \frac{\partial m}{\partial t}} + \widetilde{\nabla} \cdot \left[\bar{\bar{\mathbf{J}}} - \underline{V_\epsilon \, \bar{\bar{\mathbf{h}}} \cdot \widetilde{\nabla} \mathbf{M}} \right] - \mathbf{q}_{\mathrm{iw}} \, \widetilde{\nabla} \cdot \mathbf{M} - \boldsymbol{\tau}_{\mathrm{w}} \; = \; \mathbf{0}$$

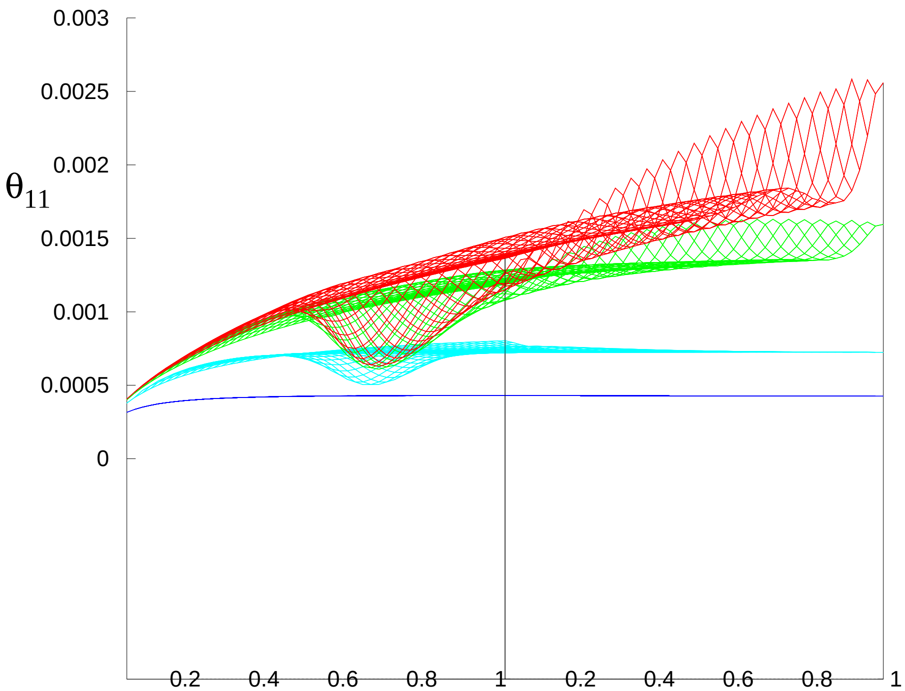
Time-Ramp Test Case – Torpedo Over Wall

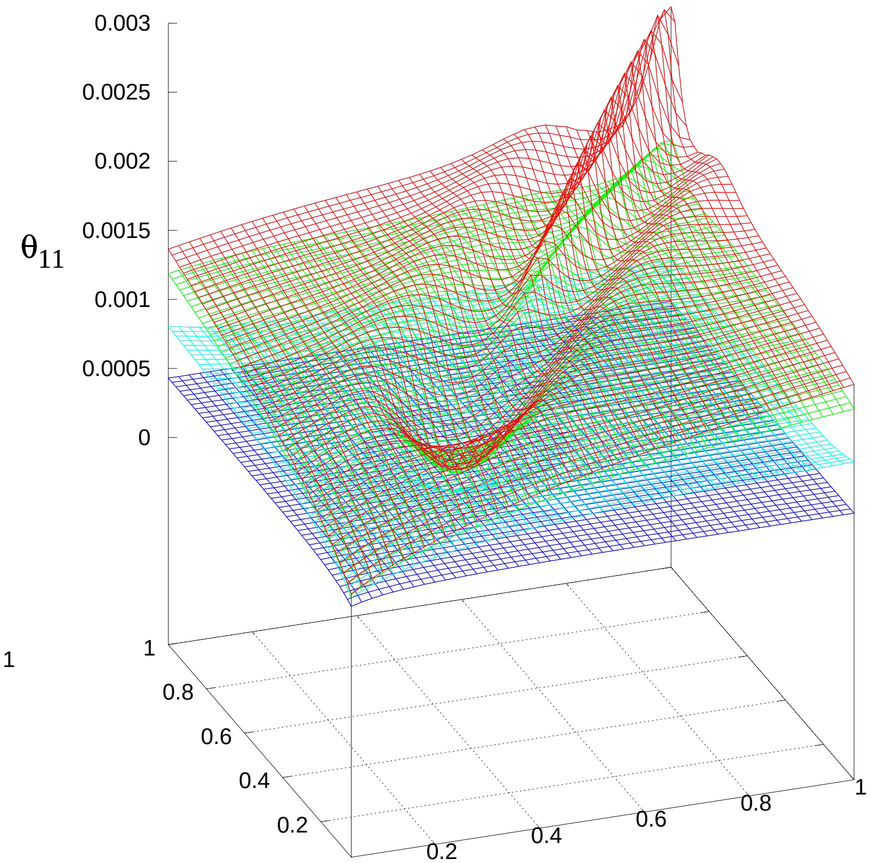
Streamwise shape parameter



Time-Ramp Test Case – Torpedo Over Wall

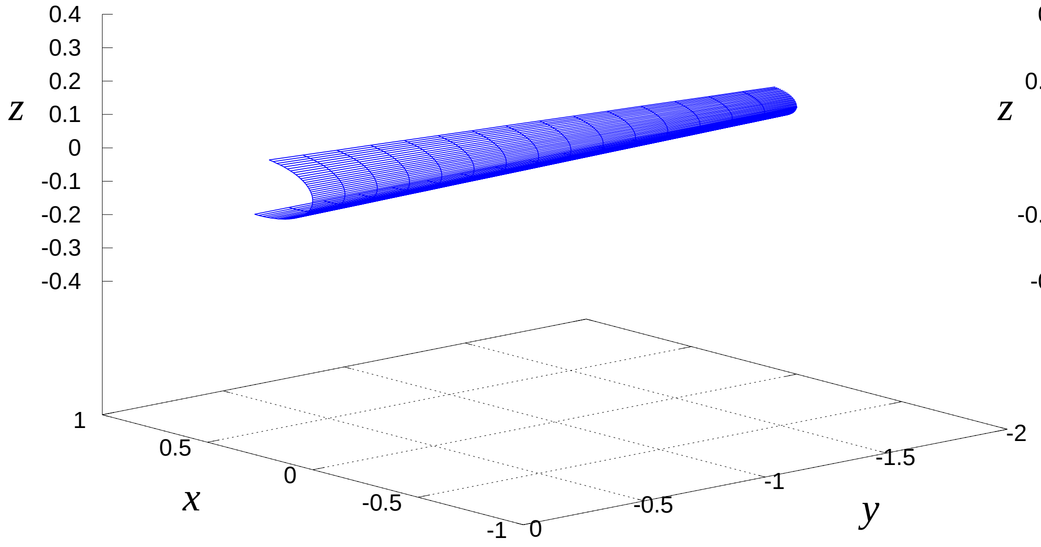
Streamwise momentum thickness



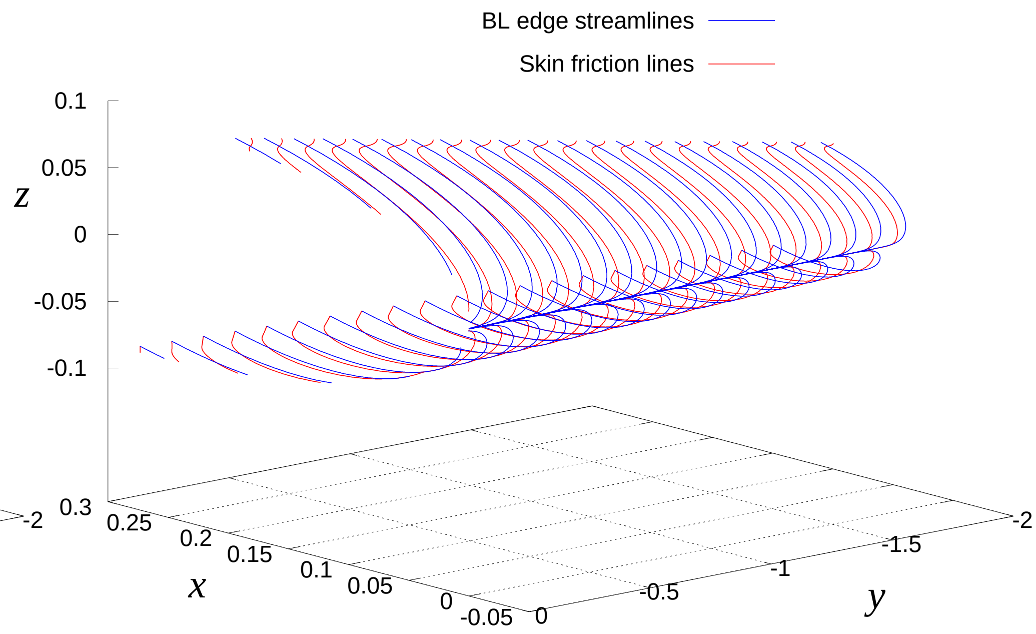


3D-Geometry Test Case – Swept Wing LE

Surface grid



Computed streamlines



- BCs needed only on upstream (inflow) edge
- Attachment line automatically captured within surface grid

Integral BL Work Tasks Underway or Planned

- Integration-based closure for turbulent profiles
- Shear-stress transport for turbulent flows
- Wake treatment
- Large sparse linear system solution
- Implementation into TRANAIR (by Boeing)